Journal of Algebra and Related Topics
Vol. 3, No 2, (2015), pp 17-25

Ω-ALMOST BOOLEAN RINGS

M.PHANI KRISHNA KISHORE

ABSTRACT. In this paper the concept of an Ω - Almost Boolean ring is introduced and illistrated how a sheaf of algebras can be constructed from an $\Omega-$ Almost Boolean ring over a locally Boolean space.

1. INTRODUCTION

Ever since Dauns and Hoffmann [2] exhibited representation of biregular rings by sheaves, several algebraists paid attention to the representation of algebraic structures by sheaves of suitable algebras over suitable topological spaces. The works of Pierce.R.S [4], Subrahmanyam.N.V [5], Comer.S.D [1], Davey.B.A [3], Wolf.A [10], Swamy. U.M [6] thrown much light on the theory of representations of algebras by sheaves. In particular, Subrahmanyam.N.V [5], Comer.S.D[1], Swamy.U.M [6] concentrated on sheaves of algebras over (locally) compact, hausdorff, and totally disconnected spaces, which are called (locally)Boolean spaces. Swamy.U.M and Rao.G.C [7] introduced the concept of an Almost Boolean Ring and observed Stone like correspondence with Almost Distributive Lattices(ADLs). Later, Swamy.U.M and Kishore.M.P.K [8] studied the prime ideal spectrum of an Almost Boolean Ring(ABR) and observed that the class of all prime ideals together with hull-kernel topology forms a locally Boolean space. Swamy.U.M et.al., [9] characterized the class of Almost Boolean Rings by sheaves of sets over locally Boolean spaces. In a quest to find equivalent characterization for sheaves of algebras, the concept of a

MSC(2010): Primary: 06E99; Secondary: 18F20
Keywords: Almost Boolean Rings, Sheaves over locally Boolean spaces, Boolean spaces.
Received: 21 September 2015, Accepted: 30 December 2015.
∗Corresponding author .

Ω-Almost Boolean rings is introduced here and observed the equivalence between these two classes. An Almost Boolean ring (ABR) R is defined as a (2,2,0) type algebraic structure that satisfies the conditions of a Boolean ring except for the associativity of addition. Instead, it satisfies (x +(y + z)). t = ((x + y) + z).t, for x,y,z,t in R. As a consequence several properties were observed [8].

The annihilator ideals and prime ideals of an ABR are defined analogous to those of a ring. It is also observed that the set X of all prime ideals of an ABR R together with the hull-kernel topology, forms a locally Boolean space in which the sets of the form $X_a = \{P \in X | a \notin P\}$ for some $a \in R$, is a base [8]. A sheaf is a triple,(S, π, X) where S and X are topological spaces and π is a surjective local homeomorphism of S onto X. For $Y \subseteq X$, a section on Y is a continuous map $f : Y \to S$ such that $\pi \circ f = Id_Y$. It can be observed that if f and g are sections on $Y(\subseteq X)$ and f (p) = g(p) for some $p \in Y$, then there exists an open set W in Y containing p such that
$f|_W = g|_W$. The class { f(U) | U is a basic open setin X and f is a section on U } is a base for the topology on S.

Any section on X is called a global section. The sheaf (S, π, X) is called a global sheaf if every element of the sheaf space S is in the image of some global section. A sheaf of algebras is a sheaf (S, π, X) in which for each
$p \in X$, the stalk S_p is an algebra and for each $\sigma \in \Omega_n$,the map
$(s_1, s_2, ..., s_n) \to \sigma(s_1, s_2, ..., s_n)$ of $S^{(n)}$ into S is continuous where,

$$S^{(n)} = \{(s_1, s_2, ..., s_n) \in S_n | \pi(s_1) = \pi(s_2) = ... = \pi(s_n)\}.$$

Suppose (S, π, X) is a sheaf and for each $p \in X$, stalk S_p is an algebra. Then (S, π, X) is a sheaf of algebras if and only if for each open set $U \subseteq X$ the set $\Gamma(U, S)$ of all sections on U is an Ω-algebra, in which for any n-ary operation $\sigma \in \Omega_n,\ f_1, f_2, ..., f_n \in \Gamma(U, S),\ \sigma(f_1, f_2, ..., f_n)(p)$ is defined point wise. That is $(f_1, f_2, ..., f_n)(p) = (f_1(p), f_2(p), ..., f_n(p))$. Two sheaves (S, π, X) and (T, η, Y) of Ω-algebras are said to be isomorphic if there exists homeomorphisms $\alpha : Y \to X$ and $\beta : T \to S$ such that $\pi \circ \beta = \alpha \circ \eta$ and for any $q \in Y$ and $p \in X$ such that $\alpha(q) = p,\ \beta|_{T_q} = T_q \to S_p$ is a Ω-isomorphism. That is the diagram,

$$\begin{array}{ccc} T & \xrightarrow{\beta} & S \\ \eta \downarrow & & \downarrow \pi \\ Y & \xrightarrow[\alpha]{} & X \end{array}$$

is commutative.

2. Ω-ALMOST BOOLEAN RING

Definition 2.1. An algebra (R, +, ., 0,Ω), where Ω is a set of finitary operational symbols different from +, and . on R, is called an Ω -Almost Boolean Ring if
(R, +, . , 0) is an ABR and for any n-ary $\sigma \in \Omega, x_1, x_2, ..., x_n$ and $a \in R$,
I. (1). $\sigma(x_1, x_2, ..., x_i + a, ..., x_n)$
$= \sigma(x_1, x_2, ..., x_i, ..., x_n) + \sigma(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n)$
(2). $a\sigma(x_1, x_2, ..., x_i, ..., x_n) = \sigma(x_1, x_2, ..., ax_i, ..., x_n)$
for all $1 \leq i \leq n$.

II. $\sigma(x_1, x_2, ..., x_n)^* = \sum_{i=1}^{n} x_i^* (n \geq 1)$, where
$\sum_{i=1}^{n} x_i^* = \{\sum_{i=1}^{n} a_i | a_i \in x_i^*\}$ and x_i^* is an annihilator of x_i.

Example 2.2. Consider the Real number system with the usual multiplication (*) and define + and . on R by

$$x + y = \begin{cases} x, \ if \ y = 0 \\ y, \ if \ x = 0 \\ 0 \ otherwise \end{cases} \quad and \ x.y = \begin{cases} 0, \ if \ x = 0 \\ y, \ if \ x \neq 0 \end{cases}$$

it can easily observed that $(R+, ., \Omega)$ is an Ω- Almost Boolean ring, where $\Omega = \{*\}$.

Lemma 2.3. *Let R be an Ω-ABR and* $\sigma \in \Omega_n, x_1, x_2, ..., x_n \in R$. *Then the following hold.*
1.If $x_i = 0$ *for some i, then* $\sigma(x_1, x_2, ..., x_n) = 0$.
2. ${x_i}^* \subseteq \sigma(x_1, x_2, ..., x_n)^*$ *for all i.*
3.$\sum_{i=1}^{n} x_i^* \subseteq \sigma(x_1, x_2, ..., x_n)^*$.
The ideals and prime ideals of an Ω-ABR are defined same as that of the underlying ABR and hence the set of all prime ideals of an ABR together with hull-kernel topology forms a locally Boolean space.

Proof. Follows easily from Definition 2.1. □

Lemma 2.4. *Let R be a Ω-ABR. For any* $x \in R$, $xR + x^* = R$.

Proof. Clearly $xR + x^* \subseteq R$.
Let $a \in R$. We have $x \in xR$ and $(a + ax)x = ax + ax = 0$
and hence $a + ax \in x^*$, so that $x + (a + ax) \in xR + x^*$.
It can be observed that,
$a = a + 0$
$= a + (xa + xa)$
$= aa + (xa + axa)$
$= (a + (x + ax))a$
$= (x + (a + ax))a \in xR + x^*$ (since $xR + x^*$ is an ideal of R)
Therefore $R \subseteq xR + x^*$
Thus $xR + x^* = R$ □

Lemma 2.5. *Let P be a prime ideal of an Ω-ABR and let $x \in R$ then $x^* \subseteq P$ iff $x \notin P$.*
Proof. *Suppose $x^* \subseteq P$. Since $xR + x^* = R$, $xR \nsubseteq P$, and as a consequence $x \notin P$. Conversely suppose $x \notin P$, then for $a \in R$, ax=0 implies $a \in P$ and hence $x^* \subseteq P$.*

Lemma 2.6. *Let R be an ABR together with an algebraic structure. Then, for any $x_1, x_2, ..., x_n \in R$ and $\sigma \in \Omega_n$ and the following are equivalent:*
1 $\sigma(x_1, x_2, ..., x_n)^* = \sum_{i=1}^{n} x_i^*$
2. For any prime ideal P of the ABR, R, $\sigma(x_1, x_2, ..., x_n) \in P$ if and only if $x_i \in P$ for some i.

Proof. Suppose (1) holds. Let P be any prime ideal of the ABR R then
$\sigma(x_1, x_2, ..., x_n) \in P \Leftrightarrow \sigma(x_1, x_2, ..., x_n)^* \notin$ P (by Lemma 2.5)
$\Leftrightarrow \sum_{i=1}^{n} {x_i}^* \notin P$ by (1)
$\Leftrightarrow {x_i}^* \notin P$ for some i
$\Leftrightarrow x_i \in P$ for some i
Conversely suppose (2) holds. Then for any prime ideal of P of the ABR R, we have
$\sigma(x_1, x_2, ..., x_n) \subseteq P \Leftrightarrow \sigma(x_1, x_2, ..., x_n) \notin P$
$\Leftrightarrow x_i \notin P$ for all i by (2)
$\Leftrightarrow {x_i}^* \subseteq P$ for all i
$\Leftrightarrow \sum {x_i}^* \subseteq P$ (by the properties of ideals)
□

3. SHEAF OF Ω-ALGEBRAS FROM A GIVEN Ω-ALMOST BOOLEAN RING

Swamy, U.M [6] gave a general construction of global sheaf from a given topological space X and a non empty set A. On the same lines sheaf of algebras can be constructed from the given ABR. The following observations can be made.

Lemma 3.1. *Let X denote the set of all prime ideals of an ABR R. For any $P \in X$, define $\phi_p = \{(x, y) \in R \times R | ax = ay \ for \ some \ a \in R - P\}$. Then ϕ_p is a congruence relation on the Ω-ABR R.*

Lemma 3.2. *Let P be a Prime ideal of R and ϕ_P be the congruence defined as in Lemma 3.1. Then $[\phi_P(x)]^* = \phi_P(x^*)$. Where $\phi_P(x^*) = \{\phi_p(a) | a \in x^*\}$ and $(\phi_p(x))^* = \{\phi_p(a) \in R/\phi_p | \phi_p(a)\phi_p(x) = \phi_p(0)\}$.*

Definition 3.3. Let R be a non empty set. Designate an arbitrary element as 0. Define the binary operations +, '.' by,

$$x + y = \begin{cases} x, \ if \ y = 0 \\ y, \ if \ x = 0 \\ 0 \ otherwise \end{cases} \quad and \ x.y = \begin{cases} 0, \ if \ x = 0 \\ y, \ if \ x \neq 0 \end{cases} \quad \text{for any x, y in}$$

R. Then (R,+,.,0) satisfies the conditions of an Almost Boolean Ring and is defined as a discrete ABR.

Definition 3.4. An Ω-ABR is said to be discrete Ω-ABR if the underlying ABR is discrete.

Lemma 3.5. *Let P be a Prime ideal of R and ϕ_p be the congruence relation defined as in Lemma 3.1. Then R/ϕ_P is a discrete Ω-ABR together with the induced operation of +, . and Ω operations.*

Theorem 3.6. *Let X be a topological space A be any non empty set. Let $p \mapsto \phi_p$ be a mapping of X into the set $\xi(A)$, of all equivalence relations on A. Let $S_p = A/\theta_p$ and $S = \bigcup^{+}_{p \in X} S_p$ the disjoint union of $S_p^{\prime}s$. For any $a \in A$, define $\hat{a} : X \to S$ by $\hat{a}(p) = \theta_p(a)$. Equip S with the largest topology with respect to which each $\hat{a}$ is continuous. Define $\pi : S \to X$ by $\pi(s) = p$ if $s \in S_p$. Then (S, π, X) is a global sheaf if and only if, for any $a, b \in A$, the set $\langle a, b \rangle = \{p \in X | (a, b) \in \theta_p\}$ is open in X.*

Note: The above theorem is restatement of similar theorm which is given in terms of congruences in [6]. However for the sake of completeness proof is given here for the construction of global sheaf of sets.

Proof. Let (S,π,X) be a global sheaf. First we prove that for $a \in$ A, $\hat{a}$ is a global section. Continuity of $\hat{a}$ is clear from the definition. Also $\pi \circ \hat{a}(p) = \pi(\eta_p(a)) = p$ for all p $\in X$. Therefore $\pi \circ \hat{a}$ is the identity and hence $\hat{a}$ is a global section.
Now we claim that X(a,b) is open in X. Let $p \in X(a,b)$ that is, $p \in X$ and $\hat{a}(p) = \hat{b}(p)$ (=s say), $s \in S$. By the definition of sheaf there exists open sets G and U in S and X respectively such that $s \in G$ and $\pi|_G : G \to U$ is a homeomorphism. Observe that $\pi(s) = \pi(\hat{a}(p)) = p, p \in U$. Now take
V= $\hat{a}^{-1}(G) \cap \hat{b}^{-1}(G) \cap U$. Since $\hat{a}, \hat{b}$, are continuous and U is open, it follows that V is open in X and $p \in V$. Now for any $q \in V$, $\hat{a}(q)$,$\hat{b}(q) \in G$ and $\pi(\hat{a}(q)) = \pi(\hat{b}(q))$. From the fact that $\pi|_G$ is one-one map, it follows that $\hat{a}(q)) = \hat{b}(q)$. Therefore $q \in X(a,b)$ and hence X(a,b) is open.
Conversely assume that X(a,b) is open in X. We now prove that (S,π,X) is a global sheaf. Let $s \in S$, then there exists $p \in X$, $a \in A$ such that $s \in \eta_p(a)$. Now since $\eta_p(a) = \hat{a}(p)$, $\hat{a}(p) \in \hat{a}(X)$ it follows that $s \in \hat{a}(X)$.
We now prove that $\pi|_{\hat{a}(X)} : \hat{a}(X) \to X$ is a homeomorphism.
Suppose, $\pi|_{\hat{a}(X)}(\eta_p(a)) = \pi|_{\hat{a}(X)}(\eta_q(a))$, by the definition of π, it follows that p =q. Thus $\eta_p(a) = \eta_q(a)$ and hence $\pi|_{\hat{a}(X)}$ is one-one.
Given $p \in X$, observe that $\pi|_{\hat{a}(X)}(\eta_p(a)) = p$ for $a \in A, \eta_p(a) \in \hat{a}(X)$. Therefore $\pi|_{\hat{a}(X)}$ is onto. Let U be open in X and $s \in (\pi|_{\hat{a}(X)})^{-1}(U)$. Then $\pi|_{\hat{a}(X)}(s) \in U$. Now since $s \in S_p$ for some p, there exists $a \in A$ such that $s = \eta_p(a)$ and hence $\pi|_{\hat{a}(X)}(\eta_p(a)) \in U$. Since $\pi|_{\hat{a}(X)}(\eta_p(a)) = p$, it follows that $p \in U$, clearly $\hat{a}(p) \in \hat{a}(U)$. From the fact that $\hat{a}$ is an open map, it is clear that $\hat{a}(U)$ is open in S.
Let $s' \in \hat{a}(U)$, then $s' = \hat{a}(q)(= \eta_q(a))$ for some $q \in U$. It can be observed that $\pi|_{\hat{a}(X)}(\eta_p(a)) \in U$ and hence $s' = \eta_q(a) \in (\pi|_{\hat{a}(X)})^{-1}(U)$. Thus $\hat{a}(U) \subseteq \pi|_{\hat{a}(X)}(U)$ and hence$\pi|_{\hat{a}(X)}$ is continuous.
Let H be an open set in $\hat{a}(X)$. By subspace toplogy induced by S, there exists an open set G in S such that $H = \hat{a}(X) \cap G$. Let $s \in H$, then there exists $q \in X$ such that $s = \hat{a}(q)(= \eta_q(a)), s \in G$. Since $q \in \hat{a}^{-1}(G)$, consider $W = \hat{a}^{-1}(G) \cap X$. Clearly $q \in W$, W is open in X. Now let $p \in W$, that is $p \in \hat{a}^{-1}(G) \cap X$, then $\hat{a}(p) \in G$ and since $\hat{a}(p) \in \hat{a}(X)$, it follows that $\hat{a}(p) \in \hat{a}(X) \cap G = H$. $p = \pi|_{\hat{a}(X)}(\hat{a}(p)) \in \pi|_{\hat{a}(X)}(H)$. Thus $\pi|_{\hat{a}(X)}$ is an open map. □

Lemma 3.7. *Let R be an Ω-ABR and X be the spectrum of R, that is, the topological space of all prime ideals of R together with the hull-kernel topology. Then, for any* $x, y \in R$, *the set* $(x,y) = \{P \in X | (x,y) \in \phi_P\}$ *in an open set in X.*

Proof. Let $P \in< x, y >$. Then $(x, y) \in \phi_P$ that is ax= ay for some $a \notin P$, so that $P \in X_a$. Now for $Q \in X_a, a \notin Q$ and ax = ay and hence $Q \in (x, y)$. Thus $P \in X_a \subseteq< x, y >$ and hence $< x, y >$ is an open set in X. □

Theorem 3.8. *Let X be the set of all prime ideals of an Ω-ABR R. For $P \in X$, let $S_P = R/\phi_P$. Consider $S = \bigcup^{+}_{P\in X} S_P$ the disjoint union of S_P s. For $x \in R$, define $\hat{x} : X \to S$ by $\hat{x}(P) = \phi_P(x)$ and equip S with the largest topology with respect to which each $\hat{x}$ is continuous. Define $\pi : S \to X$ by $\pi(s) = P$ for all $s \in S_P$ then (S, π, X) is a sheaf of Ω-algebras.*

Proof. By Theorem 3.6 and Lemma 3.7, (S, π, X) is a sheaf of sets. Each stalk $S_P = R/\phi_P$ is an Ω-algebra. Therefore it is enough to show that Ω-operations are continuous, that is, for each $\sigma \in \Omega_n$ then the map
$(s_1, s_2, ..., s_n) \mapsto \sigma(s_1, s_2, ..., s_n)$ of $S^{(n)}$ into S is continuous. Where,

$$S^{(n)} = \{(s_1, s_2, ..., s_n) \in S^n | \pi(S_1) = \pi(S_2) = ... = \pi(S_n)\}.$$

Let $(s_1, s_2, ..., s_n) \in S^{(n)}$. Then there exists $x_1, x_2, ..., x_n \in R$ such that

$s_i = \phi_p(x_i)$ $(1 \leq i \leq n)$ for some $P \in X$.

Let H be an open set in S and $\sigma(\phi_p(x_1), \phi_p(x_2), ..., \phi_p(x_p)) \in H$, which implies

$\phi_p(\sigma(x_1, x_2, ..., x_n)) \in H$, so that $\sigma(x_1, x_2\hat{,} ..., x_n)(P) \in H$.
Now, $\sigma(x_1, x_2\hat{,} ..., x_n)$ being continuous there exist open set U in X cotaining P such that
$\sigma(x_1, x_2\hat{,} ..., x_n)(U) \subseteq H$.

Consider $W = (\hat{x}_1(U), \hat{x}_2(U), ..., \hat{x}_n(U)) \bigcap S^{(n)}$. Then W is an open set containing $(s_1, s_2, ..., s_n) \in S^{(n)}$.
Let $t \in W$, where, $t = (\hat{x}_1(q), \hat{x}_2(q), ..., \hat{x}_n(q))$ for some $q \in U$. Then,

$$\sigma(t) = \sigma(\hat{x}_1(q), \hat{x}_2(q), ..., \hat{x}_n(q))$$
$$= \sigma(\theta_q(x_1), \theta_q(x_2), ..., \theta_q(x_n))$$
$$= \theta_q(\sigma(x_1, x_2, ...x_n))$$
$$= \sigma(x_1, x_2\hat{,} ..., x_n)(q) \in H$$

Therefore $\sigma(W) \subseteq H$ and hence σ is continuous and (S, π, X) is a sheaf of Ω-algebras. □

Lemma 3.9. *Let R be an Ω-ABR and let $x, y \in R$. Then for $P \in SpecR$, $\hat{x}(P) = \hat{0}(P) \Leftrightarrow x \in P$.*

Proof. Observe that $\hat{x}(P) = \hat{0}(P)$ implies $\phi_p(x) = \phi_p(0)$ and as a consequence $(x, 0) \in \phi_p$. By the definition of ϕ_P it follows that $ax = 0$ for some $a \notin P$ and hence $x \in P$ (since P is prime). Conversely, suppose $x \in P$. Choose $y \notin P$. Then $y + xy \notin P$ (since, if $y + xy \in P$, $y = ((y + xy) + xy))y \in P$ a controduction). Thus $P \in X_{y+xy}$, and $(y + xy)x = 0 = (y + xy)0$. Thus $\hat{x}(P) = \hat{0}(P)$. □

Theorem 3.10. *Let R be an Ω-ABR and let (S, π, X) be a sheaf of Ω-algebras described in Theorem 3.8. Define $S_P^o = S_P - \{\hat{0}(P)\}$ and $S^o = \bigcup_{P \in X} S_p^0$ and π^o to be the restriction of π to S^o. Then (S^o, π^o, X) is a sheaf of Ω-algebras.*

Proof. Clearly S^o can be equipped with the subspace toplology induced by that of the topology present on S. Now let $s_1, s_2, ..., s_n \in S_p^o$ i.e $S_i = \hat{x}_i(P)$ for some $x_i \in R$ $(1 \leq i \leq n)$ and $\hat{x}_i(P) \neq \hat{0}(P)$ for all i. Then by Lemma 3.9, if follows that $x_i \notin P$ for $1 \leq i \leq n$. By Lemma 2.6, $\sigma(x_1, x_2, ..., x_n) \notin P$ and again by Lemma 3.9, $\sigma(x_1, x_2\hat{,} ..., x_n)(P) \neq \hat{0}(P)$. Hence, $\sigma(s_1, s_2, ..., s_n) \in S_p^o$. Therefore S_p^o is a sub algebra of S_p and hence an Ω-algebra. Let $s \in S^o$ then there exists $x \in R$ such that $s = \hat{x}(P)(\neq \hat{o}(P))$ for some $P \in X_x$. Choose $G = \hat{x}(X_x)$ and $U = X_x$. Clearly G is open in S^o and $\pi^o/G : G \to U$ is a homeomorphism. Thus π^o is a local homeomorphism and hence (S^o, π^o, X) is a sheaf of Ω-algebras.

□

Acknowledgments

The author is greatful to Prof.U.M.Swamy, Professor (Retd), Andhra University and the anaonymous referees for the invaluable suggestions.

References

1. S. D. Comer, *Representation by algebra of sections over Boolean spaces*, Numer. Pacific J. Math, **38** (1971), 29-38.
2. J. Dauns and K. H. Hoffmann, *Representation of biregular rings by sheaves*, Math. Z., **91** (1966), 103-123.
3. B. A. Davey, *sheaf spaces and sheaves of universal algebras*, Math. Z., **134**(1973), 275-290.

4. R.S. Pierce, *Modules over commutative regular rings*, Memors. Amer. Math.Soc.,**70** (1967).
5. N.V. Subrahmanyam, *Lectures on sheaf theory*, Lecture notes, Andhra Univ.,Waltair, India (1978).
6. U.M. Swamy, *Representation of Universal algebras by sheaves*, Proc. Amer.Math. Soc., **45** (1974), 55-58.
7. U.M. Swamy, and G. C. Rao,*Almost distributive lattices*, J. Austral. Math. Soc. **31** (1981), 77-91.
8. U.M. Swamy and M.P.K. Kishore,*On Prime ideals of an Almost Boolean Ring*,Proc. of AP Academy of Sciences, Vol.8, **4** (2004), 293-298.
9. U.M. Swamy, G. C. Rao and M.P.K. Kishore, *Algebraic representation of sheaves over locally Boolean spaces*, Proc. of AP Academy of Sciences, Vol.8, **4** (2004), 299-304.
10. A. Wolf, *Sheaf representation of Arithmetical algebras*,Memors. Amer. Math.Soc., **148** (2004), (1974), 87-93.

Author

Department of Information Technology, GVP College of Engineering(A), Visakhapatnam, AP, India.

`Email: kishorempk73gvpce.ac.in`