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THE TOTAL GRAPH OF A COMMUTATIVE SEMIRING WITH RESPECT TO PROPER IDEALS

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Abstract. Let I be a proper ideal of a commutative semiring R and let $P(I)$ be the set of all elements of R that are not prime to I. In this paper, we investigate the total graph of R with respect to I, denoted by $T(\Gamma_I(R))$. It is the (undirected) graph with elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in P(I)$. The properties and possible structures of the two (induced) subgraphs $P(\Gamma_I(R))$ and $\bar{P}(\Gamma_I(R))$ of $T(\Gamma_I(R))$, with vertices $P(I)$ and $R - P(I)$, respectively are studied.

1. Introduction

As a generalization of rings, semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semirings play a central role in the structure theory and are useful for many purposes [13]. However, they do not in general coincide with the ideals of rings and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. In order to overcome this deficiency, the authors defined a more restricted class of ideals in semirings, which are called the class of "k-ideals" and the class of "Q-ideals" [2, 13, 7].

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Hence the study of the graph $T(\Gamma_I(R))$ is worth to study. One point of this paper is to extend the definition and some results given in [1] to a more general semiring case.

Among the most interesting graphs are the zero-divisor graphs, because these involve both ring theory and graph theory. By studying these graphs we can gain a broader insight into the concepts and properties that involve both graphs and rings. It was Beck (see [5]) who first introduced the notion of a zero-divisor graph for commutative ring. This notion was later redefined by D. F. Anderson and P. S. Livingston in [4]. In [15], Redmond introduced the zero-divisor graph with respect to a proper ideal. Let R be a commutative ring with $Z(R)$ its set of zero-divisors elements. The total graph of R, denoted by $T(\Gamma(R))$, is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph of a commutative ring, denoted by $\mathrm{T}(\Gamma(\mathrm{R}))$, have been introduced and studied by D.F. Anderson and A. Badawi in [3]. In [11], the notion of the total torsion element graph of a module over a commutative ring is introduced. Some other investigations into properties of zero-divisor grahs of a commutative semiring may be found in [6, 9].

Let R be a commutative semiring. We recall from [7] (also see [10]), that an element $a \in R$ is called prime to an ideal I of R if $ra \in I$ (where $r \in R$) implies that $r \in I$. Denote by $P(I)$ the set of elements of R that are not prime to I. A proper ideal I of R is said to be primal if $P(I)$ forms an ideal (so 0 is not necessarily primal); this ideal is always a prime ideal, called the adjoint ideal P of I. In this case we also say that I is a P-primal ideal of R. Moreover, $P(I)$ is not an empty set since $I \subseteq P(I)$ [7, Lemma 2.1]. Let I be a proper ideal of R. The total graph of a commutative semiring R with respect to proper ideal I, denoted by $T(\Gamma_I(R))$, is the graph which vertices are all elements of R and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in P(I)$. This definition is the same as that introduced by Abbasi and Habibi in [1]. Though their definition of total graph of a commutative ring is a generalization of the definition in [3] (An inspection will show that if $I = \{0\}$, then $\mathrm{T}(\Gamma_{\mathrm{I}}(\mathrm{R})) = \mathrm{T}(\Gamma(\mathrm{R}))$), but many of the proofs provided in that paper are essentially the same as the proofs provided in [3]. In the present paper we study a new class of graphs, called the total graph of a commutative semiring with respect to a proper ideal, and we completely characterize the possible structure of this graph. The total graph of a commutative ring with respect to proper ideal and the total graph of a commutative semiring with respect to a proper ideal are different concepts. Some of our results are analogous to the results given in [1, 3]. The corresponding results are obtained by modification as semiring structures. The study of this graph breaks naturally into two cases depending on whether or not $P(I)$ is an ideal of R. In the third section, we handle the case when $P(I)$ is not an ideal of R; in the fourth section, we do the case when $P(I)$ is an ideal of R (either k-ideal or Q-ideal).

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring, we mean a commutative semigroup $(R, \cdot)$ and a commutative monoid $(R, +, 0)$ in which 0 is the additive identity and $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$, both are connected by ring-like distributivity. In this paper, all semirings considered will be assumed to be commutative semirings with non-zero identity. A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal (= k-ideal) K is an ideal such that if $x, x + y \in K$ then $y \in K$ (so $\{0\}$ is a k-ideal of R). If I is an ideal of R, then the radical of I, denoted by $\mathrm{rad}(I)$, is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R, contains I. An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that $R = \cup\{q+I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1+I)\cap(q_2+I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q-ideal of R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the operations $\oplus$ and $\odot$ defined as follows: $(q_1+I)\oplus(q_2+I) = q_3+I$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$ and $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is the unique element such that $q_1q_2+I \subseteq q_4+I$. This semiring R/I is called the quotient semiring of R by I [2].

For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, a) = 0$ and $d(a, b) = \infty$). The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,m}$ a star graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1

and Γ_2 have no common vertices and no vertex of Γ_1 (respectively, Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (respectively, Γ_2).

2. Some basic properties of $\mathrm{T}(\Gamma_{\mathrm{I}}(\mathrm{R}))$

Let I be a proper Q-ideal of a semiring R. In this section, we explore the relationship between $\mathrm{T}(\Gamma_{\mathrm{I}}(\mathrm{R}))$ and $\mathrm{T}(\Gamma(\mathrm{R/I}))$ on basic structure (we do not need a priori assumption on whether or not $P(I)$ is an ideal of R).

Proposition 2.1. *Let I be a proper Q-ideal of a semiring R. Then*

(1) $Z(R/I) = \{q + I : q \in Q \cap P(I)\}$.

(2) $Reg(R/I) = \{q + I : q \in Q - P(I)\}$, the set of regular elements of R/I.

Proof. (1) Assume that q_0 is the unique element in Q such that $q_0 + I$ is the zero in R/I and let $\{q + I : q \in Q \cap P(I)\} = H$. Suppose that $q + I$ is a non-zero element of $Z(R/I)$ (so $q \notin I$ since every Q-ideal is a k-ideal). Then there exists $q' \in Q$ with $q' \notin I$ such that $(q + I) \odot (q' + I) = q_0 + I$, where $qq' + I \subseteq q_0 + I$. Since I is a k-ideal, $qq' \in I$ by [9, Lemma 2.3 (i)]; hence $q \in P(I)$. Therefore, $Z(R/I) \subseteq H$. For the reverse inclusion, assume that $s + I \in H$, where $s \in Q \cap P(I)$. By assumption, there exists $t \in R - I$ such that $st \in I$. As I is a Q-ideal, $t = q_1 + a$ for some $q_1 \in Q - I$ and $a \in I$, and so $st = sq_1 + sa$; thus $sq_1 \in I$ since I is a k-ideal. Let q_2 be the unique element in Q such that $(s + I) \odot (q_1 + I) = q_2 + I$, where $sq_1 + I \subseteq q_2 + I$; so $q_2 \in I$. Now by [9, Lemma 2.3 (ii)], $q_2 = q_0$; hence $s + I \in Z(R/I)$, we have equality. (2) is clear. □

Theorem 2.2. *Let I be a proper Q-ideal of a semiring R. Then $Z(R/I)$ is an ideal of R/I if and only if $P(I)$ is an ideal of R.*

Proof. First, suppose that $Z(R/I)$ is an ideal of R/I. Assume that q_0 is the unique element in Q such that $q_0 + I$ is the zero in R/I and let $a, b \in P(I)$, $r \in R$. Then there exist $c, d \notin I$ such that $ac, bd \in I$. Since I is a Q-ideal, $a = q_1 + a'$, $b = q_2 + b'$, $c = q_3 + c'$ and $r = s' + r'$ for some $q_1, q_2, q_3, s' \in Q$ and $a', b', c', r' \in I$. It follows from $ac \in I$ that $q_1q_3 \in I$ since I is a k-ideal. Let q' be the unique element in Q such that $(q_1 + I) \odot (q_3 + I) = q' + I$, where $q_1q_3 + I \subseteq q' + I$; thus $q' \in I$. As $q' \in (q' + I) \cap (q_0 + I)$, we get $q' = q_0$, and so $q_1 + I \in Z(R/I)$. Similarly, $q_2 + I \in Z(R/I)$. Therefore, $(q_1 + I) \oplus (q_2 + I) \in Z(R/I)$. Let s be the unique element in Q such that $(q_1 + I) \oplus (q_2 + I) = s + I$, where $q_1 + q_2 + I \subseteq s + I$. Since $s + I \in Z(R/I)$, there is an element

$q_5 \in Q - I$ such that $(s+I) \odot (q_5+I) = q_0+I$, where $sq_5+I \subseteq q_0+I$. Then we have

$$q_1q_5 + q_2q_5 + I \subseteq sq_5 + I \subseteq q_0 + I = I.$$

So $q_1q_5 + q_2q_5 \in I$. As $q_5a' + q_5b' \in I$, we obtain that $q_5(a+b) \in I$ with $q_5 \notin I$; hence $a+b \in P(I)$. Now we show that $ra \in P(I)$. Since $Z(R/I)$ is an ideal of R/I and $q_1+I \in Z(R/I)$, we must have $(s'+I)\odot(q_1+I) = t+I \in Z(R/I)$, where $s'q_1 + I \subseteq t + I$. By assumption, there exists a unique element $t' \in Q - I$ such that $(t+I) \odot (t'+I) = q_0 + I$, where $tt' + I \subseteq q_0 + I$. It follows that

$$s'q_1t' + I \subseteq tt' + I \subseteq q_0 + I = I,$$

and hence $s'q_1t' \in I$. As $rat' = s'q_1t' + s'a't' + r'q_1t' + r'a't' \in I$ with $t' \notin I$, we get $ra \in P(I)$. Thus $P(I)$ is an ideal of R.

Next, suppose that $P(I)$ is an ideal of R; we show that $Z(R/I)$ is an ideal of R/I. Let $t_1 + I, t_2 + I \in Z(R/I)$ and $z + I \in R/I$. Then there exists $u_1 \in Q - I$ such that $(t_1 + I) \odot (u_1 + I) = q_0 + I$, where $t_1u_1 + I \subseteq q_0 + I$; so $t_1u_1 \in I$ with $u_1 \notin I$. It follows that $t_1 \in P(I)$. Similarly, $t_2 \in P(I)$. Thus $t_1 + t_2 \in P(I)$. By assumption, there exists $e = w + f \notin I$ ($w \in Q - I$, $f \in I$) such that $e(t_1 + t_2) \in I$; hence $w(t_1+t_2) \in I$. Let $(t_1+I)\oplus(t_2+I) = t_3+I$ and $(t_3+I)\odot(w+I) = w'+I$, where $t_1 + t_2 + I \subseteq t_3 + I$ and $t_3w + I \subseteq w' + I$. Thus $w(t_1 + t_2) + I \subseteq wt_3 + I \subseteq w' + I$, so $w' \in I$ since I is a k-ideal; hence $w' = q_0$ by [9, Lemma 2.3 (ii)]. Therefore, $(t_1 + I) \oplus (t_2 + I) \in Z(R/I)$. Similarly, $(t_1 + I) \odot (z + I) \in Z(R/I)$, and this completes the proof. □

Proposition 2.3. *Assume that I is a proper Q-ideal of a semiring R and let $x = q_1 + a$, $y = q_2 + b$, where $q_1, q_2 \in Q$ and $a, b \in I$. Then*

(1) If $q_1 + I$ and $q_2 + I$ are (distinct) adjacent in $T(\Gamma(R/I))$, then x is adjacent to y in $T(\Gamma_I(R))$.

(2) If x and y are (distinct) adjacent vertices in $T(\Gamma_I(R))$ and $q_1 + I \neq q_2 + I$, then $q_1 + I$ is adjacent to $q_2 + I$ in $T(\Gamma(R/I))$.

(3) If x is adjacent to y in $T(\Gamma_I(R))$ and $q_1 = q_2$, then $2x, 2y \in P(I)$ and all distinct elements of $q_1 + I$ are adjacent in $T(\Gamma_I(R))$.

Proof. (1) By Proposition 2.1, $(q_1 + I) \oplus (q_2 + I) = q_3 + I \in Z(R/I)$, where $q_1+q_2+I \subseteq q_3+I$ and $q_3 \in P(I)\cap Q$. There exists $c = q_4+a \notin I$ (so $q_4 \notin I$) such that $q_3q_4 + q_4a \in I$; so $q_3q_4 \in I$. Since $q_4(q_1+q_2)+I \subseteq q_3q_4 + I$ and $q_4a + q_4b \in I$, we must have $q_4(x + y) \in I$ with $q_4 \notin I$; hence $x + y \in P(I)$. Thus x is adjacent to y in $T(\Gamma_I(R))$.

(2) Let $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_1 + q_2 + I \subseteq q_3 + I$. It suffices to show that $q_3 \in P(I)$. Since x is adjacent to y in $T(\Gamma_I(R))$,

$x+y \in P(I)$; thus $(x+y)e \in I$ for some $e = q_4 + f \notin I$ (so $q_4 \notin I$). It follows that $(x+y)e = (q_1+q_2)q_4 + f(q_1+q_2) + (a+b)q_4 + (a+b)f \in I$, and so $(q_1+q_2)q_4 \in I$ since I is a k-ideal. As $(q_1+q_2)q_4 + I \subseteq q_3q_4 + I$, we have $q_3q_4 \in I$ with $q_4 \notin I$; hence $q_3 \in P(I)$, as required.

(3) By assumption, $x = q_1 + a$, $y = q_1 + b$ and $x + y \in P(I)$. There exists $s \notin I$ such that $s(2q_1 + a + b) \in I$; so $(2q_1)s \in I$. As $s \notin I$ and $(2x)s = (2q_1)s + (2a)s \in I$, we obtain that $2x \in P(I)$. Similarly, $2y \in P(I)$. Finally, Let $q_1 + t_1, q_2 + t_2 \in q_1 + I$ with $q_1 + t_1 \neq q_1 + t_2$. Then $(q_1 + t_1) + (q_1 + t_2) = 2q_1 + t_1 + t_2$. Since $(2q_1)s \in I$ and $(t_1 + t_2)s \in I$, we must have $(2q_1 + t_1 + t_2)s \in I$ with $s \notin I$; hence $(q_1 + t_1) + (q_1 + t_2) \in P(I)$, as needed. □

Let I be a proper ideal of a semiring R. Let $P(\Gamma_I(R))$ be the (induced) subgraph of $T(\Gamma_I(R))$ with vertices $P(I)$, and let $\bar{P}(\Gamma_I(R))$ be the (induced) subgraph of $T(\Gamma_I(R))$ with vertices $R - P(I)$. Let I be a proper Q-ideal of R and assume that $Reg(\Gamma(R/I))$ is the (induced) subgraph of $T(\Gamma(R/I))$ with vertices $Reg(R/I)$ and let $Z(\Gamma(R/I))$ be the (induced) subgraph of $T(\Gamma(R/I))$ with vertices $Z(R/I)$. The following theorem shows that there is a basic relationship between $T(\Gamma_I(R))$ and $T(\Gamma(R/I))$.

Theorem 2.4. *Let I be a proper Q-ideal of a semiring R. Then*

(1) $T(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $T(\Gamma(R/I))$.

(2) $P(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $Z(\Gamma(R/I))$.

(3) $\bar{P}(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $Reg(\Gamma(R/I))$.

(4) $P(\Gamma_I(R))$ is complete (connected) if and only if $Z(\Gamma(R/I))$ is complete (connected).

(5) If $\bar{P}(\Gamma_I(R))$ is complete, then $Reg(\Gamma(R/I))$ is complete.

(6) $\mathrm{gr}(T(\Gamma_I(R))) \leq \mathrm{gr}(T(\Gamma(R/I)))$.

Proof. (1) Let $\{q_i\}_{i\in J} \subseteq Q$ be a set of distinct representatives of the vertices of $T(\Gamma(R/I))$. For each $k \in I$, define a graph G_k with vertices $\{q_i + k : i \in J\}$, where $q_i + k$ is adjacent to $q_t + k$ in G_k whenever $q_i + I$ is adjacent to $q_t + I$ in $T(\Gamma(R/I))$; i.e., whenever $q_i + q_t \in P(I)$; hence G_k is a subgraph of $T(\Gamma_I(R))$ by Proposition 2.3. Also, each $G_k \cong T(\Gamma(R/I))$, and G_k and G_t contains no common vertices if $k \neq t$.

(2) Let $a = q + f \in P(I)$ (where $q \in Q$ and $f \in I$), and let $k \in I$. There exists $e \in R - I$ such that $e(q + f) \in I$, so $eq \in I$ since I is a k-ideal. Thus $q \in P(I)$. It follows that $e(q+k) \in I$; hence $q+k \in P(I)$. Therefore, a graph G_k with vertices $\{q_i+k : i \in J\}$ such that $q_i \in P(I)$ is a subgraph of $P(\Gamma_I(R))$ which is isomophic to $Z(\Gamma(R/I))$.

(3) By an argument like that in (2), if $b = q' + f' \in R - P(I)$ (where $q' \in Q$ and $f' \in I$) and $k \in I$, then $q' \notin P(I)$ and $q' + k \in R - P(I)$.

So a graph G_k with vertices $\{q_t + k : t \in J\}$ such that $q_t \notin P(I)$ is a subgraph of $\bar{P}(\Gamma_I(R))$ which is isomophic to $Reg(\Gamma(R/I))$.

(4) Assume that $P(\Gamma_I(R))$ is a complete graph and let $q_1 + I$ and $q_2 + I$ be distinct elements of $Z(\Gamma(R/I))$, where $q_1, q_2 \in Q \cap P(I)$ by Proposition 2.1 and $q_1 \neq q_2$. By assumption, q_1 and q_2 are adjacent in $P(\Gamma_I(R))$, hence $q_1 + I$ and $q_2 + I$ are adjacent in $Z(\Gamma(R/I))$ by Proposition 2.3. Conversely, suppose $a = s_1 + e$ and $b = s_2 + f$ are adjacent elements of $P(\Gamma_I(R))$, where $s_1, s_2 \in Q \cap P(I)$ and $e, f \in I$. If $s_1+I = s_2+I$, then $s_1 = s_2$ and $a+b = 2s_1+e+f$. By assumption, there is an element $r \in R-I$ such that $rs_1 \in I$; hence $r(a+b) = 2(rs_1)+r(e+f) \in I$. Thus a and b are adjacent in $P(\Gamma_I(R))$. If $s_1+I \neq s_2+I$, then s_1+I and s_2+I are adjacent in $Z(\Gamma(R/I))$, so $(s_1+I)\oplus(s_2+I) = s_3+I$ for some $s_3 \in Q \cap P(I)$, where $s_1 + s_2 + I \subseteq s_3 + I$. Then there exists $t \in R - I$ such that $ts_3 \in I$. As $t(s_1 + s_2) + I \subseteq ts_3 + I$, we must have $t(s_1 + s_2) \in I$ since I is a k-ideal. It follows that $t(a + b) \in I$ since $te, tf \in I$. So $a + b \in P(I)$, as needed.

(5) Assume that $\bar{P}(\Gamma_I(R))$ is a complete graph and let $q_1 + I$ and $q_2 + I$ be distinct elements of $Reg(\Gamma(R/I))$, where $q_1, q_2 \in Q - P(I)$ by Proposition 2.1 and $q_1 \neq q_2$. So q_1 and q_2 are not adjacent in $\bar{P}(\Gamma_I(R))$; so $q_1 + I$ and $q_2 + I$ are adjacent in $Reg(\Gamma(R/I))$.

(6) We may assume that $\mathrm{gr}(T(\Gamma(R/I))) = n < \infty$. Let $q_1 + I - q_2 + I - ... - q_n + I - q_1 + I$ be a cycle in $T(\Gamma(R/I))$ through n distinct vertices. Thus $q_1 - q_2 - ... - q_n - q_1$ is a cycle in $T(\Gamma_I(R))$ of legh n by Proposition 2.3; hence $\mathrm{gr}(T(\Gamma_I(R))) \leq n$, as needed. □

Example 2.5. Let R be the semiring of non-negative integers with the usual addition and multiplication. Then clearly $I = \{3k : k \in R\}$ is a partitioning ideal of R with respect to $Q = \{0, 1, 2\}$; we show that $P(I) = I$. Since the inclusion $I \subseteq P(I)$ is clear, we will prove the reverse inclusion. Let $a \in P(I)$. Then $ra \in I$ for some $r \in R - I$; so $a \in I$, and we have equality. Thus the converse of Theorem 2.4 (5) is not necessarily true.

3. $P(I)$ is not an ideal of R

Let R be a commutative semiring. In this section, we study the total graph $T(\Gamma_I(R))$ when $P(I)$ is not an ideal of R.

Lemma 3.1. *Let I be a proper ideal of a semiring R such that $P(I)$ is not an ideal of R. Then*

(1) There exist distinct $a, b \in P(I)^*$ such that $a + b \in R - P(I)$.

(2) $|P(I)| \geq 3$.

Proof. (1) It suffices to show that $P(I)$ is always closed under scalar multiplication of its elements by elements of R. Let $x \in P(I)$ and $r \in R$. There is an element $t \in R - I$ such that $tx \in I$; hence $r(tx) = t(rx) \in I$. Thus $rx \in P(I)$. This completes the proof.

(2) By (1), there are distinct $x, y \in P(I)^*$ such that $x+y \in R-P(I)$; hence $|P(I)| \geq 3$. □

Theorem 3.2. *Let I be a proper ideal of a semiring R such that $P(I)$ is not an ideal of R. Then*

(1) $P(\Gamma_I(R))$ is connected with $\mathrm{diam}(P(\Gamma_I(R))) = 2$.

(2) $\mathrm{gr}(P(\Gamma_I(R))) = 3$ *or* $\mathrm{gr}(P(\Gamma_I(R))) = \infty$

Proof. (1) Let $a \in P(I)^*$. Then a is adjacent to 0. Thus $a - 0 - b$ is a path in $P(\Gamma_I(R))$ of length two between any two distinct $a, b \in P(I)^*$. In addition to, there exist nonadjacent $x, y \in P(I)^*$ by Lemma 3.1; hence $\mathrm{diam}(P(\Gamma_I(R))) = 2$.

(2) If $a+b \in P(I)$ for some distinct $a, b \in P(I)^*$, then $0-a-b-0$ is a 3-cycle in $P(\Gamma_I(R))$; so $\mathrm{gr}(P(\Gamma_I(R))) = 3$. Otherwise, $a+b \in R-P(I)$ for all distinct $a, b \in P(I)$. Therefore, in this case, each $a \in P(I)^*$ is adjacent to 0, and no two distinct $a, b \in P(I)^*$ are adjacent. Thus $P(\Gamma_I(R))$ is a star graph with center 0; hence $\mathrm{gr}(P(\Gamma_I(R))) = \infty$. □

Theorem 3.3. *Let I be a proper ideal of a semiring R such that $P(I)$ is not an ideal of R. Then* $\mathrm{gr}(\bar{P}(\Gamma_I(R))) = 3$ *or* ∞.

Proof. We may assume that $\bar{P}(\Gamma_I(R))$ contains a cycle. So there is a path $x - y - z$ in $\bar{P}(\Gamma_I(R))$. If $x + z \in P(I)$, then we have a 3-cycle in $\bar{P}(\Gamma_I(R))$. So we may assume that $x + z \notin P(I)$. There exist $a, b \in P(I)$ such that $a + b \notin P(I)$ by Lemma 3.1. So there are $e, f \in R - I$ such that $ae, bf \in I$ and then $ef \in I$ since $ef(a + b) \in I$. Therefore $ex + ez \in P(I)$ since $f(ex + ez) \in I$. Thus $ex - ey - ez - ex$ is a 3-cycle in $\bar{P}(\Gamma_I(R))$ and the proof is complete. □

Example 3.4. Let $\mathcal{X} = \{a, b, c\}$ and $R = (\mathcal{P}(X), \cup, \cap)$ a semiring, where $\mathcal{P}(X) =$ the set of all subsets of $\mathcal{X}$. An inspection will show that $I = \{\emptyset, \{a\}\}$ is an ideal of R, but $P(I) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ is not an ideal of R. Moreover, $P(\Gamma_I(R))$ is disjoint from $\bar{P}(\Gamma_I(R))$, and $\bar{P}(\Gamma_I(R))$ is a totally disconnected subgraph of $T(\Gamma_I(R))$. Hence $T(\Gamma_I(R))$ is disconnected.

4. $P(I)$ IS AN IDEAL OF R

The main goal of this section is a general structure theorem (Theorem 4.6) for $\bar{P}(\Gamma_I(R))$ when either $P(I)$ is a k-ideal of R or $P(I)$ is a Q-ideal. But first, we record the basic observation that if $P(I)$ is a k-ideal of (resp. $P(I)$ is not a k-ideal), then the subgraph $P(\Gamma_I(R))$ is disjoint from $\bar{P}(\Gamma_I(R))$ (resp. $P(\Gamma_I(R))$ is not disjoint from $\bar{P}(\Gamma_I(R))$. Thus we will concentrate on the subgraph $\bar{P}(\Gamma_I(R))$ throughout this section.

Lemma 4.1. *Assume that I be a proper ideal of a semiring R and let $r \in R - P(I)$. Then $2 = 1_R + 1_R \in P(I)$ if and only if $2r \in P(I)$.*

Proof. Let $2 \in P(I)$. Then $2s \in I$ for some $s \in R - I$; hence $(2r)s = r(2s) \in I$. Thus $2r \in P(I)$. Conversely, assume that $(2r)t = 2(rt) \in I$ for some $t \in R - I$. By assumption, $rt \notin I$; hence $2 \in P(I)$. $\square$

Example 4.2. (1) An ideal of a semiring in general does not need to be either a k-ideal or a Q-ideal. Let R be the set of all real numbers x satisfying $0 < x \leq 1$, and define $a+b = a.b = \min\{a, b\}$ for all $a, b \in R$. Then $(R, +, .)$ is easily checked to be a commutative semiring with 1 as identity. Each real number r such that $0 < r < 1$ defines an ideal $I_r = \{t \in R : t \leq r\}$ of R. However, $r + 1 = r$ together $r \in I_r$ and $1 \notin I_r$ show that I_r is not a k-ideal of R. In particular, I_r is not a Q-ideal of R since every Q-ideal is a k-ideal.

(2) Let R denote the semiring of non-negative integers with the usual operations of addition and multiplication. If $m \in R - \{0\}$, the ideal

$$I_m = \{km : k \in R\}$$

is a Q-ideal of R when $Q = \{0, 1, \cdots, m-1\}$. In particular, I_m is a k-ideal.

(3) Assume that R denote the semiring of non-negative integers. Define $x + y = \gcd(x, y)$ and $x.y = \mathrm{lcm(x, y)}$. It is easy to see that R is a semiring in which every element is idempotent. The ideal $I = \{0, 2, 4, \cdots\}$ is a k-ideal of R but is not a Q-ideal.

Example 4.3. Consider the set $S = \{0, 1\}$. On S we define addition and multiplication as follows: $0 + 0 = 1 + 1 = 0, 1 + 0 = 0 + 1 = 1$ and $0.0 = 0.1 = 1.0 = 1.1 = 0$. It is easy to see that $(S, +, .)$ is a commutative semiring. Assume that J_4 is the ring integers modulo 4 and let $R = J_4 \oplus S$ denote the direct sum of semirings J_4 and S. Then R is a commutative semiring. Set $I = \{(\bar{0}, a) : a \in S\}$. An inspection will show that I and $P(I) = \{(\bar{0}, 0), (\bar{0}, 1), (\bar{2}, 0), (\bar{2}, 1)\}$ are k-ideals of R.

Theorem 4.4. *Let I be a proper ideal of a semiring R such that $P(I)$ is an ideal of R. Then*

(1) $P(\Gamma_I(R))$ is a complete (induced) subgraph of $T(\Gamma_I(R))$.

(2) If $P(I)$ is a k-ideal of R, then $P(\Gamma_I(R))$ is disjoint from $\bar{P}(\Gamma_I(R))$.

(3) If $P(I)$ is not a k-ideal of R, then $P(\Gamma_I(R))$ is not disjoint from $\bar{P}(\Gamma_I(R))$.

Proof. (1) This is clear according to definition.

(2) If $P(\Gamma_I(R))$ is not disjoint from $\bar{P}(\Gamma_I(R))$, then there exist $a \in P(I)$ and $b \in R - P(I)$ such that $a + b \in P(I)$. Thus $b \in P(I)$ since $P(I)$ is a k-ideal of R which is a contradiction. Thus $P(\Gamma_I(R))$ is disjoint from $\bar{P}(\Gamma_I(R))$.

(3) Since $P(I)$ is not a k-ideal, there exist $a \in P(I)$ and $b \in R - P(I)$ such that $a+b \in P(I)$. Let $u \in R$. We define the subset $T(u)$ as follows: $T(u) = \{t \in P(I) :$ there is a path of finite length between u and $t\}$. Clearly, if $u \in P(I)$, then $P(I) \subseteq T(u)$, and so $T(u) \neq \emptyset$. Set $J = \{u \in R : T(u) \neq \emptyset\}$. Thus $P(I) \subsetneqq J$ since $b \in J - P(I)$. Now we show that J is an ideal of R. Let $r_1, s_1 \in J$. Then there exist $t_1, t'_1 \in P(I)$, $r_1, r_2, \cdots, r_n \in R$ and $s_1, s_2, \cdots, s_k \in R$ such that $r_1 - r_2 - \cdots - r_n - t_1$ and $s_1 - s_2 - \cdots - s_k - t'_1$ are paths of finite lengths between r_1, t_1 and s_1, t'_1, and so we have $r_i + r_{i+1}, s_j + s_{j+1}, r_n + t_1, s_k + t'_1, t_1 + t'_1 \in P(I)$ for each $1 \leq i \leq n-1$ and $1 \leq j \leq k-1$. We may assume that $n \leq k$. So $(r_i + s_i) + (r_{i+1} + s_{i+1}) \in P(I)$ for each $1 \leq i \leq n-1$. Then $(r_1 + s_1) - (r_2 + s_2) - \cdots - (r_n + s_n) -$

$$(t_1 + s_{n+1}) - (t'_1 + s_{n+2}) - (t_1 + s_{n+3}) - \cdots - t_1$$

is a path of finite length between $r_1 + s_1$ and t_1. Hence $T(r_1 + s_1) \neq \emptyset$, and so $r_1 + s_1 \in J$. Now, let $r \in R$. Therefore, $rr_1 - rr_2 - \cdots - rr_n - rt_1$ is a path between rr_1 and rt_1 of finite length, and so $T(rr_1) \neq \emptyset$. Thus J is an ideal of R and $P(I) \subset J$. It is easy to see that $P(\Gamma_I(J))$ is a connected subgraph of $T(\Gamma_I(R))$ containing $P(\Gamma_I(R))$. Hence, $P(\Gamma_I(R))$ is not disjoint from $\bar{P}(\Gamma_I(R))$. □

Theorem 4.5. *Let I be a proper ideal of a semiring R such that $P(I)$ is an ideal of R. Then*

(1) The (induced) subgraph $P(\Gamma_I(R))$ with vertices $\mathrm{rad}(I)$ is complete and each vertex of this graph is adjacent to each vertex of $P(\Gamma_I(R))$ and disjoint from $\bar{P}(\Gamma_I(R))$.

(2) If $\{0\} \neq \mathrm{rad}(I) \subsetneqq P(I)$, then $\mathrm{gr}(P(\Gamma_I(R))) = 3$.

(3) Assume that $\mathcal{G}$ is an induced subgraph of $\bar{P}(\Gamma_I(R))$ and let r and r' be distinct vertices of $\mathcal{G}$ such that are connected by a path in $\mathcal{G}$. If $P(I)$ is a k-ideal of R, then there exists a path in $\mathcal{G}$ of length 2 between r and r'. In particular, if $\bar{P}(\Gamma_I(R))$ is connected, then $\mathrm{diam}(\bar{P}(\Gamma_I(R))) \leq 2$.

Proof. (1) It suffices to show that $\mathrm{rad}(I) \subseteq P(I)$ (so $\mathrm{rad}(I) + P(I) \subseteq P(I)$). Let $x \in \mathrm{rad}(I)$. If $x \in I$, then $x \in I \subseteq P(I)$; otherwise there is an integer $n \geq 2$ such that $x^n \in I$ and $x^{n-1} \notin I$. Now we have $x.x^{n-1} \in I$; hence $x \in P(I)$, and this completes the proof.

(2) Let a be a non-zero element of $\mathrm{rad}(I)$ and $b \in P(I) - \mathrm{rad}(I)$. Then $0 - a - b - 0$ is a 3-cycle in $P(\Gamma_I(R))$ by part (1), as needed.

(3) It suffices to show that if r_1, r_2, r_3 and r_4 are distinct vertices of $\mathcal{G}$ and there is a path $r_1 - r_2 - r_3 - r_4$ from r_1 to r_4, then r_1 and r_4 are adjacent. Now we have $r_1 + r_2 + r_3 + r_4 \in P(I)$. Then $P(I)$ being k-ideal of R gives $r_1 + r_4 \in P(I)$, and so r_1 and r_4 are adjacent. So if $\bar{P}(\Gamma_I(R))$ is connected, then $\mathrm{diam}(\bar{P}(\Gamma_I(R))) \leq 2$. □

Theorem 4.6. *Assume that I is a proper ideal of a semiring R such that $P(I)$ is an ideal of R and let $|P(I)| = \alpha$. Then*

(1) If $P(I)$ is a k-ideal of R and $2 \in P(I)$, then $\bar{P}(\Gamma_I(R))$ is the union of disjoint complete subgraphs.

(2) If $P(I)$ is a k-ideal of R and $2 \notin P(I)$, then $\bar{P}(\Gamma_I(R))$ is the union of totally disconnected subgraphs and some connected subgraphs.

(3) If $P(I)$ is a Q-ideal of R, $|Q - P(I)| = \beta$ and $2 \in P(I)$, then $\bar{P}(\Gamma_I(R))$ is the union of β disjoint K^λ's such that $\lambda \leq \alpha$.

(4) If $P(I)$ is a Q-ideal of R, $|Q - P(I)| = \beta$ and $2 \notin P(I)$, then $\bar{P}(\Gamma_I(R))$ is the union of totally disconnected subgraphs and complete bipartite subgraphs.

Proof. (1) Let $2 \in P(I)$. We set up an equivalence relation $\sim$ on $R - P(I)$ as follows: for $r, r' \in R - P(I)$, we write $r \sim r'$ if and only if $r + r' \in P(I)$. It is straightforward to check that $\sim$ is an equivalence relation on $R - P(I)$: for $r \in R - P(I)$, we denote the equivalence class which contains r by $[r]$. Now let $r \in R - P(I)$. If $[r] = \{r\}$, then $(r + a) + (r + b) = 2r + (a + b) \in P(I)$ for every $a, b \in P(I)$ by Lemma 4.1. So $r + P(I)$ is a complete subgraph with at most α vertices. If $|[r]| = \gamma > 1$, then for every $r' \in [r]$ we have $(r + a) + (r' + b) = (r + r') + a + b \in P(I)$, where $a, b \in P(I)$. Thus $r + P(I)$ is a part of a complete graph K^ν with $\nu \leq \alpha\gamma$ vertices. Therefore, $\bar{P}(\Gamma_I(R))$ is the union of disjoint complete subgraphs.

(2) Let $2 \notin P(I)$ and $r \in R - P(I)$. Set

$$T(r) = \{r' \in R - P(I) : r + r' \in P(I)\}.$$

If $T(r) = \emptyset$, then $r + r' \notin P(I)$ for every $r' \in R - P(I)$. In this case, we show that $r + P(I)$ is a totally disconnected subgraph of $\bar{P}(\Gamma_I(R))$. If $(r + a) + (r + b) \in P(I)$ for some $a, b \in P(I)$, then $2r + a + b \in P(I)$; so $2r \in P(I)$, which is a contradiction by Lemma 4.1. Therefore, $r + P(I)$ is a totally disconnected subgraph of $\bar{P}(\Gamma_I(R))$. We may assume that $T(r) \neq \emptyset$. Then $r + r' \in P(I)$ for some $r' \in R - P(I)$. Thus $(r + a) + (r' + b) = (r + r') + (a + b) \in P(I)$ for every $a, b \in P(I)$; hence each element of $r + P(I)$ is adjacent to each element of $r' + P(I)$. If $|T(r)| = \nu$, then we have a connected subgraph of $\bar{P}(\Gamma_I(R))$ with at most $\alpha\nu$ vertices. Hence, If $2 \notin P(I)$, then $\bar{P}(\Gamma_I(R))$ is the union of totally disconnected subgraphs and some connected subgraphs.

(3) First, we show that $q+P(I) \subseteq R-P(I)$ for every $q \in Q-P(I)$. If $q+a \notin R-P(I)$ for some $a \in P(I)$, then $q+a \in P(I)$; hence $q \in P(I)$ since $P(I)$ is a k-ideal which is a contradiction. Let $2 \in P(I)$ and $q \in Q-P(I)$. Then each coset $q+P(I)$ is a complete subgraph of $\bar{P}(\Gamma_I(R))$ with λ vertices such that $\lambda \leq \alpha$ (note that $(q_1+P(I)) \cap (q_2+P(I)) \neq \emptyset$ if and only if $q_1 = q_2$) since $(q+a)+(q+b) = 2q+(a+b) \in P(I)$ for all $a, b \in P(I)$ by Lemma 4.1 and $P(I)$ is an ideal. Next, we show that distinct cosets form disjoint subgraphs of $\bar{P}(\Gamma_I(R))$. If q_1+a and q_2+b are adjacent for some $q_1, q_2 \in Q-P(I)$ and $a, b \in P(I)$, then $(q_1+a)+(q_2+b) \in P(I)$ gives $q_1+q_2 \in P(I)$ since $P(I)$ is a k-ideal of R. So $q_2+2q_1 = q_1+(q_1+q_2) \in q_1+P(I)$. Likewise, $q_2+2q_1 \in q_2+P(I)$ by Lemma 4.1. So $q_2+2q_1 \in (q_1+P(I)) \cap (q_2+P(I))$; hence $q_1 = q_2$. Thus $\bar{P}(\Gamma_I(R))$ is the union of β disjoint induced subgraphs $q+P(I)$, each of which is a K^λ such that $\lambda \leq \alpha$.

(4) Assume that $2 \notin P(I)$ and let $q \in Q-P(I)$. If $q+q' \notin P(I)$ for every $q' \in Q-P(I)$, then $T(q) = \emptyset$. Then by (ii), $q+P(I)$ is a totally disconnected subgraph of $\bar{P}(\Gamma_I(R))$. So we may assume that $q+q' \in P(I)$ for some $q' \in Q-P(I)$. Then by (ii) each element of $q+P(I)$ is adjacent to each element of $q'+P(I)$. Now we show that q' is the unique element. Let $q+q'' \in P(I)$ for some $q'' \in Q-P(I)$. Therefore, $q+q'+q'' = q'+(q+q'') \in q'+P(I)$. Likewise, $q+q'+q'' = q''+(q+q') \in q''+P(I)$. Thus $(q'+P(I)) \cap (q''+P(I)) \neq \emptyset$ gives $q' = q''$. Therefore $(q+P(I)) \cup (q'+P(I))$ is a complete bipartite subgraph of $\bar{P}(\Gamma_I(R))$. So $\bar{P}(\Gamma_I(R))$ is the union of totally disconnected subgraphs and complete bipartite subgraphs. □

Proposition 4.7. *Let I be a proper ideal of a semiring R such that $P(I)$ is an ideal of R. Then*

(1) If $P(I)$ is a k-ideal of R and $\bar{P}(\Gamma_I(R))$ is complete, then $|R-P(I)| = 1$ or $|R-P(I)| = 2$.

(2) If $P(I)$ is a Q-ideal of R and $\bar{P}(\Gamma_I(R))$ is complete, then $|R/P(I)| = 2$ or $|R/P(I)| = 3$.

(3) If $P(I)$ is a Q-ideal of R, $|R/P(I)| = 2$ and $2 \in P(I)$, then $\bar{P}(\Gamma_I(R))$ is complete.

Proof. (1) If $2 \in P(I)$, then $2r \in P(I)$ for every $r \in R-P(I)$. Then $r+P(I)$ is a complete subgraph of $\bar{P}(\Gamma_I(R))$; hence $|R-P(I)| = 1$ since $\bar{P}(\Gamma_I(R))$ is complete. If $2 \notin P(I)$, then for each $r \in R-P(I)$, there exists $r' \in R-P(I)$ such that $r+r' \in P(I)$. So $|R-P(I)| = 2$ since $\bar{P}(\Gamma_I(R))$ is complete. In this case, $\bar{P}(\Gamma_I(R))$ is a complete bipartite graph (see Theorem 4.6).

(2) Since every Q-ideal is a k-ideal, the part (1) gives $|R-P(I)|=1$ or $|R-P(I)|=2$. If $|R-P(I)|=1$, then $R=P(I)\cup(q+P(I))$ for $q\in R-P(I)$ and hence $|R/P(I)|=2$. Similarly, if $|R-P(I)|=2$, then $R=P(I)\cup(q+P(I))\cup(q'+P(I))$ for $q,q'\in R-P(I)$ with $q\neq q'$, and hence $|R/P(I)|=3$.

(3) By assumption, $R=P(I)\cup(q+P(I))$ for some $q\in Q-P(I)$; so $2q\in P(I)$ by Lemma 4.1. Let $r,r'\in R-P(I)$. Then $r,r'\in q+P(I)$. So $r+r'=(q+a)+(q+b)=2q+(a+b)\in P(I))$ for some $a,b\in P(I)$. Thus $\bar{P}(\Gamma_I(R))$ is complete. □

Proposition 4.8. *Let R be a commutative semiring such that $P(I)$ is a proper Q-ideal of R. Then*

(1) If $\bar{P}(\Gamma_I(R))$ is connected, then $|R/P(I)|=2$ or $|R/P(I)|=3$.

(2) If $|R/P(I)|=2$ and $2\in P(I)$, then $\bar{P}(\Gamma_I(R))$ is connected.

Proof. (1) Let $\bar{P}(\Gamma_I(R))$ be a connected graph. Then $\bar{P}(\Gamma_I(R))$ is a single complete graph K^λ or a bipartite graph by Theorem 4.6. Hence $\bar{P}(\Gamma_I(R))$ is a complete graph. Now the assertion follows from Proposition 4.7.

(2) Apply Proposition 4.7. □

Theorem 4.9. *Let I be a proper ideal of a semiring R such that $P(I)$ is an ideal of R. Then*

(1) If $P(I)$ is a k-ideal of R, then $\operatorname{diam}(\bar{P}(\Gamma_I(R)))=0$ *if and only if $P(I)=\{0\}$ and $|R|=2$.*

(2) Let $P(I)$ be a Q-ideal of R. Then:

(a) $\operatorname{diam}(\bar{P}(\Gamma_I(R))=1$ *if and only if $2\in P(I)$ and $|R/P(I)|=2$.*

(b) $\operatorname{diam}(\bar{P}(\Gamma_I(R))=2$ *if and only if $|R/P(I)|=3$, $2\notin P(I)$ and $q+q'\in P(I)$ for every $q,q'\in Q-P(I)$.*

(c) Otherwise $\operatorname{diam}(\bar{P}(\Gamma_I(R))=\infty$.

Proof. (1) If $\operatorname{diam}(\bar{P}(\Gamma_I(R))=0$, then $\bar{P}(\Gamma_I(R))$ is a complete graph K^1, and so $|P(I)|=|R-P(I)|=1$ by Theorem 4.6. Hence $P(I)=\{0\}$ and $|R|=2$. The other implication is clear.

(2) (a) If $\operatorname{diam}(\bar{P}(\Gamma_I(R))=1$, then $\bar{P}(\Gamma_I(R))$ is a complete graph K^λ with $\lambda\leq|P(I)|$ by Theorem 4.6. Therefore, $2\in P(I)$ and $|Q-P(I)|=1$. Thus $R=P(I)\cup(q+P(I))$ for some $q\in Q-P(I)$; hence $|R/P(I)|=2$. The converse follows from Theorem 4.6.

(2) (b) If $\operatorname{diam}(\bar{P}(\Gamma_I(R)))=2$, then $\bar{P}(\Gamma_I(R))$ is a complete bipartite graph $K^{1,2}$ or $K^{2,2}$; thus $2\notin P(I)$ and $|Q-P(I)|=2$ by Theorem 4.6. Since $\bar{P}(\Gamma_I(R))$ has not any totally disconnected subgraph, we must have $q+q'\in P(I)$ for every $q,q'\in Q-P(I)$. □

Proposition 4.10. *Let I be a proper ideal of a semiring R such that $P(I)$ is a k-ideal of R. Then* $\mathrm{gr}(\bar{P}(\Gamma_I(R)) = 3, 4$ *or* ∞. *In particular, if* $\bar{P}(\Gamma_I(R))$ *contains a cycle,* $\mathrm{gr}(\bar{P}(\Gamma_I(R)) \leq 4$.

Proof. Let $\bar{P}(\Gamma_I(R))$ contains a cycle. Then $\bar{P}(\Gamma_I(R))$ is not a totally disconnected graph, so by the proof of Theorem 4.6, $\bar{P}(\Gamma_I(R)))$ has either a complete or a complete bipartite subgraph. Therefore, it must contain either a 3-cycle or a 4-cycle. Thus $\mathrm{gr}(\bar{P}(\Gamma_I(R)) \leq 4$. □

Theorem 4.11. *Let I be a proper ideal of a semiring R such that $P(I)$ is a k-ideal of R. Then*

(1) $\mathrm{gr}(\bar{P}(\Gamma_I(R)) = 3$ *if and only if* $2 \in P(I)$ *and* $|r + P(I)| \geq 3$ *for some* $r \in R - P(I)$.

(2) $\mathrm{gr}(\bar{P}(\Gamma_I(R)) = 4$ *if and only if* $2 \notin P(I)$ *and* $r + r' \in P(I)$ *for some* $r, r' \in R - P(I)$.

Proof. (1) Assume that $\mathrm{gr}((\bar{P}(\Gamma_I(R)) = 3$. Then by Theorem 4.6, $(\bar{P}(\Gamma_I(R))$ is a complete graph K^{λ} with $3 \leq \lambda$. Therefore, $2 \in P(I)$ and $|r + P(I)| \geq 3$ for some $r \in R - P(I)$.

(2) If $\mathrm{gr}((\bar{P}(\Gamma_I(R)) = 4$, then by Theorem 4.6, $(\bar{P}(\Gamma_I(R))$ has a complete bipartite subgraph; hence $2 \notin P(I)$ and $r + r' \in P(I)$ for some $r, r' \in R - P(I)$ by Theorem 4.6. The other implications of (1) and (2) follows directly from Theorem 4.6. □

Theorem 4.12. *Let I be a proper ideal of a semiring R such that $P(I)$ is a k-ideal of R. Then*

(1) $\mathrm{gr}(T(\Gamma_I(R))) = 3$ *if and only if* $|P(I)| \geq 3$.

(2) $\mathrm{gr}(T(\Gamma_I(R))) = 4$ *if and only if* $2 \notin P(I)$, $|P(I)| < 3$ *and* $r + r' \in P(I)$ *for some* $r, r' \in R - P(I)$.

(3) Otherwise, $\mathrm{gr}(T(\Gamma_I(R))) = \infty$.

Proof. (1) This follows from Theorem 4.4.

(2) Since $\mathrm{gr}(P(\Gamma_I(R))) = 3$ or ∞, then $\mathrm{gr}(\bar{P}(\Gamma_I(R))) = 4$. Therefore, $2 \notin P(I)$ and $r + r' \in P(I)$ for some $r, r' \in R - P(I)$ by Theorem 4.11. On the other hand, $\mathrm{gr}(T(\Gamma_I(R))) \neq 3$; so $|P(I)| < 3$. The other implication follows from Theorem 4.6. □

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