ON THE FITTING IDEALS OF A COMULTIPLICATION MODULE

S. KARIMZADEH * AND S. HADJIREZAEI

ABSTRACT. Let $R$ be a commutative ring. In this paper we assert some properties of finitely generated comultiplication modules and Fitting ideals of them.

1. Introduction

Let $R$ be a commutative ring with identity and $M$ be a finitely generated $R$-module. For a set $\{x_1, \ldots, x_n\}$ of generators of $M$ there is an exact sequence $0 \rightarrow N \rightarrow R^n \xrightarrow{\varphi} M \rightarrow 0$ where $R^n$ is a free $R$-module with the set $\{e_1, \ldots, e_n\}$ of basis, the $R$-homomorphism $\varphi$ is defined by $\varphi(e_j) = x_j$ and $N$ is the kernel of $\varphi$. Let $N$ be generated by $u_\lambda = a_{1\lambda}e_1 + \ldots + a_{n\lambda}e_n$, with $\lambda$ in some index set $\Lambda$. Let $\text{Fitt}_i(M)$ be the ideal of $R$ generated by the minors of size $n - i$ of the matrix

$$
\begin{pmatrix}
\cdots & a_{1\lambda} & \cdots \\
\vdots & \vdots & \vdots \\
\cdots & a_{n\lambda} & \cdots
\end{pmatrix}.
$$

For $i > n$, $\text{Fitt}_i(M)$ is defined to be $R$, and for $i < 0$, $\text{Fitt}_i(M)$ is defined to be the zero ideal. It is known that $\text{Fitt}_i(M)$ is an invariant ideal determined by $M$, that is, it is determined uniquely by $M$ and it does not depend on the choice of the set of generators of $M$ [10]. The ideal $\text{Fitt}_i(M)$ will be called the $i$-th Fitting ideal of the module $M$. It follows from the definition of $\text{Fitt}_i(M)$ that $\text{Fitt}_i(M) \subseteq \text{Fitt}_{i+1}(M)$.

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Moreover, it is shown that $\text{Fitt}_0(M) \subseteq \text{Ann}_R(M)$ and $(\text{Ann}_R(M))^n \subseteq \text{Fitt}_0(M)$ ($M$ is generated by $n$ elements) and $\text{Fitt}_i(M)_P = \text{Fitt}_i(M_P)$, for every prime ideal $P$ of $R$ [9]. The most important Fitting ideal of $M$ is the first of the $\text{Fitt}_j(M)$ that is nonzero. We shall denote this Fitting ideal by $I(M)$. Note that if $I(M)$ contains a nonzerodivisor, then $I(M_P) = I(M)_P$ for every prime ideal $P$ of $R$. Fitting ideals are strong tools to identify properties of modules and sometimes to characterize modules. For example Buchsbaum and Eisenbud have shown in [8] that for a finitely generated $R$-module $M$, $I(M) = R$ if and only if $M$ is a projective of constant rank module. A lemma of Lipman asserts that if $R$ is a local ring and $M = R^m/K$ and $I(M)$ is the $(m - q)$th Fitting ideal of $M$, then $I(M)$ is a regular principal ideal if and only if $K$ is finitely generated free and $M/T(M)$ is free of rank $m - q$ ([14]). Finally it is shown in [11] that if $M$ is a finitely generated module over a Noetherian local UFD $(R, P)$, then $I(M) = P$ if and only if

1. $M$ is isomorphic to $R^n/\langle (a_1, \ldots, a_n)^t \rangle$, where $P = \langle a_1, \ldots, a_n \rangle$ and $n$ is a positive integer if $M$ is torsionfree, and
2. $M$ is isomorphic to $R^n \oplus R/P$, for some positive integer $n$ if $M$ is not torsionfree.

An $R$-module $M$ is said to be a comultiplication module if for any submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$ [5]. Ansari-Toroghy and Farshadifar have shown in [5] that an $R$-module $M$ is a comultiplication module if and only if for each submodule $N$ of $M$, $N = (0 :_M \text{Ann}_R(N))$. An $R$-module $M$ satisfies the double annihilator conditions (DAC for short) if for each ideal $I$ of $R$, we have $I = \text{Ann}_R(0 :_M I)$. $M$ is said to be a strong comultiplication module if $M$ is a comultiplication $R$-module which satisfies the double annihilator conditions [4].

1.1. Comultiplication Module.

**Lemma 1.1.** Let $R$ be an integral domain. If $R$ is a comultiplication $R$-module, then $R$ is a field.

**Proof.** Let $I$ be a nonzero ideal of $R$. Since $R$ is a domain, $\text{Ann}_R(I) = 0$. Since $R$ is a comultiplication $R$-module, $I = (0 :_R \text{Ann}_R(I)) = R$. So $R$ is a field. □

**Lemma 1.2.** Let $M$ be a finitely generated comultiplication $R$-module. If there exists a submodule $N$ of $M$ such that $\text{Ann}_R(N) = \text{Ann}_R(M)$, then $N = M$.

**Proof.** [3, Proposition 3.2] □
Theorem 1.3. Let $M$ be a finitely generated comultiplication $R$-module. If $R$ is an integral domain, then $I(M) = \text{Fitt}_0(M)$ or $M \cong R$.

Proof. Let $M$ be generated by $\{x_1, \ldots, x_n\}$. Consider the exact sequence $0 \rightarrow N \rightarrow R^n \xrightarrow{\varphi} M \rightarrow 0$, where $\varphi(e_j) = x_j$ and $N = \text{Ker}(\varphi)$. Let $r_i \in \text{Ann}_R(x_i)$ for $i = 1, \ldots, n$. Consider the matrix

\[
\begin{bmatrix}
  r_1 & 0 & \cdots & 0 \\
  0 & r_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & r_n
\end{bmatrix}
\]

Since each column of this matrix belongs to $N$, so we have

$\text{Ann}_R(x_1) \ldots \text{Ann}_R(x_n) \subseteq \text{Fitt}_0(M)$. If $\text{Ann}_R(x_1) \ldots \text{Ann}_R(x_n) \neq 0$, then $\text{Fitt}_0(M) = I(M)$. If there is an integer $i$, $1 \leq i \leq n$, such that $\text{Ann}_R(x_i) = 0$, then $\text{Ann}_R(M) = \cap_{i=1}^n \text{Ann}_R(x_i) = \text{Ann}_R(x_i) = 0$. Thus by Lemma 1.2, $M = Rx_i \cong R$. \hfill $\Box$

An $R$-module $M$ is said to be a prime module if $\text{Ann}_R(N) = \text{Ann}_R(M)$, for every non-zero submodule $N$ of $M$ [17].

Proposition 1.4. Let $M$ be a finitely generated comultiplication $R$-module. If $M$ is a prime module, then $M$ is a simple $R$-module.

Proof. [3, Proposition 3.18] \hfill $\Box$

Corollary 1.5. Let $M$ be a finitely generated faithful comultiplication $R$-module. If $R$ is an integral domain, then $R$ is a field and $M \cong R$.

Proof. [3, Theorem 3.3] \hfill $\Box$

Proposition 1.6. Let $M$ be a finitely generated comultiplication module over an integral domain $R$. If $I(M)$ is a prime ideal of $R$, then $M$ is a simple $R$-module.

Proof. Let $M = \langle x_1, \ldots, x_n \rangle$. By Theorem 1.3, we have $M \cong R$ or $I(M) = \text{Fitt}_0(M)$. If $M \cong R$, then by Corollary 1.5, $R$ is a field and hence $M$ is simple. If $I(M) = \text{Fitt}_0(M)$, then as the proof of Theorem 1.3, we can conclude that $\text{Ann}_R(x_1) \ldots \text{Ann}_R(x_n) \subseteq \text{Fitt}_0(M)$. So there exists some $x_i$, $1 \leq i \leq n$, such that $\text{Ann}_R(x_i) \subseteq \text{Fitt}_0(M)$. Since $\text{Ann}_R(x_1) \subseteq \text{Fitt}_0(M) \subseteq \text{Ann}_R(M) \subseteq \text{Ann}_R(x_i)$, $I(M) = \text{Ann}_R(M) = \text{Ann}_R(x_i)$. Thus by Lemma 1.2, $M = Rx_i$. Let $N$ be a nonzero submodule of $M$ and $0 \neq n \in N$. So there exists $a \in R$ such that $n = ax_i$. Suppose that $r \in \text{Ann}_R(N)$, $0 = rn = rax_i$. Thus $ra \in \text{Ann}_R(M)$. Since $\text{Ann}_R(M)$ is a prime ideal of $R$, $r \in \text{Ann}_R(M)$ or $a \in \text{Ann}_R(M)$. Since $n$ is a nonzero element of $N$, $r \in \text{Ann}_R(M)$.
Hence $\text{Ann}_R(N) = \text{Ann}_R(M)$. So $M$ is a prime module. Therefore by Proposition 1.4, $M$ is simple.

**Proposition 1.7.** Every finitely generated comultiplication module over a valuation ring is cyclic.

**Proof.** Let $M = \langle x_1, ..., x_n \rangle$. Since $R$ is a valuation ring, there exists a positive integer $i$, $1 \leq i \leq n$, such that $\text{Ann}_R(x_i) \subseteq \text{Ann}_R(x_j)$ for all $1 \leq j \leq n$. Hence $\text{Ann}_R(M) = \text{Ann}_R(x_i)$. Thus by Lemma 1.2, $M = \langle x_i \rangle$.

**Lemma 1.8.** Let $M$ be a finitely generated comultiplication $R$-module. If $R$ is a Dedekind domain, then $M$ is cyclic.

**Proof.** Let $M = \langle x_1, ..., x_n \rangle$ and $P$ be a maximal ideal of $R$. By [1, Corollary 2.6], $M_P$ is a comultiplication module. Since $R_P$ is a valuation ring, by Proposition 1.7, $M_P$ is a cyclic module. So by [6, Proposition 5], $M$ is a multiplication module. By [1, Corollary 2.2], $R/\text{Ann}_R(M)$ is a semi-local ring. Therefore by [6, Proposition 4], $M$ is a cyclic.

**Lemma 1.9.** Let $M$ be a finitely generated comultiplication $R$-module. If $M = \langle x_1, ..., x_n \rangle$ and $\cap_{i=1}^n Rx_i = 0$, then $\text{Fitt}_{n-1}(M) = R$.

**Proof.** Consider the exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$, where $\varphi(e_j) = x_j$ and $N = \text{Ker}(\varphi)$. Let $r_i \in \text{Ann}_R(x_i)$ for $i = 1, ..., n$. Consider the matrix

$$
\begin{bmatrix}
  r_1 & 0 & \cdots & 0 \\
  0 & r_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & r_n
\end{bmatrix}
$$

Since each column of this matrix belongs to $N$, so we have $\Sigma_{i=1}^n \text{Ann}_R(x_i) \subseteq \text{Fitt}_{n-1}(M)$. By [1, lemma 2.3], $\Sigma_{i=1}^n \text{Ann}_R(x_i) = R$. Thus $\text{Fitt}_{n-1}(M) = R$.

**Lemma 1.10.** Let $M$ be a comultiplication $R$-module generated by $n$ elements. If $M$ is a decomposable $R$-module, then $\text{Fitt}_{n-1}(M) = R$.

**Proof.** Let $M$ be a decomposable $R$-module. Since $M$ is a finitely generated decomposable module, there exist finitely generated submodules $N_1, ..., N_k$ of $M$ such that $M = \oplus_{i=1}^k N_i$ and $N_i = \langle x_{i_1}, ..., x_{i_{m_i}} \rangle$, for some elements $x_{ij} \in M$, $1 \leq j \leq m_i$, $1 \leq i \leq k$. By [1, Lemma 2.3], $\Sigma_{i=1}^k \text{Ann}_R(N_i) = R$. So $M = \langle x_{11}, ..., x_{1m_1}, x_{21}, ..., x_{kn_k} \rangle$. Put $n = n_1 + ... + n_k$. Therefore $\Sigma_{i,j} \text{Ann}_R(x_{ij}) \subseteq \text{Fitt}_{n-1}(M)$. Since $\Sigma_{i=1}^k \text{Ann}_R(N_i) \subseteq \Sigma_{i,j} \text{Ann}_R(x_{ij})$, $\text{Fitt}_{n-1}(M) = R$. □
Proposition 1.11. Let $M$ be a decomposable comultiplication module. If $M$ is generated by two elements, then $\text{Fitt}_0(M) = \text{Ann}_R(M)$.

Proof. By Lemma 1.10, $\text{Fitt}_n(M) = R$. By [9, Proposition 20.7], $\text{Ann}_R(M)\text{Fitt}_1(M) \subseteq \text{Fitt}_0(M)$. So $\text{Fitt}_0(M) = \text{Ann}_R(M)$. □

Theorem 1.12. Let $M$ be a finitely generated $R$-module.

(i) If $I(M)$ is a prime ideal of $R$, then $\text{Ann}_R(M) \subseteq I(M)$.

(ii) If $I(M) = Q_1\ldots Q_n$ such that $Q_i$ are distinct maximal ideals of $R$, then $\text{Ann}_R(M) \subseteq I(M)$.

(iii) If $\text{Ann}_R(M) = Q^n$, for some maximal ideal $Q$ of $R$ and positive integer $n$, then $I(M) = R$ or $I(M)$ is a $Q$-primary ideal of $R$.

Proof. Let $M$ be generated by $n$ elements.

(i) By [9, Proposition 20.7], $(\text{Ann}_R(M))^n \subseteq \text{Fitt}_0(M) \subseteq I(M)$. So $\text{Ann}_R(M) \subseteq I(M)$.

(ii) We shall show that $\text{Ann}_R(M) \subseteq Q_i$, for all $i = 1,\ldots,n$. Assume that $\text{Ann}_R(M) \nsubseteq Q_i$, for some $i = 1,\ldots,n$. So $\text{Ann}_R(M) + Q_i = R$. Hence there is a $q \in Q_i$ such that $1-q \in \text{Ann}_R(M)$. By [9, Proposition 20.7], $(\text{Ann}_R(M))^n \subseteq \text{Fitt}_0(M) \subseteq I(M)$. So $(1-q)^n \in I(M) \subseteq Q_i$. Therefore $1-q \in Q_i$, a contradiction. Thus $\text{Ann}_R(M) \subseteq Q_1\cap\ldots\cap Q_n = Q_1\ldots Q_n = I(M)$.

(iii) By [9, Proposition 20.7], $(\text{Ann}_R(M))^m = Q^m \subseteq \text{Fitt}_0(M) \subseteq I(M)$. So $Q \subseteq \sqrt{I(M)}$. Since $Q$ is a maximal ideal of $R$, $\sqrt{I(M)} = Q$ or $\sqrt{I(M)} = R$. This implies that $I(M)$ is a primary ideal or $I(M) = R$. □

Proposition 1.13. Let $M$ be a comultiplication $R$-module. If $M$ is a decomposable module and $M = \langle x_1,\ldots,x_n \rangle$, then $(\text{Ann}_R(M))^{n-1} \subseteq \text{Fitt}_0(M)$.

Proof. By Lemma 1.10, $\text{Fitt}_{n-1}(M) = R$. Hence by [9, Proposition 20.7], $(\text{Ann}_R(M))^{n-1} \subseteq \text{Fitt}_0(M)$. □

Theorem 1.14. Let $M$ be a finitely generated comultiplication module. If $\text{Ann}_R(M) = Q_1\ldots Q_n$, where $Q_i$, $1 \leq i \leq n$, are distinct maximal ideals of $R$, then $M \cong R/Q_1 \oplus \ldots \oplus R/Q_n$.

Proof. Assume that $(0 : Q_j) = 0$, for some $1 \leq j \leq n$. By [1, Lemma 2.1], $Q_j M = M$. So $\prod_{i \neq j} Q_i M = \prod_{i = 1}^n Q_i M = 0$. Hence $\prod_{i \neq j} Q_i \subseteq \text{Ann}_R(M) \subseteq Q_j$ which is a contradiction since $Q_1,\ldots,Q_n$ are distinct maximal ideals of $R$. Therefore by [1, Lemma 2.1], $(0 :_M Q_i)$ is a simple module for all $i = 1,\ldots,n$. Hence for all $i = 1,\ldots,n$ there exists $x_i \in M$ such that $(0 :_M Q_i) = R x_i$. We shall show that $R x_j \cap \sum_{i \neq j} R x_i = 0$ for all
Assume that \( Rx_j \cap \Sigma_{i=1}^{j} Rx_i \neq 0 \) for some \( j = 1, ..., n \). Since \( Rx_j \) is simple, \( Rx_j \cap \Sigma_{i=1}^{j} Rx_i = Rx_j \). So, \( Rx_j \subseteq \Sigma_{i=1}^{j} Rx_i \) and hence \( Q_j = \text{Ann}_R(Rx_j) \supseteq \text{Ann}_R(\Sigma_{i=1}^{j} Rx_i) \supseteq \Pi_{i=1}^{n} Q_i \). This implies that there exists \( i \neq j, 1 \leq i \leq n \), such that \( Q_i = Q_j \) which is a contradiction since \( Q_1, ..., Q_n \) are distinct maximal ideals of \( R \). Therefore \( N = Rx_1 \oplus ... \oplus Rx_n \) is a submodule of \( M \) and \( \text{Ann}_R(N) = \text{Ann}_R(M) \). Since \( M \) is a comultiplication module, by Lemma 1.2, \( N = M \). Hence \( M = Rx_1 \oplus ... \oplus Rx_n \cong R/Q_1 \oplus ... \oplus R/Q_n \). □

Von Neumann regular ring is a ring \( R \) such that for every \( a \in R \) there exists an element \( b \in R \) such that \( a = aba \).

**Proposition 1.15.** Let \( M \) be a finitely generated comultiplication \( R \)-module. If \( R \) is a von Neumann regular ring, then \( I(M) = Q_1...Q_n \), where \( Q_i \) are maximal ideals of \( R, 1 \leq i \leq n \).

**Proof.** By [2, Corollary 1.7], \( M \) is a semisimple module. Hence there exist maximal ideals \( Q_1, ..., Q_n \) of \( R \) such that \( M \cong R/Q_1 \oplus ... \oplus R/Q_n \). So \( I(M) = Q_1...Q_n \). □

**Theorem 1.16.** Let \( M \) be a finitely generated comultiplication module. If \( \text{Fitt}_0(M) = Q_1...Q_n \), where \( Q_i, 1 \leq i \leq n \), are distinct maximal ideals of \( R \), then \( M \) is a semisimple module.

**Proof.** Similar to the proof of Theorem 1.14, \( (0 :_M Q_i) \) are simple modules for all \( i = 1, ..., n \) and for all \( 1 \leq j \leq n \), \( Rx_j \cap \Sigma_{i \neq j} Rx_i = 0 \), where \( Rx_i = (0 :_M Q_i) \) for all \( 1 \leq i \leq n \). Put \( N = Rx_1 \oplus ... \oplus Rx_n \). We have \( Q_1...Q_n \subseteq \text{Ann}_R(M) \subseteq \text{Ann}_R(N) = Q_1...Q_n \). Thus by Lemma 1.2, \( M = N \). □

**Lemma 1.17.** Let \( M \) be a finitely generated module. If \( \text{Ann}_R(M) = \langle e \rangle \), where \( e \) is a non-zero idempotent element of \( R \), then \( I(M) = \text{Ann}_R(M) \).

**Proof.** Let \( M = \langle x_1, ..., x_n \rangle \). By [9, Proposition 20.7], \( (\text{Ann}_R(M))^n \subseteq \text{Fitt}_0(M) \). So \( e = e^n \in \text{Fitt}_0(M) \). Hence \( \text{Fitt}_0(M) = \langle e \rangle \). □

**Theorem 1.18.** Let \( M \) be a finitely generated comultiplication module. If there is a submodule \( N \) of \( M \) such that \( \text{Ann}_R(N) = \langle e \rangle \), where \( e \) is an idempotent element of \( R \), then \( N \) is a direct summand of \( M \) and \( I(M) \subseteq \langle e \rangle \).

**Proof.** Assume that \( N \) is a proper submodule of \( M \) and \( \text{Ann}_R(N) = \langle e \rangle \). Put \( K = \{ m \in M : (1-re)m = 0 \text{ for some } r \in R \} \). It is clear that \( K \) is a submodule of \( M \) and \( N \cap K = 0 \). For \( m \in M \) we have
\[ m = (1 - e)m + em \text{ and } em \in K, \ (1 - e)m \in N. \text{ So } M = N \oplus K. \]

By \[7, \text{ p.174}, \ I(M) = I(N)I(K) \text{ and by Lemma 1.17, } I(N) = \langle e \rangle, \text{ so } I(M) \subseteq \langle e \rangle. \]

**Corollary 1.19.** Let \( M \) be a finitely generated strong comultiplication \( R \)-module. If \( e \) is an idempotent element of \( R \), then \( e \in \text{Ann}_R(M) \) or \( 1 - e \in \text{Ann}_R(M) \).

**Proof.** Let \( e \) be an idempotent element of \( R \). If \( (0 :_M e) = M \), then \( e \in \text{Ann}_R(M) \). If \( (0 :_M e) = 0 \), then \( (0 :_M 1 - e) = M \). Hence \( 1 - e \in \text{Ann}_R(M) \). If \( (0 :_M e) \) is neither \( M \) nor \( 0 \), then by Theorem 1.18, \( M = (0 :_M e) \oplus L \) where \( L = \{ m : (1-re)m = 0 \text{ for some } r \in R \} \). If \( m \in L \), then there is some \( r \in R \) such that \( m = rem \). So \( em = r(e)^2m = rem = m \). Hence \( 1 - e \in \text{Ann}_R(L) \). Thus \( L \subseteq (0 :_M 1 - e) \). It's clear that \( (0 :_M 1 - e) \subseteq L \). Since \( M \) is a strong comultiplication, \( \text{Ann}_R(L) = \langle 1-e \rangle \) and \( \text{Ann}_R(0 :_M e) = \langle e \rangle \). By Theorem 1.18, \( I(M) \subseteq \langle e \rangle \) and \( I(M) \subseteq \langle 1 - e \rangle \). So \( I(M) = 0 \) and it's contradiction. \( \square \)

**Proposition 1.20.** Let \( M \) be a finitely generated module over a Prüfer domain \( R \) and \( Q \) be a maximal ideal of \( R \). Then \( \text{Ann}_R(M) = Q^n \), for some positive integer \( n \) if and only if \( \text{Fitt}_0(M) = Q^k \), for some \( k \in \mathbb{N} \).

**Proof.** Let \( M \) be generated by \( m \) elements. By \[9, \text{ Proposition 20.7}, \ Q^{nm} = \text{Ann}_R(M)^m \subseteq \text{Fitt}_0(M) \). So \( \text{Fitt}_0(M) \) is a \( Q \)-primary ideal of \( R \). By \[13, \text{ Proposition 6.9}, \] there exists some \( k \in \mathbb{N} \) such that \( \text{Fitt}_0(M) = Q^k \). Hence \( \text{Fitt}_0(M) = Q^k \). Conversely, suppose that \( \text{Fitt}_0(M) = Q^k \), for some \( k \in \mathbb{N} \). Since \( Q^k = \text{Fitt}_0(M) \subseteq \text{Ann}_R(M) \), \( \text{Ann}_R(M) \) is a \( Q \)-primary ideal of \( R \). Hence By \[13, \text{ Proposition 6.9}, \] there exists some \( n \in \mathbb{N} \) such that \( \text{Ann}_R(M) = Q^n \). \( \square \)

**Theorem 1.21.** Let \( M \) be a finitely generated comultiplication module over a Prüfer domain \( R \). If \( \text{Fitt}_0(M) = Q^n \), where \( Q \) is a maximal ideal of \( R \) and \( n \) is a positive integer, then \( M \) is cyclic.

**Proof.** Let \( M = \langle x_1, ..., x_n \rangle \). Since \( Q^n = \text{Fitt}_0(M) \subseteq \text{Ann}_R(M) \subseteq \text{Ann}_R(x_i) \), \( \text{Ann}_R(x_i) \) is a \( Q \)-primary ideal of \( R \). By \[13, \text{ Proposition 6.9}, \] there exist some \( k_i \in \mathbb{N}, \) \( 1 \leq i \leq n, \) such that \( \text{Ann}_R(x_i) = Q^{k_i} \). Put \( k = \max\{k_1, ..., k_n\} \). Let \( \text{Ann}_R(x_j) = Q^k \), for some \( 1 \leq j \leq n \). So \( \text{Ann}_R(x_j) \subseteq \text{Ann}_R(x_i) \) for all \( 1 \leq i \leq n \). Hence \( Rx_i \subseteq Rx_j \), for all \( 1 \leq i \leq n \). This implies that \( M = Rx_j \). \( \square \)

**Theorem 1.22.** Let \( M \) be a finitely generated comultiplication module. Then \( R/\text{Fitt}_0(M) \) is a semilocal ring.

**Proof.** Let \( M \) be generated by \( n \) elements. By \[9, \text{ Proposition 20.7}, \] \( \text{Ann}_R(M)^n \subseteq \text{Fitt}_0(M) \). If \( Q \) is a maximal ideal of \( R \) such that
Fitt_0(M) \subseteq Q$, then $\text{Ann}_R(M) \subseteq Q$. Since $R/\text{Ann}_R(M)$ is a semilocal ring, $R/Fitt_0(M)$ is a semilocal ring. \hfill \square

**Corollary 1.23.** Let $M$ be a finitely generated comultiplicatin module. If $R$ is not a semilocal ring, then $I(M) = \text{Fit}t_0(M)$.

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