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n^{th}-ROOTS AND n-CENTRALITY OF FINITE 2-GENERATOR p-GROUPS OF NILPOTENCY CLASS 2

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ABSTRACT. Here we consider all finite non-abelian 2-generator p-groups (p an odd prime) of nilpotency class two and study the probability of having n^{th}-roots of them. Also we find integers n for which, these groups are n-central.

1. Introduction

Let $n > 1$ be an integer. An element a of group G is said to have an n^{th}-root b in G, if $a = b^n$. The probability that a randomly chosen element in G has an n^{th}-root, is given by

$$P_n(G) = \frac{|G^n|}{|G|}$$

where $G^n = \{a \in G | a = b^n, for\ some\ b \in G\} = \{x^n | x \in G\}$. A. Sadeghieh and H. Doostie in [3] computed the probability $P_n(G)$ for Dihedral groups D_{2m} and Quaternion groups Q_{2^m} for every integer $m \geq 3$. Also, in [2] the probability that Hamiltonian groups may have n^{th}-roots have been calculated.

For $n > 1$, a group G is said to be n-central if $[x^n, y] = 1$ for all $x, y \in G$. In [4], some relations between n-abelian and n-central groups have been investigated.

Suppose that $H \lhd G$ and there is subgroup K such that $G = HK$ and $H \cap K = \{e\}$, then G is said to be the semidirect product of H by K; in symbol $G = H \rtimes K$. Clearly if $K \lhd G$, then $H \rtimes K \cong H \times K$.

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First, we state the following Lemma without proof.

Lemma 1.1. *If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$:*
(i) $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$;
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$;
(iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.

The following theorem classifies all finite non-abelian 2-generator p-groups of nilpotency class two ($p \neq 2$).

Theorem 1.2. [1] *Let G be a finite non-abelian 2-generator p-group of nilpotency class two (p an odd prime). Then G is isomorphic to exactly one of the following three types of groups:*
(1) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$;
(2) $G \cong \langle a\rangle \rtimes \langle b\rangle$, *where* $[a, b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$;
(3) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a, b] = a^{p^{\alpha-\gamma}} c$, $[c, b] = a^{-p^{2(\alpha-\gamma)}} c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\sigma$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$.

Remark 1.3. By the relators given in each case, every element x of the above classes of groups can be uniquely presented as $x = c^k a^i b^j$ where $0 \leq k <| c |$, $0 \leq i < p^\alpha$ and $0 \leq j < p^\beta$.

In Section 2, we consider all finite nonabelian 2-generator p-groups ($p \neq 2$) of nilpotency class two and study the probability of having n^{th}-roots of them. Section 3 is devoted to investigating n-centrality of these groups.

2. The probability of having n^{th}-roots

In this section for each class of finite non-abelian 2-generator p-groups ($p \neq 2$) of nilpotency class two, we find the probability of having n^{th}-roots. Here for $m \in \mathbb{Z}$, by m^* we mean the arithmetic inverse of m.

Theorem 2.1. *Let $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$. Then*

$$P_n(G) = \frac{1}{p^{s+t+w}}$$

where $(n, p^\alpha) = p^s$, $(n, p^\beta) = p^t$ *and* $(n, p^\gamma) = p^w$.

Proof. Let $x = c^k a^i b^j$ be an element of G^n where $0 \leq k < p^\gamma$, $0 \leq i < p^\alpha$ and $0 \leq j < p^\beta$. If $x = (x_1)^n$ when $x_1 = c^{k_1} a^{i_1} b^{j_1} \in G$, $0 \leq k_1 < p^\gamma$, $0 \leq i_1 < p^\alpha$ and $0 \leq j_1 < p^\beta$, then we must have

$$\begin{aligned} c^k a^i b^j &= (c^{k_1} a^{i_1} b^{j_1})^n \\ &= c^{nk_1 - \frac{n(n-1)}{2} i_1 j_1} a^{ni_1} b^{nj_1}. \end{aligned}$$

By uniqueness of presentation of elements of G, we obtain

$$\begin{cases} ni_1 \equiv i \ (mod\ p^\alpha) \\ nj_1 \equiv j \ (mod\ p^\beta) \ (1) \\ nk_1 - \frac{n(n-1)}{2} i_1 j_1 \equiv k \ (mod\ p^\gamma). \end{cases}$$

Now let $(n, p^\alpha) = p^s$. The first congruence of the system (1) has the solution

$$i_1 \equiv (\frac{n}{p^s})^* (\frac{i}{p^s}) \ (mod\ p^{\alpha - s})$$

if and only if $p^s \mid i$. Then

$$i \in \{p^s, 2p^s, \ \ldots, p^{\alpha-s} p^s\}.$$

This means that i has $p^{\alpha-s}$ choices. Similarly if $(n, p^\beta) = p^t$, then by the second equation of System (1) we get

$$j \in \{p^t, 2p^t, \ \ldots, p^{\beta-t} p^t\}.$$

So j admits $p^{\beta-t}$ values.

Now suppose $(n, p^\gamma) = p^w$. Since $p \neq 2$, clearly for all $n \in \mathbb{N}$ we have $p^w \mid \frac{n(n-1)}{2}$. Hence from the third equation of system (1), we obtain

$$k_1 \equiv (\frac{n}{p^w})^* (\frac{n^2 - n}{2p^w}) i_1 j_1 + (\frac{n}{p^w})^* (\frac{k}{p^w}) \ (mod\ p^{\gamma - w})$$

provided that

$$k \in \{p^w, 2p^w, \ \ldots, p^{\gamma-w} p^w\}.$$

Therefore we have $p^{\gamma-w}$ choices for k. By the above facts, $\mid G^n \mid$ is equal to

$$\mid \{c^k a^i b^j \mid i \in \{p^s, ..., p^{\alpha-s} p^s\}, j \in \{p^t, ..., p^{\beta-t} p^t\}, k \in \{p^w, ..., p^{\gamma-w} p^w\}\} \mid .$$

Thus

$$\mid G^n \mid = p^{\alpha-s} \times p^{\beta-t} \times p^{\gamma-w} = p^{\alpha+\beta+\gamma-s-t-w}$$

and

$$\mid G \mid = \mid a \mid \times \mid b \mid \times \mid c \mid = p^{\alpha+\beta+\gamma}.$$

So

$$P_n(G) = \frac{\mid G^n \mid}{\mid G \mid} = \frac{1}{p^{s+t+w}}.$$

$\square$

To continue, we find the probability of having n^{th}-root for second class of groups of Theorem 1.2.

Theorem 2.2. *Let* $G \cong \langle a \rangle \rtimes \langle b \rangle$, *where* $[a,b] = a^{p^{\alpha-\gamma}}$, $|a| = p^{\alpha}$, $|b| = p^{\beta}$, $|[a,b]| = p^{\gamma}$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$. *Then*

$$P_n(G) = \frac{1}{p^{s+t}}$$

where $(n, p^{\alpha}) = p^s$ *and* $(n, p^{\beta}) = p^t$.

Proof. Let $x = a^i b^j \in G^n$ where $0 \leq i < p^{\alpha}$ and $0 \leq j < p^{\beta}$. If $x_1 = a^{i_1} b^{j_1} \in G$, $0 \leq i_1 < p^{\alpha}$ and $0 \leq j_1 < p^{\beta}$ such that $x = (x_1)^n$, then by uniqueness of presentation of elements of G (See Remark 1.3) we must have

$$\begin{aligned} a^i b^j &= (a^{i_1} b^{j_1})^n \\ &= a^{n i_1 - \frac{n(n-1)}{2} i_1 j_1} b^{n j_1}. \end{aligned}$$

So

$$\begin{cases} n j_1 \equiv j \ (mod\ p^{\beta}) \\ n i_1 - \frac{n(n-1)}{2} i_1 j_1 \equiv i \ (mod\ p^{\alpha}). \ (2) \end{cases}$$

Now, we consider two cases:

Case 1. Suppose $p^{\beta} \mid n$. Then the above system changes to

$$\begin{cases} j = 0 \\ n i_1 \equiv i \ (mod\ p^{\alpha}). \end{cases}$$

If $(n, p^{\alpha}) = p^s$, then

$$i_1 \equiv (\frac{n}{p^s})^* (\frac{i}{p^s}) \ (mod\ p^{\alpha - s})$$

is the solution of system (2) if and only if $p^s \mid i$. So

$$i \in \{p^s, 2p^s, \ \ldots, p^{\alpha-s} p^s\}.$$

Therefore in this case

$$\begin{aligned} P_n(G) &= \frac{\mid G^n \mid}{\mid G \mid} \\ &= \frac{\mid \{(i,j) \mid i \in \{p^s, 2p^s, \ \ldots, p^{\alpha-s}p^s\}, j = 0\} \mid}{\mid a \mid \times \mid b \mid} \\ &= \frac{p^{\alpha-s}}{p^{\alpha+\beta}} = \frac{1}{p^{s+\beta}} = \frac{1}{p^{s+t}}. \end{aligned}$$

Case 2. Let $p^\beta \nmid n$ and $(n, p^\beta) = p^t$. Then the first equation of the System (2) has solution

$$j_1 \equiv (\frac{n}{p^t})^*(\frac{j}{p^t}) \ (mod \ p^{\beta-t}) \ (3)$$

if $p^t \mid j$. Then

$$j \in \{p^t, 2p^t, \ . \ . \ . \ , p^{\beta-t}p^t\}.$$

Now let $(n, p^\alpha) = p^s$. For finding the number of choices of i, we have to consider two subcases:

Subcase 2.a. Let n be an even integer, then in second congruence of system (2) we have

$$\frac{n}{2}i_1 \, (2 - (n-1)j_1) \equiv i \ (mod \ p^\alpha).$$

Since $(p^\alpha, \frac{n}{2}) = p^s$,

$$i_1(2 - (n-1)j_1) \equiv (\frac{n}{2p^s})^*(\frac{i}{p^s}) \ (mod \ p^{\alpha-s}).$$

Now by replacing $j = p^{t+1}$ in Congruence (3), we get

$$j_1 \equiv p(\frac{n}{p^t})^* \ (mod \ p^{\beta-t}).$$

Then $2 - (n-1)j_1$ and $p^{\alpha-s}$ are prime to each other. So we can write

$$i_1 \equiv (\frac{n}{2p^s})^*(2 - (n-1)j_1)^*(\frac{i}{p^s}) \ (mod \ p^{\alpha-s})$$

provided that

$$i \in \{p^s, 2p^s, \ . \ . \ . \ , p^{\alpha-s}p^s\}.$$

This means that there are $p^{\alpha-s}$ solutions for i.

Subcase 2.b. Let n be an odd integer, then

$$ni_1(1 - \frac{(n-1)}{2}j_1) \equiv i \ (mod \ p^\alpha).$$

So by considering $j = p^{t+1}$, we get that

$$j_1 \equiv p(\frac{n}{p^t})^* \ (mod \ p^{\beta-t}).$$

Hence we can write

$$i_1 \equiv (\frac{n}{p^s})^*(1 - \frac{(n-1)}{2}j_1)^*(\frac{i}{p^s}) \ (mod \ p^{\alpha-s}).$$

This obtained i_1 is a solution of the second equation of system (2) if and only if

$$i \in \{p^s, 2p^s, \ . \ . \ . \ , p^{\alpha-s}p^s\}.$$

Now since in both subcases we have $p^{\alpha-s}$ choices for i, we get

$$\begin{aligned} P_n(G) &= \frac{\mid G^n \mid}{\mid G \mid} \\ &= \frac{\mid \{(i,j) \mid i \in \{p^s, \ . \ . \ . \ , p^{\alpha-s}p^s\}, j \in \{p^t, \ . \ . \ . \ , p^{\beta-t}p^t\}\} \mid}{\mid a \mid \times \mid b \mid} \\ &= \frac{p^{\alpha+\beta-s-t}}{p^{\alpha+\beta}} = \frac{1}{p^{s+t}}. \end{aligned}$$

□

Finally for third class of groups of Theorem 1.2, we have the following theorem.

Theorem 2.3. *Let* $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b] = a^{p^{\alpha-\gamma}}c$, $[c,b] = a^{-p^{2(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\sigma$, $|[a,b]| = p^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$. *Then*

$$P_n(G) = \frac{1}{p^{s+t+u}}$$

where $(n, p^\alpha) = p^s$, $(n, p^\beta) = p^t$ *and* $(n, p^\sigma) = p^u$.

Proof. Let $x = c^k a^i b^j$ be an element of G^n where $0 \leq k < p^\sigma$, $0 \leq i < p^\alpha$ and $0 \leq j < p^\beta$. If $x_1 = c^{k_1}a^{i_1}b^{j_1} \in G$ where $0 \leq k_1 < p^\sigma$, $0 \leq i_1 < p^\alpha$, $0 \leq j_1 < p^\beta$ and $x = (x_1)^n$, then we must have

$$\begin{aligned} c^k a^i b^j &= (c^{k_1}a^{i_1}b^{j_1})^n \\ &= c^{nk_1 - \frac{n(n-1)}{2}i_1 j_1 + \frac{n(n-1)}{2}p^{\alpha-\gamma}k_1 j_1} a^{ni_1 - \frac{n(n-1)}{2}p^{\alpha-\gamma}i_1 j_1 + \frac{n(n-1)}{2}p^{2(\alpha-\gamma)}k_1 j_1} b^{nj_1}. \end{aligned}$$

So by uniqueness of presentation of elements of G (See Remark 1.3), we obtain

$$\begin{cases} nj_1 \equiv j \ (mod \ p^\beta) \\ nk_1 - \frac{n(n-1)}{2}i_1 j_1 + \frac{n(n-1)}{2}p^{\alpha-\gamma}k_1 j_1 \equiv k \ (mod \ p^\sigma) \ (4) \\ ni_1 - \frac{n(n-1)}{2}p^{\alpha-\gamma}i_1 j_1 + \frac{n(n-1)}{2}p^{2(\alpha-\gamma)}k_1 j_1 \equiv i \ (mod \ p^\alpha). \end{cases}$$

For solution of this system, we consider two cases:

Case I. Let $(n, p^\beta) = p^t$ and $t \neq \beta$. Then the first congruence of System (4) has the solution

$$j_1 \equiv (\frac{n}{p^t})^*(\frac{j}{p^t}) \ (mod \ p^{\beta-t})$$

if and only if $p^t \mid j$. So

$$j \in \{p^t, 2p^t, \ . \ . \ . \ , p^{\beta-t}p^t\}$$

and consequently we have $p^{\beta-t}$ choices for j. Now let $(n, p^\alpha) = p^s$ and $(n, p^\sigma) = p^u$. For solving congruences, we consider two cases. First let n be an even integer, then we can write

$$\frac{n}{2}\left(2k_1 - (n-1)i_1j_1 + (n-1)p^{\alpha-\gamma}k_1j_1\right) \equiv k \ (mod \ p^\sigma)$$

Since $p \neq 2$, we have $(\frac{n}{2}, p^\sigma) = p^u$. Therefore

$$2k_1 - (n-1)i_1j_1 + (n-1)p^{\alpha-\gamma}k_1j_1 \equiv \frac{k}{p^u}(\frac{n}{2p^u})^* \ (mod \ p^{\sigma-u})$$

provided that $p^u \mid k$. So

$$k_1(2 + (n-1)p^{\alpha-\gamma}j_1) \equiv \frac{k}{p^u}(\frac{n}{2p^u})^* + (n-1)i_1j_1 \ (mod \ p^{\sigma-u}).$$

Since $p \mid j$, we have

$$\begin{aligned} k_1 \equiv\ & (2 + (n-1)p^{\alpha-\gamma}j_1)^* \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* \\ & + (2 + (n-1)p^{\alpha-\gamma}j_1)^*(n-1)i_1j_1 \ (mod \ p^{\sigma-u}) \qquad (5) \end{aligned}$$

if

$$k \in \left\{p^u, 2p^u, \ . \ . \ , p^{\sigma-u}p^u\right\}.$$

Hence there are at most $p^{\sigma-u}$ choices for k. On the other hand, we write

$$\frac{n}{2}\left(2i_1 - (n-1)p^{\alpha-\gamma}i_1j_1 + (n-1)p^{2(\alpha-\gamma)}k_1j_1\right) \equiv i \ (mod \ p^\alpha).$$

Since $(\frac{n}{2}, p^\alpha) = p^s$, we obtain

$$2i_1 - (n-1)p^{\alpha-\gamma}i_1j_1 + (n-1)p^{2(\alpha-\gamma)}k_1j_1 \equiv (\frac{n}{2p^s})^* \frac{i}{p^s} \ (mod \ p^{\alpha-s}).$$

provided that $p^s \mid i$. By replacing the obtained k_1, in the above congruence we get

$$\begin{aligned} & 2i_1 - (n-1)p^{\alpha-\gamma}i_1j_1 + (n-1)p^{2(\alpha-\gamma)}j_1(2 + (n-1)p^{\alpha-\gamma}j_1)^* \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* \\ & + (n-1)^2p^{2(\alpha-\gamma)}i_1j_1^2(2 + (n-1)p^{\alpha-\gamma}j_1)^*) \equiv (\frac{n}{2p^s})^* \frac{i}{p^s} \ (mod \ p^{\alpha-s}). \end{aligned}$$

Therefore

$$\begin{aligned} & i_1\left(2 - (n-1)p^{\alpha-\gamma}j_1 + (n-1)^2p^{2(\alpha-\gamma)}j_1^2(2 + (n-1)p^{\alpha-\gamma}j_1)^*\right) \equiv \\ & (\frac{n}{2p^s})^* \frac{i}{p^s} - (n-1)p^{2(\alpha-\gamma)}j_1(2+(n-1)p^{\alpha-\gamma}j_1)^* \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* \ (mod \ p^{\alpha-s}). \end{aligned}$$

Since $p \mid (n-1)p^{\alpha-\gamma}j_1$ and $p \mid (n-1)^2p^{2(\alpha-\gamma)}j_1^2$, we can write

$$i_1 \equiv \left(2-(n-1)p^{\alpha-\gamma}j_1+(n-1)p^{2(\alpha-\gamma)}j_1^2(2+(n-1)p^{\alpha-\gamma}j_1)^*\right)^* \frac{i}{p^s}$$
$$\times \left(\frac{n}{2p^s}\right)^* - \left(2-(n-1)p^{\alpha-\gamma}j_1+(n-1)p^{2(\alpha-\gamma)}j_1^2(2+(n-1)p^{\alpha-\gamma}j_1)^*\right)^*$$
$$\times (n-1)p^{2(\alpha-\gamma)}j_1\left(2+(n-1)p^{\alpha-\gamma}\right)^* \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* \pmod{p^{\alpha-s}}$$

provided that $p^s \mid i$. Now clearly i_1 is a solution of this system if and only if

$$i \in \left\{p^s, 2p^s, \ \ldots, p^{\alpha-s}.p^s\right\}.$$

Hence we must have exactly $p^{\alpha-s}$ choices for i. By replacing i_1 in congruence (5), we get

$$k_1 \equiv \left(2+(n-1)p^{\alpha-\gamma}j_1\right)^* \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* + \left(2+(n-1)p^{\alpha-\gamma}j_1\right)^*(n-1)j_1$$
$$\times \left(2-(n-1)p^{\alpha-\gamma}j_1+(n-1)^2p^{2(\alpha-\gamma)}j_1^2(2+(n-1)p^{\alpha-\gamma}j_1)^*\right)^*\left(\frac{n}{2p^s}\right)^*$$
$$\times \frac{i}{p^s} - \left(2+(n-1)p^{\alpha-\gamma}j_1\right)^*(n-1)^2j_1^2p^{2(\alpha-\gamma)}\left(2+(n-1)p^{\alpha-\gamma}\right)^*$$
$$\times \left(2-(n-1)p^{\alpha-\gamma}j_1+(n-1)^2p^{2(\alpha-\gamma)}j_1^2(2+(n-1)p^{\alpha-\gamma}j_1)^*\right)^*$$
$$\times \frac{k}{p^u}\left(\frac{n}{2p^u}\right) \qquad (mod\, p^{\sigma-u}).$$

So we conclude that k can be chosen in exactly $p^{\sigma-u}$ ways. Therefore

$$\mid G^n \mid = p^{\alpha-s} \times p^{\beta-t} \times p^{\sigma-u} = p^{\alpha+\beta+\sigma-s-t-u}$$

and

$$\mid G \mid = \mid a \mid \times \mid b \mid \times \mid c \mid = p^{\alpha+\beta+\sigma}.$$

Then we get the desired result. When n is an odd integer, the theorem can be proved similarly.

Case II. Let $(n, p^\beta) = p^t$. Then clearly $p^\beta \mid j$ and since $0 \leq j < p^\beta$, $j = 0$. Then the second and third congruence of System (4) will be proved similar to the proof of Case I. In this case we obtain

$$\mid G^n \mid = \mid \{(i,j,k) \mid i \in \{p^s, \ldots, p^{\alpha-s}p^s\}, j=0, k \in \{p^u, \ldots, p^{\sigma-u}p^u\}\} \mid .$$

Hence

$$P_n(G) = \frac{\mid G^n \mid}{\mid G \mid} = \frac{p^{\alpha+\sigma-s-u}}{p^{\alpha+\beta+\sigma}} = \frac{1}{p^{\beta+s+u}} = \frac{1}{p^{s+t+u}}.$$

□

3. n-CENTRALITY

In this section, we again consider all finite non-abelian 2-generator p-groups ($p \neq 2$) of nilpotency class two and this time we investigate n-centrality for them.

Theorem 3.1. *Let G be a finite non-abelian 2-generator p-group of nilpotency class two. Then for $n > 1$, the group G is n-central if and only if $p^\gamma \mid n$.*

Proof. According to the Theorem 1.2, we consider three cases:

Case 1. Let $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a,b] = c$, $[a,c] = [b,c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$. Also let $x = c^{k_1}a^{i_1}b^{j_1}$ and $y = c^{k_2}a^{i_2}b^{j_2}$ be two elements of G where $0 \leq k_1, k_2 < p^\gamma$, $0 \leq i_1, i_2 < p^\alpha$ and $0 \leq j_1, j_2 < p^\beta$. Then by Lemma 1.1, we get

$$x^n = c^{nk_1 - \frac{n(n-1)}{2}i_1j_1}a^{ni_1}b^{nj_1}$$

and

$$x^n y = c^{nk_1 + k_2 - \frac{n(n-1)}{2}i_1j_1 - ni_2j_1}a^{ni_1+i_2}b^{nj_1+j_2}.$$

Also we obtain

$$yx^n = c^{nk_1 + k_2 - \frac{n(n-1)}{2}i_1j_1 - ni_1j_2}a^{ni_1+i_2}b^{nj_1+j_2}.$$

We know that G is n-central if and only if $x^n y = yx^n$, for all $x, y \in G$. Furthermore by uniqueness of presentation of $x^n y$ and yx^n, we see that $x^n y = yx^n$ if and only if

$$nk_1 + k_2 - \frac{n(n-1)}{2}i_1j_1 - ni_2j_1 \equiv nk_1 + k_2 - \frac{n(n-1)}{2}i_1j_1 - ni_1j_2 \; (mod\; p^\gamma).$$

This is equivalent to

$$n(i_1j_2 - i_2j_1) \equiv 0 \; (mod\; p^\gamma).$$

Now since this holds for all $x, y \in G$, $p^\gamma \mid n$.

Case 2. Let $G \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a,b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a,b]| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$. Also, let $x = a^{i_1}b^{j_1}$, $y = a^{i_2}b^{j_2}$ be two elements of G, where $0 \leq i_1, i_2 < p^\alpha$ and $0 \leq j_1, j_2 < p^\beta$. By using Lemma 1.1, we get

$$x^n y = a^{ni_1 + i_2 - \frac{n(n-1)}{2}p^{\alpha-\gamma}i_1j_1 - np^{\alpha-\gamma}i_2j_1}b^{nj_1+j_2}$$

and

$$yx^n = a^{ni_1 + i_2 - \frac{n(n-1)}{2}p^{\alpha-\gamma}i_1j_1 - np^{\alpha-\gamma}i_1j_2}b^{nj_1+j_2}.$$

Hence by uniqueness of presentation of $x^n y$ and yx^n, the statement $x^n y = yx^n$ is equal to

$$n(i_1 j_2 - i_2 j_1) \equiv 0 \ (mod \ p^\gamma)$$

for all $x, y \in G$. So, we get the desired result.

Case 3. Let $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a,b] = a^{p^{\alpha-\gamma}} c$, $[c,b] = a^{-p^{2(\alpha-\gamma)}} c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\sigma$, $|[a,b]| = p^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$. By the presentation of elements of G, we have $x = c^{k_1} a^{i_1} b^{j_1}$ and $y = c^{k_2} a^{i_2} b^{j_2}$ where $0 \leq k_1, k_2 < p^\sigma$, $0 \leq i_1, i_2 < p^\alpha$ and $0 \leq j_1, j_2 < p^\beta$.

$$x^n y = c^{nk_1 + k_2 + \frac{n(n-1)}{2} p^{\alpha-\gamma} k_1 j_1 - \frac{n(n-1)}{2} i_1 j_1 + np^{\alpha-\gamma} k_2 j_1 - n i_2 j_1}$$

$$\times a^{ni_1 + i_2 + \frac{n(n-1)}{2} p^{2(\alpha-\gamma)} k_1 j_1 - \frac{n(n-1)}{2} p^{\alpha-\gamma} i_1 j_1 + np^{2(\alpha-\gamma)} k_2 j_1 - np^{\alpha-\gamma} i_2 j_1}$$

$$\times b^{nj_1 + j_2}.$$

and

$$yx^n = c^{nk_1 + k_2 + \frac{n(n-1)}{2} p^{\alpha-\gamma} k_1 j_1 - \frac{n(n-1)}{2} i_1 j_1 + np^{\alpha-\gamma} k_1 j_2 - n i_1 j_2}$$

$$\times a^{ni_1 + i_2 + \frac{n(n-1)}{2} p^{2(\alpha-\gamma)} k_1 j_1 - \frac{n(n-1)}{2} p^{\alpha-\gamma} i_1 j_1 + np^{2(\alpha-\gamma)} k_1 j_2 - np^{\alpha-\gamma} i_1 j_2}$$

$$\times b^{nj_1 + j_2}.$$

By the above facts, we see that for all $x, y \in G$; $x^n y = yx^n$ if and only if the following system holds

$$\begin{cases} n(p^{\alpha-\gamma}(k_1 j_2 - k_2 j_1) + i_2 j_1 - i_1 j_2) \equiv 0 \ (mod \ p^\sigma) \\ n(p^{\alpha-\gamma}(k_1 j_2 - k_2 j_1) + i_2 j_1 - i_1 j_2) \equiv 0 \ (mod \ p^\gamma). \ (6) \end{cases}$$

Now let $p^\gamma | n$, then surely $p^\sigma | n$ and the above congruence system holds. Hence G will be n-central.
Conversely let G be an n-central group. So the system (6) must hold for all $x, y \in G$ such as $x = c^3 ab$ and $y = c^2 a^2 b$. Then we get

$$\begin{cases} n(p^{\alpha-\gamma} - 1) \equiv 0 \ (mod \ p^\sigma) \\ n(p^{\alpha-\gamma} - 1) \equiv 0 \ (mod \ p^\gamma). \end{cases}$$

Hence $p^\gamma | n$.

$\square$

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