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CASTELNUOVO-MUMFORD REGULARITY OF PRODUCTS OF MONOMIAL IDEALS

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Abstract. Let $R = k[x_1, x_2, \cdots, x_N]$ be a polynomial ring over a field k. We prove that for any positive integers m, n,

$$\mathrm{reg}(I^m J^n K) \leq m\mathrm{reg}(I) + n\mathrm{reg}(J) + \mathrm{reg}(K)$$

if $I, J, K \subseteq R$ are three monomial complete intersections (I, J, K are not necessarily proper ideals of the polynomial ring R), and I, J are of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \cdots, x_{i_l}^{a_l})$.

1. Introduction

Through this paper, let $R = k[x_1, x_2, \cdots, x_N]$ be a polynomial ring over a field k and I a graded ideal. Let M be a finitely generated graded R-module and $\mathfrak{m}$ be the maximal graded ideal of R. Let $a_i(M) := \max\{j \mid H^i_{\mathfrak{m}}(M)_j \neq 0\}$ if $H^i_{\mathfrak{m}}(M) \neq 0$, and $a_i(M) := -\infty$ otherwise. The Castelnuovo-Mumford regularity (or regularity for short) of M is defined as the invariant

$$\mathrm{reg}(M) := \max\{a_i(M) + i : i \geq 0\}.$$

The regularity is one of the most important invariants of a finitely genetated graded module M over a polynomial ring R. In particular, it is of great interest to have a good bound for this invariant. For

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example, it is natural to ask whether the following formula holds:

$$\operatorname{reg}(IM) \leq \operatorname{reg}(I) + \operatorname{reg}(M)$$

This is not the case in general. There are several counterexamples already with $M = I$ such that $\operatorname{reg}(I^2) > 2\operatorname{reg}(I)$, see Sturmfels [9] and Terai [10]. On the other hand, for a graded ideal I with $\dim R/I \leq 1$,, Geramita, Gimigliano and Pitteloud [6] and Chandler [2], have shown that

$$\operatorname{reg}(I^k) \leq k\operatorname{reg}(I).$$

This inequality was generalized by Conca and Herzog [5] to the result (if $\dim R/I \leq 1$):

$$\operatorname{reg}(IM) \leq \operatorname{reg}(I) + \operatorname{reg}(M).$$

At the same time, a question was raised by them.
Question. ([5, Question 3.6]): Is it true that

$$\operatorname{reg}(I_1 I_2 \cdots I_d) \leq \operatorname{reg}(I_1) + \operatorname{reg}(I_2) + \cdots + \operatorname{reg}(I_d)$$

for any set of complete intersections $I_1, I_2, \cdots, I_d$?

The question has a negative answer for non-monomial complete intersections $I_1, I_2, \cdots, I_d$ (see [3, Example 4.5]). On the other hand, along the positive direction, it holds in some cases:

(i) Each of $I_1, I_2, \cdots, I_d$ is generated by linear forms ([5, Theorem 3.1]).

(ii) $d = 2$ and I_1, I_2 are monomial complete intersections ([[3, Theorem 1.1]).

In [4] , Cimpoeaş proved that for two monomial ideals of Borel type I, J,

$$\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J).$$

These results (specially, [3, Theorem 1.1] and Cimpoeaş's result) give us an inspiration to consider the regularity of the products of ideals of Borel type and monomial complete intersections. In fact, it is not hard to find that

$$\operatorname{reg}(IK) \leq \operatorname{reg}(I) + \operatorname{reg}(K),$$

for a monomial ideal of Borel type I and a monomial complete intersection K (Corollary 2.7). Following the idea of this result, we prove the result below:

Let $I, J, K \subseteq R$ be three monomial complete intersections (I, J, K are not necessarily proper ideals of the polynomial ring R). Assume *that I, J are generated by some powers of some varibles, i.e., I, J are of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \cdots, x_{i_l}^{a_l})$. Then for any positive integers m, n,*

$$\mathrm{reg}(I^m J^n K) \leq m\mathrm{reg}(I) + n\mathrm{reg}(J) + \mathrm{reg}(K).$$

2. Results

We begin this section with some remarks.

Remark 2.1. (i) ([8]) Let I, J be two monomial ideals of R. Then

$$\mathrm{reg}(I + J) \leq \mathrm{reg}(I) + \mathrm{reg}(J) - 1,$$

$$\mathrm{reg}(I \cap J) \leq \mathrm{reg}(I) + \mathrm{reg}(J).$$

(ii) Let $I = (u_1, u_2, \cdots, u_r) \subseteq R$ be a monomial ideal. Then I is a complete intersection if and only if $Supp(u_i) \cap Supp(u_j) = \emptyset, i \neq j$, where $Supp(u) = \{j \mid x_j | u\}$ for a monomial $u \in R$. Hence, for a ring isomorphism φ: $R \longrightarrow R$ given by a reordering of variables, $\varphi(I)$ is still a complete intersection if I is a monomial complete intersection.

Remark 2.2. Let $I, J, K \subseteq R$ be three monomial complete intersections, and I, J be of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \cdots, x_{i_l}^{a_l})$. We can define a ring isomorphism φ: $R \longrightarrow R$ given by a reordering of variables, such that

$$\varphi(I) = (x_1^{a_1}, x_2^{a_2}, \cdots, x_s^{a_s})$$

and

$$\varphi(J) = (x_1^{b_1}, x_2^{b_2}, \cdots, x_r^{b_r}, y_1^{b_{r+1}}, \cdots, y_t^{b_{r+t}}),$$

$r \leq s, \{y_1, \cdots, y_t\} \subseteq \{x_{r+1}, x_{r+2}, \cdots, x_n\}$. Since the Castelnuovo-Mumford regularity is an invariant, it follows that for any positive integers m, n,

$$\mathrm{reg}(I^m J^n K) = \mathrm{reg}(\varphi(I)^m \varphi(J)^n \varphi(K)),$$

and $\mathrm{reg}(I) = \mathrm{reg}(\varphi(I))$, $\mathrm{reg}(J) = \mathrm{reg}(\varphi(J))$, $\mathrm{reg}(K) = \mathrm{reg}(\varphi(K))$.

Lemma 2.3. *Let I, J be two arbitrary graded ideals. Then*

$$reg(I \cap J) \leq max\{reg(I), reg(J), reg(I + J) + 1\}.$$

We consider the monomial ideal $I = (x_1^{l_1}, x_2^{l_2}, \cdots, x_s^{l_s}) \subseteq R$, where $s \leq N$ is a positive integer and $l_1 \geq l_2 \geq \cdots \geq l_s \geq 1$ are integers. Let $\bar{R} = K[x_1, x_2, \cdots, x_s]$ and $\bar{I} = \bar{R} \cap I$. By the definition of the regularity, $\mathrm{reg}(I) = \mathrm{reg}(\bar{I}) = \mathrm{reg}(\bar{R}/\bar{I}) + 1 = \max\{j \mid H^0_{(x_1, x_2, \cdots, x_s)}(\bar{R}/\bar{I})_j \neq 0\} + 1 = \max\{j \mid (\bar{R}/\bar{I})_j \neq 0\} + 1$. It is clear that the monomial $x_1^{l_1 - 1} x_2^{l_2 - 1} \cdots x_s^{l_s - 1} \in \bar{R}$ is the monomial of the maximal degree which is not in $\bar{I}$. Then

$$\mathrm{reg}(I) = (l_1 - 1) + (l_2 - 1) + \cdots + (l_s - 1) + 1 = l_1 + l_2 + \cdots + l_s - s + 1.$$

In addition, note that $(\bar{I})^m = \bar{R} \cap I^m$, and then, as above, $\mathrm{reg}(I^m) = \max\{j \mid (\bar{R}/(\bar{I})^m))_j \neq 0\} + 1$ We can easily see that the monomial $x_1^{ml_1-1}x_2^{l_2-1}\cdots x_s^{l_s-1} \in \bar{R}$ is the monomial of the maximal degree which is not in $(\bar{I})^m$. Therefore,

$$\mathrm{reg}(I^m) = (ml_1-1)+(l_2-1)+\cdots+(l_s-1)+1 = ml_1+l_2+\cdots+l_s-s+1,$$

and

$$\mathrm{reg}(I^m) \leq m\mathrm{reg}(I).$$

Generally, for the monomial ideal of complete intersection $I = (u_1, u_2, \cdots, u_s) \subseteq R$, where $s \leq N$ is a positive integer, $u_1, u_2, \cdots, u_s$ is an $R-$regular sequence and $\mathrm{deg} u_i = l_i$, $i = 1, 2, \cdots, s$, by the main result of [7], the graded Betti numbers of I^m depend only on the numbers $\{m, l_1, l_2, \cdots, l_s\}$. Then the following result is given.

Remark 2.4. Let $I = (u_1, u_2, \cdots, u_s)$ be a monomial complete intersection, where $u_1, u_2, \cdots, u_s$ is an $R-$regular sequence and $\mathrm{deg} u_i = l_i$, $i = 1, 2, \cdots, s$, with $l_1 \geq l_2 \geq \cdots \geq l_s \geq 1$. Then

$$\mathrm{reg}(I^m) = (ml_1-1)+(l_2-1)+\cdots+(l_s-1)+1 = ml_1+l_2+\cdots+l_s-s+1.$$

In particular, $\mathrm{reg}(I^m) \leq m\mathrm{reg}(I)$. Moreover, if $m \geq 2$ is an integer, then $\mathrm{reg}(I^m) = m\mathrm{reg}(I)$ if and only if $l_2 = \cdots = l_s = 1$.

Lemma 2.5. *([1, Corollary 3.17]) Let I be an ideal of Borel type with a irredundant irreducible decomposition $I = q_1 \cap q_2 \cap \cdots \cap q_r$. Then*

$$reg(I) = \max\{reg(q_1), \cdots, reg(q_r)\}.$$

Proposition 2.6. *Let I, J be two monomial ideals of Borel type in R and Q an arbitrary monomial ideal in S (J, Q are not necessarily proper ideals of the polynomial ring R). Then $(IJ : Q)$ is also of Borel type. Moreover,*

$$reg(IJ : Q) \leq reg(I) + reg(J).$$

In particular, $reg(IJ) \leq reg(I) + reg(J)$ and $reg(I^m) \leq m reg(I)$.

Proof. It is easy to prove that $(IJ : Q)$ is of Borel type by the known results (for example, [4, Proposition 1.1]). For the second statement, let $IJ = q_1 \cap q_2 \cap \cdots \cap q_r$ be a irredundant irreducible decomposition of IJ. Then $(IJ : Q) = (q_1 : Q) \cap (q_2 : Q) \cap \cdots \cap (q_r : Q)$ is a irreducible decomposition of $(IJ : Q)$. Note that $\mathrm{reg}(q_i : Q) \leq \mathrm{reg}(q_i)$. By Lemma 2.5, $\mathrm{reg}(IJ : Q) \leq \max\{\mathrm{reg}(q_1 : Q), \cdots, \mathrm{reg}(q_r : Q)\} \leq \max\{\mathrm{reg}(q_1), \cdots, \mathrm{reg}(q_r)\} = \mathrm{reg}(IJ)$. Then by [4, Theorem 1.7],

$$\mathrm{reg}(IJ : Q) \leq \mathrm{reg}(IJ) \leq \mathrm{reg}(I) + \mathrm{reg}(J).$$

Corollary 2.7. *Let I be a monomial ideal of Borel type and K a monomial complete intersection. Then*

$$reg(IK) \leq reg(I) + reg(K).$$

Proof. We follow the method of [3, Theorem 3.1]. As stated in [3, Remark 3.2], we only need to prove that $\mathrm{reg}(I : Q) \leq \mathrm{reg}(I)$ for an arbitrary monomial ideal Q. Then the result follows by Proposition 2.6.

Proposition 2.8. *Assume that I, J are generated by some powers of some varibles, i.e., I, J are of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \cdots, x_{i_l}^{a_l})$. Then for any positive integers m, n, and an arbitrary monomial ideal Q,*

$$reg(I^m J^n : Q) \leq m reg(I) + n reg(J).$$

In particular, $reg(I^m J^n) \leq m reg(I) + n reg(J)$.

Proof. By Remark 2.2, we can assume that $I = (x_1^{a_1}, x_2^{a_2}, \cdots, x_s^{a_s})$ and $J = (x_1^{b_1}, x_2^{b_2}, \cdots, x_r^{b_r}, y_1^{b_{r+1}}, \cdots, y_t^{b_{r+t}})$, $r \leq s$, $\{y_1, \cdots, y_t\} \subseteq \{x_{r+1}, x_{r+2}, \cdots, x_n\}$. Set $J_1 = (x_1^{b_1}, x_2^{b_2}, \cdots, x_r^{b_r})$, $J_2 = (y_1^{b_{r+1}}, \cdots, y_t^{b_{r+t}})$. Since I, J are two monomial ideals, we have that $I^m J^n = I^m(J_1 + J_2)^n = I^m(J_1^n + J_1^{n-1}J_2 + \cdots + J_1 J_2^{n-1} + J_2^n) = I^m J_1^n + I^m J_1^{n-1} J_2 + \cdots + I^m J_1 J_2^{n-1} + I^m J_2^n$ $= I^m J_1^n + I^m J_1^{n-1} J_2 + \cdots + I^m J_1 J_2^{n-1} + I^m \cap J_2^n = I^m \cap (I^m J_1^n + I^m J_1^{n-1} J_2 + \cdots + I^m J_1 J_2^{n-1} + J_2^n)$
$= I^m \cap (I^m J_1^n + I^m J_1^{n-1} + \cdots + I^m J_1 J_2^{n-1} + J_2^n) \cap (I^m J_1^n + J_2 + \cdots + I^m J_1 J_2^{n-1} + J_2^n)$
$= I^m \cap (I^m J_1^{n-1} + \cdots + I^m J_1 J_2^{n-1} + J_2^n) \cap (I^m J_1^n + J_2) = I^m \cap (I^m J_1^{n-2} + \cdots + I^m J_1 J_2^{n-1} + J_2^n) \cap (I^m J_1^{n-1} + J_2^2) \cap (I^m J_1^n + J_2) = \cdots = I^m \cap (I^m J_1 + J_2^n) \cap (I^m J_1^2 + J_2^{n-1}) \cap \cdots \cap (I^m J_1^{n-1} + J_2^2) \cap (I^m J_1^n + J_2)$.

Therefore, $(I^m J^n : Q) = (I^m \cap (I^m J_1 + J_2^n) \cap (I^m J_1^2 + J_2^{n-1}) \cap \cdots \cap (I^m J_1^{n-1} + J_2^2) \cap (I^m J_1^n + J_2) : Q) = (I^m : Q) \cap ((I^m J_1 + J_2^n) : Q) \cap ((I^m J_1^2 + J_2^{n-1}) : Q)) \cap \cdots \cap ((I^m J_1^{n-1} + J_2^2) : Q) \cap ((I^m J_1^n + J_2) : Q)$.

Let $L_i = (I^m J_1^{n-i+1} + J_2^i) : Q$, $i = 1, 2, \cdots, n$. we want to prove that $\mathrm{reg}(\cap_{i=1}^n L_i) \leq m\mathrm{reg}(I) + n\mathrm{reg}(J)$ by induction.

When $i = 1$, since I, J_1 are of Borel type, by Remark 2.1(i) and Proposition 2.6, $\mathrm{reg}(L_1) = \mathrm{reg}((I^m J_1^n + J_2) : Q) = \mathrm{reg}(I^m J_1^n : Q + J_2 : Q) \leq \mathrm{reg}(I^m J_1^n : Q) + \mathrm{reg}(J_2 : Q) - 1 \leq m\mathrm{reg}(I) + n\mathrm{reg}(J_1) + \mathrm{reg}(J_2) - 1 \leq m\mathrm{reg}(I) + n\mathrm{reg}(J)$ (Note that $\mathrm{reg}(J) = \mathrm{reg}(J_1) + \mathrm{reg}(J_2) - 1$). In fact, by the same reason, for all $L_i's$, $\mathrm{reg}(L_i) \leq m\mathrm{reg}(I) + n\mathrm{reg}(J)$.

Assume that $\mathrm{reg}(\cap_{i=1}^{t-1} L_i) \leq m\mathrm{reg}(I) + n\mathrm{reg}(J)$. In order to prove that $\mathrm{reg}(\cap_{i=1}^{t} L_i) \leq m\mathrm{reg}(I) + n\mathrm{reg}(J)$, by Lemma 2.3, we only check that $\mathrm{reg}(L_t + \cap_{i=1}^{t-1} L_i) \leq m\mathrm{reg}(I) + n\mathrm{reg}(J) - 1$. Note that $L_t + \cap_{i=1}^{t-1} L_i = \cap_{i=1}^{t-1}(L_i + L_t) = \cap_{i=1}^{t-1}((I^m J_1^{n-i+1} + J_2^i + I^m J_1^{n-t+1} + J_2^t) : Q) = (I^m J_1^{n-t+1} +$

$J_2^{t-1}):Q$, by Remark 2.1 (i) and Proposition 2.6, we have that $\mathrm{reg}((I^mJ_1^{n-t+1}+J_2^{t-1}):Q) = \mathrm{reg}((I^mJ_1^{n-t+1}):Q+J_2^{t-1}:Q) \le \mathrm{reg}((I^mJ_1^{n-t+1}):Q)+\mathrm{reg}(J_2^{t-1}:Q)-1 \le m\mathrm{reg}(I)+(n-t+1)\mathrm{reg}(J_1)+(t-1)\mathrm{reg}(J_2)-1 \le m\mathrm{reg}(I)+n\mathrm{reg}(J)-1$. Then the result follows.

Again by Lemma 2.3, in order to prove that $\mathrm{reg}((I^m:Q)\cap(\cap_{i=1}^n L_i)) \le m\mathrm{reg}(I)+n\mathrm{reg}(J)$, we only check that $\mathrm{reg}(I^m:Q+(\cap_{i=1}^n L_i)) \le m\mathrm{reg}(I)+n\mathrm{reg}(J)-1$. This can follow by the same reason as above from the fact that $(I^m:Q+(\cap_{i=1}^n L_i)) = (I^m+J_2^n):Q$. The proof is completed.

Example 2.9. Let $I=(x_{i_1}^{\alpha}, x_{i_2}, x_{i_3}, \cdots, x_{i_s})$, $J=(y_{i_1}^{\beta}, y_{i_2}, y_{i_3}, \cdots, y_{i_t})$ with $\alpha, \beta \ge 1$ and $(y_{i_1}, x_{i_j})=1, j=2,3,\cdots,s$. Note that $\mathrm{reg}(I)=\alpha$, $\mathrm{reg}(J)=\beta$. Then by Proposition 2.8, $m(\alpha+\beta)=\deg(IJ)^m \le \mathrm{reg}(IJ)^m \le m(\alpha+\beta)$, and $\mathrm{reg}(IJ)^m = m(\alpha+\beta)$. In this situation, $\mathrm{reg}(IJ)^m = m\mathrm{reg}(IJ)$.

Theorem 2.10. *Let $I, J, K \subseteq R$ be three monomial complete intersections. Assume that I, J are generated by some powers of some varibles, i.e., I, J are of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \cdots, x_{i_l}^{a_l})$. Then for any positive integers m, n,*

$$\mathit{reg}(I^mJ^nK) \le m\,\mathit{reg}(I)+n\,\mathit{reg}(J)+\mathit{reg}(K).$$

Proof. The proof is following the method of [3, Theorem 3.1]. In fact, similarly as stated in [3, Remark 3.2], we only need to prove that $\mathrm{reg}(I^mJ^n:Q) \le m\mathrm{reg}(I)+n\mathrm{reg}(J)$ for an arbitrary monomial ideal Q. Then the result follows by Proposition 2.8, Remark 2.1 (ii) and Remark 2.2.

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