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SMALL SUBMODULES WITH RESPECT TO AN ARBITRARY SUBMODULE

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Abstract. Let R be an arbitrary ring and T be a submodule of an R-module M. A submodule N of M is called T-small in M provided for each submodule X of M, $T \subseteq X + N$ implies that $T \subseteq X$. We study this mentioned notion which is a generalization of the small submodules and we obtain some related results.

1. Introduction

In this paper, all rings have identity elements and all modules are right unitary. We use the notations "$\subseteq$" and "$\leq$" to denote inclusion and submodule, respectively. For two integers n and m, we denote $n \mid m$ in case n divides m and $\gcd(n, m)$ denotes the greatest common divisor of n and m.

Let R be a ring and M be an R-module. Recall that a submodule N of M is *small*, denoted by $N \ll M$, if for any submodule X of M, $X + N = M$ implies that $X = M$. More details about small submodules can be found in [2, 3, 4]. The concept of small submodule has been extended by some researchers, for this see [1, 6]. In [5], the authors extended the concept of essential submodule with respect to an arbitrary submodule. This motivates us to define a new generalization of small submodules. Let T be an arbitrary submodule of M. We say that a submodule N of M is an T-*small* submodule of M provided for each submodule $X \leq M$, $T \subseteq X + N$ implies that $T \subseteq X$. Note that

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the notions of smallness and T-smallness coincide if $T = M$. In the first section, we investigate the basic properties of T-small submodules. In the second section, we introduce T-maximal submodules and the T-radical submodule of M, denoted by $\mathrm{Rad}_T M$, and we show that if T is a finitely generated submodule of M, then $\mathrm{Rad}_T M$ is equal to the sum of the certain T-small submodules of M (Theorem 3.2). Also if M and N are right R-modules and $f : M \to N$ is an R-epimorphism such that $\mathrm{Ker} f \subseteq \mathrm{Rad}_T M$, then $f(\mathrm{Rad}_T M) = \mathrm{Rad}_{f(T)} N$ (Theorem 3.6). Finally, T-cosemisimple modules are introduced and a characterization of this class of modules is given in Theorem 3.10.

2. T-small submodules

Definition 2.1. Let R be a ring and T be a submodule of an R-module M. A submodule N of M is called T-small (in M), denoted by $N \ll_T M$, in case for any submodule $X \leq M$, $T \subseteq X + N$ implies that $T \subseteq X$.

Under the notations of the above definition, if $T = 0$, then every submodule of M is T-small in M. Also if $T \neq 0$, then $N \ll_T M$ implies that $T \nsubseteq N$, for if not, $T \subseteq N + (0)$ and hence $T \subseteq (0)$, a contradiction. If $T = M$, then $N \ll_T M$ if and only if $N \ll M$.

Example 2.2. (a) Let $\mathbb{Z}$ be the ring of integers. It is easy to see that (0) is the only small submodule of $\mathbb{Z}$ and also for any nonzero integer m, the submodule (0) is the only $m\mathbb{Z}$-small submodule of $\mathbb{Z}$.

(b) Let $\mathbb{Z}_n$ be the ring of integers modulo n and $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_t}^{\alpha_t}$ where $p_i^{,}s$ be distinct prime numbers and $\alpha_i \geq 0$. One can verify that $k\mathbb{Z}_n \ll \mathbb{Z}_n$ if and only if $k = q{p_1}^{\beta_1} {p_2}^{\beta_2} \dots {p_t}^{\beta_t}$, where $\gcd(q, n) = 1$ and for any $1 \leq i \leq t$, $1 \leq \beta_i \leq \alpha_i$.

(c) Let n, m and k be positive integers. Then $k\mathbb{Z}_n \ll_{m\mathbb{Z}_n} \mathbb{Z}_n$ if and only if $\gcd(n, k) \nmid m$ and for any $w \in \mathbb{Z}$, $\gcd(\gcd(k, w), n) \mid m$ implies that $\gcd(w, n) \mid m$.

(d) $4\mathbb{Z}_{24} \ll_{3\mathbb{Z}_{24}} \mathbb{Z}_{24}$ but $4\mathbb{Z}_{24}$ is not small in $\mathbb{Z}_{24}$.

(e) In $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$, where p is a prime number, we set $H_n =< 1/p^n + \mathbb{Z} >$ and $H_m =< 1/p^m + \mathbb{Z} >$. Then $m > n$ if and only if $H_n \ll_{H_m} \mathbb{Z}_{p^\infty}$.

Proposition 2.3. *Let M be an R-module, $L \leq T \leq M$ and $K \leq M$. Then*
(1) If $K \ll_T M$, then $K \cap T \ll M$;
(2) $L \ll_T M$ if and only if $L \ll T$.

Proof. (1) Suppose that $(K \cap T) + X = M$ for some $X \leq M$. Then $T \subseteq (K \cap T) + X \subseteq K + X$ and since $K \ll_T M$, we have $T \subseteq X$. Thus $K \cap T \subseteq X$ and hence $X = (K \cap T) + X = M$.
(2) Suppose that $L \ll_T M$ and $L + X = T$ for some $X \leq T$. Then $T \subseteq L + X$ and so $T \subseteq X$. Thus $X = T$. Conversely, suppose that $L \ll T$ and $T \subseteq L + X$ for some $X \leq M$. Then $T = (L + X) \cap T = L + (X \cap T)$ and hence $X \cap T = T$. Thus $T \subseteq X$, as desired. □

Proposition 2.4. *Let M be an R-module with submodules $N \leq K \leq M$ and $T \leq K$. If $N \ll_T K$, then $N \ll_T M$*

Proof. Suppose that $T \subseteq N + X$, for some $X \leq M$. Then $T \subseteq (N + X) \cap K = N + (X \cap K)$. Since $N \ll_T K$, we have $T \subseteq X \cap K \subseteq X$. □

Proposition 2.5. *Let M be an R-module with submodules N_1, N_2 and T. Then $N_1 \ll_T M$ and $N_2 \ll_T M$ if and only if $N_1 + N_2 \ll_T M$*

Proof. Clear. □

Theorem 2.6. *Let M be an R-module with submodules $K \leq N \leq M$ and $K \leq T$. Then $N \ll_T M$ if and only if $K \ll_T M$ and $N/K \ll_{T/K} M/K$.*

Proof. Suppose that $N \ll_T M$ and $T \subseteq K + X$ for some $X \leq M$. Then $T \subseteq N + X$ and by hypothesis, $T \subseteq X$. Thus $K \ll_T M$. Now assume that $T/K \subseteq N/K + X/K = (N + X)/K$ for some $K \leq X \leq M$. Then $T \subseteq N + X$ and so $T \subseteq X$. Thus $T/K \subseteq X/K$. Conversely, suppose that $K \ll_T M$ and $N/K \ll_{T/K} M/K$ and also $T \subseteq N + X$ for some $X \leq M$. Then $T/K \subseteq (N + X)/K = N/K + (X + K)/K$. Since $N/K \ll_{T/K} M/K$, $T/K \subseteq (X + K)/K$ and so $T \subseteq X + K$. Since $K \ll_T M$, we have $T \subseteq X$, as desired. □

Proposition 2.7. *Let M be an R-module with $K_1 \leq M_1 \leq M$ and $K_2 \leq M_2 \leq M$ such that $T \subseteq M_1 \cap M_2$. Then $K_1 \ll_T M_1$ and $K_2 \ll_T M_2$ if and only if $K_1 + K_2 \ll_T M_1 + M_2$.*

Proof. First assume that $K_1 \ll_T M_1$ and $K_2 \ll_T M_2$. By Proposition 2.4, $K_1 \ll_T M_1 + M_2$ and $K_2 \ll_T M_1 + M_2$. Also by Proposition 2.5, $K_1 + K_2 \ll_T M_1 + M_2$. The other direction is clear. □

Theorem 2.8. *Let $\{T_i\}_{i \in I}$ be an indexed set of submodules of an R-module M and K be a submodule of M. If for each $i \in I$, $K \ll_{T_i} M$, then $K \ll_{\sum_{i \in I} T_i} M$.*

Proof. Suppose that $\sum_{i \in I} T_i \subseteq K + X$, for some $X \leq M$. Then for each $i \in I$, $T_i \subseteq K + X$ and by hypothesis, $T_i \subseteq X$. Thus $\sum_{i \in I} T_i \subseteq X$. □

Corollary 2.9. *Let K_1 and K_2 be submodules of an R-module M such that $K_1 \ll_{K_2} M$ and $K_2 \ll_{K_1} M$. Then $K_1 \cap K_2 \ll_{K_1+K_2} M$.*

Proof. Since $K_1 \ll_{K_2} M$ and $K_2 \ll_{K_1} M$, by Theorem 2.6, $K_1 \cap K_2 \ll_{K_2} M$ and $K_1 \cap K_2 \ll_{K_1} M$. Also by Theorem 2.8, $K_1 \cap K_2 \ll_{K_1+K_2} M$. $\square$

Let M, N be two right R-modules and $0 \neq T \leq M$. An R-epimorphism $f : M \to N$ is called *T-small* in case $\mathrm{Ker} f \ll_T M$.

Proposition 2.10. *Let K and $0 \neq T$ be two submodules of a right R-module M. The following statements are equivalent:*

(1) $K \ll_T M$;

(2) The natural map $P_K : M \to M/K$ is T-small;

(3) For every right R-module N and R-homomorphism $h : N \to M$, $T \subseteq K + Imh$ implies that $T \subseteq Imh$.

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are clear by the definition.

(3) $\Rightarrow$ (1). Suppose that $T \subseteq K + X$ for some $X \leq M$. Let $i : X \to M$ be the inclusion map. Then $T \subseteq K + Imi = K + X$ and by (3), $T \subseteq X$. $\square$

Lemma 2.11. *Let M and N be right R-modules and $f : M \to N$ be an R-homomorphism. If K and T are submodules of M such that $K \ll_T M$, then $f(K) \ll_{f(T)} N$. In particular, if $K \ll_T M \leq N$, then $K \ll_T N$.*

Proof. We may assume that $f(T) \neq 0$. Let $f(T) \subseteq f(K) + X$, for some $X \leq N$. We claim that $T \subseteq K + f^{-1}(X)$. Let $t \in T$. Then $f(t) = f(k) + x$ for some $k \in K$ and $x \in X$. Thus $f(t-k) \in X$ and so $t - k \in f^{-1}(X)$. This implies that $t \in K + f^{-1}(X)$. Since $K \ll_T M$, we have $T \subseteq f^{-1}(X)$ and hence $f(T) \subseteq X$. $\square$

Now we have the following evident result.

Corollary 2.12. *Let M and N be right R-modules and $f : M \to N$ be an R-monomorphism. If K and T are submodules of M, then $K \ll_T M$ if and only if $f(K) \ll_{f(T)} N$.*

Let M and N be R-modules and $f : M \to N$ is an R-homomorphism. If $N_1 \ll N$, then we do not conclude that $f^{-1}(N_1) \ll M$. For example, consider $f : \mathbb{Z}_{10} \to \mathbb{Z}_{20}$ with $f(\overline{x}) = \overline{2x}$. Then $10\mathbb{Z}_{20} \ll \mathbb{Z}_{20}$ but

$f^{-1}(10\mathbb{Z}_{20}) = 5\mathbb{Z}_{10}$ is not small in $\mathbb{Z}_{10}$.

Let M be an R-module and $N \leq M$. If $N' \leq M$ is minimal with respect to $N + N' = M$, then N' is called a *supplement* of N in M.

Proposition 2.13. *Let N and T be submodules of an R-module M and N' be a supplement of N in M. If $N \ll_T M$, then $T \subseteq N'$. Moreover, if $N \ll_T M$ and $N + T = M$, then $N' = T$.*

Proof. Clear. □

Theorem 2.14. *Let K be a submodule of an R-module M and K' is a supplement of K in M. The following are equivalent:*
(1) $K \ll_{K'} M$;
(2) For each submodule N of M, the relation $K + N = M$ implies that $K' \subseteq N$.

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1). Suppose that $K' \subseteq K + X$ some $X \leq M$. Since $M = K + K' \subseteq K + X$, we have $M = K + X$ and by hypothesis, $K' \subseteq X$. □

3. The T-radical of a module

Definition 3.1. Let M be an R-module and $T \leq M$. A submodule K of M is called T-maximal (in M) if $(T + K)/K$ is a simple R-module.

Theorem 3.2. *Let M be a right R-module and $0 \neq T$ be a proper finitely generated submodule of M. Then*

$$\sum_{L \in A} L = \bigcap_{K \in B} K,$$

where

$$A = \{L \leq M \mid L \ll_T M \text{ and } L + K \subseteq T + K, \text{ for all } T\text{-maximal submodule } K \text{ of } M\}$$

and

$$B = \{K \leq M \mid K \text{ is an } T\text{-maximal submodule of } M\}.$$

Proof. Suppose that $L \in A$ and $K \in B$. We show that $L \subseteq K$. If $L \nsubseteq K$, then $K \subsetneqq L + K \leq T + K$. Since K is T-maximal, $L + K = T + K$ and so $T \subseteq L + K$. Since $L \ll_T M$, we have $T \subseteq K$, a contradiction (note that $(T + K)/K \neq 0$). Thus $\sum_{L \in A} L \subseteq \bigcap_{K \in B} K$. Conversely, let $x \in \cap_{K \in B} K$. We show that $xR \in A$. Suppose that N is a submodule of M such that $T \subseteq xR + N$. If $T \nsubseteq N$, then we set $S = \{K \leq M \mid T \nsubseteq K$ and $N \subseteq K\}$. We show that S has a maximal element. Since $N \in S$,

$S \neq \emptyset$. Let $T = \Sigma_i^n x_i R$ for some $\{x_1, \dots, x_n\} \subseteq M$. Assume that Λ is a chain in S. Clearly $N \subseteq \bigcup_{K \in \Lambda} K \leq M$. If $T \subseteq \bigcup_{K \in \Lambda} K$, then there exists $\{K_1, \dots, K_n\} \subseteq \Lambda$ such that for any $1 \leq i \leq n$, $x_i \in K_i$. Since Λ is chain, we may assume that $K_i \subseteq K_n$ for $1 \leq i \leq n$. Thus $T \subseteq K_n$, a contradiction. Therefore $T \nsubseteq \bigcup_{K \in \Lambda} K$ and so $\bigcup_{K \in \Lambda} K$ is an upper bounded for Λ. Now by Zorn's lemma, S has a maximal element, say, K. We claim that K is a T-maximal submodule of M. Note that $(T + K)/K \neq 0$. Suppose that $K \leq W \leq T + K$ such that $K \subsetneqq W$. By the maximality of K, we have $T \subseteq W$ and hence $W = T + K$. Thus K is T-maximal and $x \in K$, a contradiction because $x \in \bigcap_{K \in B} K$. Therefore $xR \ll_T M$. On the other hand, for any T-maximal submodule K of M, $K = xR + K \subseteq T + K$ and so $xR \in A$. Thus $\bigcap_{K \in B} K \subseteq \sum_{L \in A} L$. □

We have not found any examples of a module M with a proper submodule T for which $\sum_{L \in A} L \neq \bigcap_{K \in B} K$, where A and B are the same as in Theorem 3.2. The lack of such counterexamples together with Theorem 3.2 motivates the following conjecture.

Cojecture 3.3. *Let M be a right R-module and $0 \neq T$ be a proper submodule of M. Then $\sum_{L \in A} L = \bigcap_{K \in B} K$, where A and B are the same as in* Theorem 3.2.

Lemma 3.4. *Let M and N be right R-modules and $f : M \to N$ be an R-homomorphism. If T is a submodule of M and K is an T-maximal submodule of M such that $\ker f \subseteq K$, then $f(K)$ also is an $f(T)$-maximal submodule of N.*

Proof. We show that $(f(K) + f(T))/f(K)$ is simple. First we claim that $f(T) \nsubseteq f(K)$. If $f(T) \subseteq f(K)$, then $f(t) \in f(K)$ for some $t \in T \setminus K$ because $T \nsubseteq K$. Thus $f(t) = f(k)$ for some $k \in K$ and so $t - k \in \ker f \leq K$. Therefore $t \in K$, a contradiction. So $(f(K) + f(T))/f(K) \neq 0$. Now let W be a submodule of N such that $f(K) \subsetneqq W \leq f(K) + f(T)$. Then $K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(W) \subseteq f^{-1}(f(K + T))$. Since $\ker f \subseteq K$, we have $f^{-1}(f(K + T)) = K + T$. On the other hand, $K \subsetneqq f^{-1}(W)$, for if not, $K = f^{-1}(W)$ and since $f(K) \subsetneqq W$, there exists $x \in W \setminus f(K)$. Now $W \leq f(K) + f(T)$ implies that $x = f(k+t)$ for some $k \in K$ and $t \in T$. Thus $k+t \in f^{-1}(W) = K$ and so $t \in K$. It follows that $x \in f(K)$, a contradiction. Since K is T-maximal, we have $f^{-1}(W) = K + T$ and hence $W = f(K) + f(T)$. This proves that $f(K)$ is an $f(T)$-maximal in N. □

Lemma 3.5. *Let M and N be right R-modules and $f : M \to N$ be an R-epimorphism. If T is a submodule of M and K is an $f(T)$-maximal submodule of N, then $f^{-1}(K)$ also is an T-maximal submodule of M.*

Proof. First we note that $(f^{-1}(K) + T)/f^{-1}(K) \neq 0$, for if not, $T \subseteq f^{-1}(K)$ and so $f(T) \subseteq f(f^{-1}(K)) \subseteq K$, a contradiction. Now suppose that $f^{-1}(K) \subseteq W \subseteq f^{-1}(K) + T$ for some $W \leq M$. Then $K = f(f^{-1}(K)) \subseteq f(W) \subseteq f(f^{-1}(K) + T) = K + f(T)$. Since K is $f(T)$-maximal, we have $f(W) = K$ or $f(W) = K + T$. If $f(W) = K$, then $W \subseteq f^{-1}(f(W)) = f^{-1}(K)$ and so $W = f^{-1}(K)$. Thus we assume that $f(W) = K + T$ and let $a \in f^{-1}(K)$ and $t \in T$. Then $f(a) + f(t) = f(w)$ for some $w \in W$ and hence $a + t - w \in \mathrm{Ker} f \subseteq f^{-1}(K) \subseteq W$. Therefore $a + t \in W$ and this implies that $f^{-1}(K) + T \subseteq W$. Thus $f^{-1}(K) + T = W$ and this means $f^{-1}(K)$ is an T-maximal submodule of M. □

Let M be an R-module and $T \leq M$. We denote the intersection of all T-maximal submodules in M by $\mathrm{Rad}_T M$.

Theorem 3.6. *Let M and N be right R-modules and $f : M \to N$ be an R-epimorphism such that $Kerf \subseteq Rad_T M$. Then $f(Rad_T M) = Rad_{f(T)} N$.*

Proof. Since f is epic, by Lemma 3.4 and Lemma 3.5, we have

$$f(\mathrm{Rad}_T M) = f(\cap_A K) = \cap_B f(K) = \mathrm{Rad}_{f(T)} N,$$

where,

$$A = \{K \leq M \mid K \text{ is an } T\text{-maximal submodule of } M\}$$

and

$$B = \{f(K) \leq N \mid f(K) \text{ is an } f(T)\text{-maximal submodule of } N\}.$$

□

Proposition 3.7. *Let M be an R-module and $T \leq M$. If every proper submodule X of M with $T \nsubseteq X$ is contained in an T-maximal submodule of M, then $Rad_T M$ is an T-small submodule of M*

Proof. Suppose that $T \subseteq \mathrm{Rad}_T M + X$ for some $X \leq M$. If $T \nsubseteq X$, then by hypothesis there exists an T-maximal submodule K of M containing X. Then $T \subseteq \mathrm{Rad}_T M + X \subseteq K$, which contradicts the T-maximality of K. Thus $T \subseteq X$. □

We have the following evident result.

Corollary 3.8. *Let T be a finitely generated submodule of an R-module M. Then $Rad_T M \ll_T M$.*

Proposition 3.9. *Let M be an R-module and T be a semisimple submodule of M. Then $Rad_T M = 0$.*

Proof. Suppose $T = \oplus_I T_i$, where T_i is a simple submodule of M for all $i \in I$. Since $\oplus_{i \neq j \in I} T_j$ is T-maximal and $\bigcap_{i \in I}(\oplus_{i \neq j \in I} T_j) = 0$, we have $\mathrm{Rad}_T M = 0$. □

Let M be an R-module and T be a nonzero submodule of M. We say that M is T-cosemisimple if every submodule of M is the intersection of T-maximal submodules. We conclude the paper with the following interesting theorem.

Theorem 3.10. *Let M be an R-module and T be a nonzero submodule of M. Then*
(1) *M is T-cosemisimple if and only if $Rad_{\frac{T+K}{K}}(M/K) = 0$, for all $K \leq M$.*
(2) *If M is T-cosemisimple, then every submodule of M containing T is T-cosemisimple module and also M/N is $(T+K)/K$-cosemisimple module for all $N \leq M$.*
(3) *Let $\{N_\alpha\}_{\alpha \in A}$ be an indexed set of simple submodules of M and $M = \oplus_A N_\alpha$. Let $N \leq M$, $T \leq M$ and $B \subseteq A$ such that $N \cong \oplus_B N_\alpha$ and $\oplus_{A \setminus B} N_\alpha \leq T \leq M$. Then N is the intersection of T-maximal submodules.*

Proof. (1) Suppose that M is T-cosemisimple and $K \leq M$. By the hypothesis, $K = \bigcap_B S$, where B is a set of T-maximal submoduls of M. Thus $\mathrm{Rad}_{\frac{T+K}{K}}(M/K) = \bigcap_A S/K = \frac{\bigcap_{A'} S}{K} = \frac{\bigcap_{A'} S}{\bigcap_B S} = 0$, where

$$A = \{S/K \leq M/K \mid S/K \text{ is an } \tfrac{T+K}{K}\text{-maximal submodule of } M/K\},$$
$$A' = \{K \leq S \leq M \mid S \text{ is an } T\text{-maximal submodule of } M\}$$

and we note that $B \subseteq A'$ and $S/K \in A$ if and only if $S \in A'$.
Conversely, Suppose that $\mathrm{Rad}_{\frac{T+K}{K}}(M/K) = 0$, for all $K \leq M$ and K is a submodule of M. Then $\mathrm{Rad}_{\frac{T+K}{K}}(M/K) = \bigcap_A S/K = \frac{\bigcap_{A'} S}{K} = 0$, where

$$A = \{S/K \leq M/K \mid S/K \text{ is an } \tfrac{T+K}{K}\text{-maximal submodule of } M/K\}$$

and

$$A' = \{K \leq S \leq M \mid S \text{ is an } T\text{-maximal submodule of } M\}.$$

This means that $K = \bigcap_{A'} S$.
(2) Suppose that $T \subseteq N \leq M$ and M is T-cosemisimple. If $L \leq N$, then $L = L \cap N = (\bigcap_A S) \cap N = \bigcap_A (S \cap N)$, where A is a set of

T-maximal submodules of M. Note that $\frac{(S\cap N)+T}{S\cap N} \cong \frac{T}{(S\cap N)\cap T} = \frac{T}{S\cap T} \cong \frac{S+T}{S}$ is a simple R-module. Thus N is T-cosemisimple. Now assume that $N \leq M$ and $L/N \leq M/N$. Then $L/N = \frac{\bigcap_A S}{N} = \bigcap_A S/N$, where A is a set of T-maximal submodules of M. We note that if S is T-maximal, then S/N is $(T+N)/N$-maximal. Thus M/N is $(T+N)/N$-cosemisimple.

(3) We have $N \cong \oplus_B N_\alpha = \bigcap_{\alpha\in A\setminus B}(\oplus_{\beta\in A\setminus\{\alpha\}} N_\beta)$. We show that for any $\alpha \in A\setminus B$, the maximal submodule $\oplus_{A\setminus\{\alpha\}} N_\beta$ is T-maximal. Since $\frac{\oplus_{A\setminus\{\alpha\}} N_\beta + T}{\oplus_{A\setminus\{\alpha\}} N_\beta} \leq \frac{M}{\oplus_{A\setminus\{\alpha\}} N_\beta} \cong N_\alpha$ and N_α is simple, the proof is complete if $\frac{\oplus_{A\setminus\{\alpha\}} N_\beta + T}{\oplus_{A\setminus\{\alpha\}} N_\beta} \neq 0$. In otherwise, $T \subseteq \oplus_{A\setminus\{\alpha\}} N_\beta$ and so $\oplus_{A\setminus B} N_\beta \subseteq T \subseteq \oplus_{A\setminus\{\alpha\}} N_\beta$, a contradiction. $\square$

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