Journal of Algebra and Related Topics
Vol. 4, No 1, (2016), pp 21-32

THE SMALL INTERSECTION GRAPH RELATIVE TO MULTIPLICATION MODULES

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Abstract. Let R be a commutative ring and let M be an R-module. We define the small intersection graph $G(M)$ of M with all non-small proper submodules of M as vertices and two distinct vertices N, K are adjacent if and only if $N \cap K$ is a non-small submodule of M. In this article, we investigate the interplay between the graph-theoretic properties of $G(M)$ and algebraic properties of M, where M is a multiplication module.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Let M be an R-module. We denote the set of all maximal submodules of M by $Max(M)$ and the intersection of all maximal submodule of M by $Rad(M)$. A submodule N of M is called *small* in M (denoted by $N \ll M$), in case for every submodule L of M, $N + L = M$ implies that $L = M$. A module M is said to be *hollow* module if every proper submodule of M is a small submodule.

A *graph* G is defined as the pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. For two distinct vertices a and b denoted by $a - b$ means that a and b are adjacent. The *degree* of a vertex a of graph G which denoted by $deg(a)$ is the number of edges incident on a. If $|V(G)| \geqslant 2$, a *path* from a to b is

MSC(2010): Primary: 05C75; Secondary: 13A99, 05C99
Keywords: Graph, non-small submodule, multiplication module.
Received: 18 February 2016, Accepted: 13 April 2016.
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a series of adjacent vertices $a - v_1 - v_2 - ... - v_n - b$. In a graph G, the distance between two distinct vertices a and b, dented by $d(a, b)$ is the length of the shortest path connecting a and b. If there is not a path between a and b, $d(a, b) = \infty$. The *diameter* of a graph G is $diam(G) = sup \{d(a, b) \mid a, b \in V(G)\}$. A graph G is called *connected*, if for any vertices a and b of G there is a path between a and b. If not, G is *disconnected*. The *girth* of G, is the length of the shortest cycle in G and it is denoted by $g(G)$. If G has no cycle, we define the girth of G to be infinite. An *r-partite* graph is one whose vertex set can be partitioned into r subsets such that no edge has both ends in any one subset. A *complete* r-partite graph is one each vertex is jointed to every vertex that is not in the same subset. The *complete bipartite* (i.e, 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph $G = (V, E)$, a set $S \subseteq V$ is an *independent* if no two vertices in S are adjacent. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G. The (open) *neighbourhood* $N(a)$ of a vertex $a \in V$ is the set of vertices which are adjacent to a. For each $S \subseteq V$, $N(S) = \bigcup_{a \in S} N(a)$ and $N[S] = N(S) \bigcup S$. A set of vertices S in G is a *dominating set*, if $N[S] = V$. The *dominating number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G ([9]). Note that a graph whose vertices-set is empty is a *null graph* and a graph whose edge-set is empty is an *empty* graph.

The idea of zero divisor graph of a commutative ring was introduced by I. Beck in 1988 [2]. The zero-divisor graph of a commutative ring has also been studied by several other authors. One of the most important graphs which has been studied is the intersection graph. Bosak [4] in 1964 defined the intersection graph of semigroups. In 1964, Csákány and PolláK [10], studied the graph of subgroups of a finite groups. In 2009, the intersection graph of ideals of ring was considered by Chakrabarty, Ghosh, Mukherjee and San [5]. The intersection graph of ideal of rings and submodules of modules have been investigated by several other authors (e.g., [1, 10, 14]).

An R-module M is said to be a *multiplication* R-module if for each submodule N of M there exists an ideal I of R such that $N = IM$.

In [7], the authors introduced and studied the small intersection graph of a commutative ring. In this article, we give a generalization of this concept and obtain some results similar to those of in [7] when M is a multiplication module. Also we provide some examples and remarks which show that the similarly doesn't go parallel in general when M

is not a multiplication R-module. This graph helps us to consider algebraic properties submodules of M by using graph theoretical tools.

2. Basic properties of G(M)

Definition 2.1. Let M be an R-module. We define the *small intersection graph $G(M)$ of M* with all non-small proper submodules of M as vertices and two distinct vertices N, K are adjacent if and only if $N \cap K \not\ll M$. Clearly when $M = R$, we get the small intersection graph $G(R)$ of R introduced in [7].

A proper submodule N of an R-module M is said to be a *prime submodule* of M if $ax \in N$ for $a \in R$ and $x \in M$, then either $aM \subseteq N$ or $x \in N$. We remark that if N is a prime submodule of M, then $P = (N : M)$ is necessarily a prime ideal of R. Moreover, every maximal submodule of M is a prime submodule by [11, Proposition 4].

The next lemma plays a key role in the sequel.

Lemma 2.2. *Let M be a non-zero multiplication R-module.*

(a) *Every proper submodule of M is contained in a maximal submodule of M. In particular, $Max(M) \neq \varnothing$.*
(b) *If N is a submodule of M, then $N \ll M$ if and only if $N \subseteq Rad(M)$.*
(c) *If N, K are submodules of M and P a prime submodule of M with $P \supseteq N \cap K$, then $P \supseteq N$ or $P \supseteq K$.*

Proof. (a) See [8, Theorem 2.5].

(b) Let N be a small submodule of M. If $N \nsubseteq Rad(M)$, then there exists $M_j \in Max(M)$ such that $N \nsubseteq M_j$. This implies that $N + M_j = M$. Since N is a small submodule, $M_j = M$, a contradiction. Conversely, if $N \not\ll M$, then there exists a proper submodule K of M such that $N + K = M$. Since M is a multiplication module, by part (a), there exists $M_t \in Max(M)$ such that $K \subseteq M_t$. It follows that $M = K + N \subseteq M_t + N$ and hence $M = M_t + N$. Since $N \subseteq M_t$, $M_t = M$, a contradiction.

(c) Let $P \subset M$ be a prime submodule with $P \supseteq N \cap K$. Then $(P :_R M) \supseteq (N \cap K :_R M) = (N :_R M) \cap (K :_R M)$. Since $(P :_R M)$ is a prime ideal, $(P :_R M) \supseteq (N :_R M)$ or $(P :_R M) \supseteq (K :_R M)$. Thus $(P :_R M)M \supseteq (N :_R M)M$ or $(P :_R M)M \supseteq (K :_R M)M$. It follows that $P \supseteq N$ or $P \supseteq K$ because M is a multiplication module. □

Remark 2.3. The parts (a) and (b) of Lemma 2.2 are also true when M is replaced by a coatomic R-module (we recall that an R- module M is a *coatomic* if every proper submodule of M is contained in a maximal submodule).

In the rest of this paper, we assume that M is a non-zero multiplication R-module. We recall that $Max(M) \neq \varnothing$ by Lemma 2.2 part (a).

Lemma 2.4. *Let M be an R-module with $Max(M) = \{M_i\}_{i \in I}$, where $|I| > 1$, and let Λ be a non-empty proper finite subset of I. Then $\bigcap_{\lambda \in \Lambda} M_\lambda$ is not a small submodule of M.*

Proof. Let $\bigcap_{\lambda \in \Lambda} M_\lambda$ be a small submodule of M and let $j \in I \setminus \Lambda$. Then by Lemma 2.2 (b), $\bigcap_{\lambda \in \Lambda} M_\lambda \subseteq M_j$. Hence by Lemma 2.2 (c), $M_\lambda \subseteq M_j$ for some $\lambda \in \Lambda$, a contradiction. □

Proposition 2.5. *Let M be an R-module. Then $G(M)$ is a null graph if and only if M is a local module.*

Proof. The necessity is clear and the sufficiency follows from Lemma 2.2 (b). □

All definitions of graph theory are for non-null graphs ([3]). So in this paper, all considered graphs are non-null.

Theorem 2.6. *Let M be an R-module. Then $G(M)$ is an empty graph if and only if $Max(M) = \{M_1, M_2\}$, where M_1 and M_2 are finitely generated hollow R-modules.*

Proof. Let $G(M)$ be an empty graph. If $|Max(M)| = 1$, then $G(M)$ is a null graph by Proposition 2.5, a contradiction. If $|Max(M)| \geq 3$, then by choosing $M_1, M_2 \in Max(M)$, we have $M_1 \cap M_2$ is a non-small submodule of M by Lemma 2.4. Thus M_1 and M_2 are adjacent, a contradiction. Hence, $|Max(M)| = 2$. Suppose that $Max(M) = \{M_1, M_2\}$. We claim that M_1, M_2 are hollow R-modules. $M_1 \cap M_2$ is a maximal submodule of M_1 because $\frac{M}{M_2}$ is a simple R-module and $\frac{M}{M_2} = \frac{M_1+M_2}{M_2} \cong \frac{M_1}{M_1 \cap M_2}$. We show that this is the only maximal submodule of M_1. Let K be a maximal submodule of M_1. If $K \not\ll M$, then $K \cap M_1 = K$ implies that K and M_1 are adjacent, a contradiction. Thus $K \ll M$. So by Lemma 2.2 (b), $K \subseteq M_1 \cap M_2 \subseteq M_1$ which implies that $K = M_1 \cap M_2$ by maximality of K. Therefore, M_1 is a local R-module. Thus M_1 is a hollow R-module. Now, we show that M_1 is a finitely generated R-module. Choose $x \in M_1 \setminus M_2$, so $Rx \not\ll M$. If $Rx \neq M_1$, then $Rx \cap M_1 = Rx$ which shows that Rx and M_1 are adjacent, a contradiction. Hence M_1 is a finitely generated local R-module. We have similar argument for M_2. Hence M_1 and M_2 are finitely generated local R-module. Conversely, let $Max(M) = \{M_1, M_2\}$ where M_1, M_2 are finitely generated hollow R-modules. We can see $M_1 \cap M_2$ is a maximal submodule of M_1 and M_2. By [13, page 352], $M_1 \cap M_2$ is the only maximal submodule of M_1 and M_2. Suppose that $N \neq M_1$ and M_2 is a non-small submodule of M. Then $N \subseteq M_1$ or $N \subseteq M_2$. Without loss of generality, we can assume that $N \subseteq M_1$. By Lemma 2.2 (b), N is a small submodule, a contradiction. Hence, M_1 and M_2 are the only non-small submodules of M which are not adjacent. □

The following example shows that the condition "M is a multiplication module" can not be removed in Theorem 2.6.

Example 2.7. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be a $\mathbb{Z}$-module. Then $V(G(M)) = Max(M) = \{(1,0)\mathbb{Z}, (0,1)\mathbb{Z}, (1,1)\mathbb{Z}\}$. But $G(M)$ is an empty graph.

Theorem 2.8. *Let M be an R-module. The following statements are equivalent.*

(a) *$G(M)$ is not connected.*
(b) *$|Max(M)| = 2$.*
(c) *$G(M) = G_1$ and G_2, where G_1, G_2 are two disjoint complete subgraphs.*

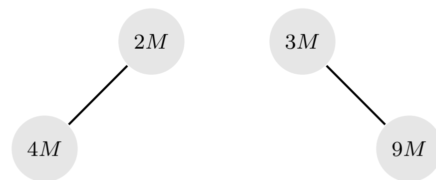
Proof. $(a) \Rightarrow (b)$ Suppose that $G(M)$ is not connected and $|Max(M)| > 2$. Let G_1, G_2 be two components of $G(M)$ and N, K be submodules of M such that $N \in G_1$ and $K \in G_2$. Consider M_1, M_2 be maximal submodules of M and $N \subseteq M_1$ and $K \subseteq M_2$. If $M_1 = M_2$, then $N - M_1 - K$ is a path in $G(M)$, which is a contradiction. So assume that $M_1 \neq M_2$. Since $|Max(M)| > 2$, we have $M_1 \cap M_2 \neq 0$ and is a non-small submodule of M. Thus $N - M_1 - M_2 - K$ is a path between G_1 and G_2, a contradiction. Therefore, $|Max(M)| = 2$.
$(b) \Rightarrow (c)$ Let $|Max(M)| = \{M_1, M_2\}$ where M_1 and M_2 are two maximal submodules of M. Let $G_j = \{M_k < M \mid M_k \subseteq M_j \, and \, M_k \not\ll M\}$ for $j = 1, 2$. Consider $N, K \in G_1$. We claim that N and K are adjacent. Otherwise, if $N \cap K \ll M$, then by Lemma 2.2 (b), $N \cap K \subseteq M_1 \cap M_2$ which implies that $N \subseteq M_2$ or $K \subseteq M_2$ by Lemma 2.2 (c). This implies that $N \ll M$ or $K \ll M$, a contradiction. Thus G_1 is a complete subgraph and by similar arguments G_2 is a complete subgraph too. We show that there is no path between G_1 and G_2. Assume to the contrary that there are $N \in G_2$ and $K \in G_2$ which are adjacent. We have $N \cap K \subseteq M_1 \cap M_2$. So $N \cap K$ is a small submodule of M by Lemma 2.2 (b), a contradiction. Hence $G = G_1 \cup G_2$ which G_1 and G_2 are complete subgraphs.
$(c) \Rightarrow (a)$ This is clear.

□

Example 2.9. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{36}$. Then $V(G(M)) = \{3M, 9M, 2M, 4M\}$. We can see $G(M)$ is not connected and $G(M) = G_1 \cup G_2$

where $G_1 = \{3M, 9M\}$ and $G_2 = \{2M, 4M\}$ are complete subgraphs (Figure 1).

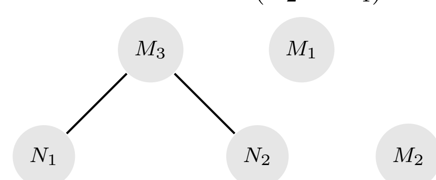
FIGURE 1. $G(\mathbb{Z}_{36})$.



The following example shows that the condition "M is a multiplication module" can not be dropped in Theorem 2.8.

Example 2.10. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ be a $\mathbb{Z}$-module. This is clear that M is not a multiplication module. Also we have $Max(M) = \{M_1, M_2, M_3\}$ and $V(G) = \{N_1, N_2, M_1, M_2, M_3\}$, where $N_1 := (1,0)\mathbb{Z}$, $N_2 := (1,2)\mathbb{Z}$, $M_1 := (0,1)\mathbb{Z}$, $M_2 := (1,1)\mathbb{Z}$, and $M_3 := (0,1)\mathbb{Z} + (1,2)\mathbb{Z}$. We see that $| Max(M) | \geq 3$ but $G(M)$ is not connected (Figure 2).

FIGURE 2. $G(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$.



Theorem 2.11. *Let $G(M)$ be a connected graph. Then $diam(G(M)) \leqslant 2$.*

Proof. Suppose that N and K are two vertices of $G(M)$ which are not adjacent. Thus $N \cap K \ll M$. Then by Lemma 2.2 (a), there exists two maximal submodules M_1, M_2 of M such that $N \subseteq M_1$ and $K \subseteq M_2$. If $N \cap M_2 \not\ll M$, then $N - M_2 - K$ is a path so that $d(N, K) = 2$. Similarly, if $K \not\ll M_1$, then $d(N, K) = 2$. Now assume that $N \cap M_2 \ll M$ and $K \cap M_1 \ll M$. By Theorem 2.8 and our assumption $Max(M) \geq 3$. Let $M_3 \in Max(M)$. Then by Lemma 2.2 (b), $N \cap K \subseteq Rad(M) \subseteq M_3$. Thus $N \subseteq M_3$ or $K \subseteq M_3$. Without loss of generality, we can assume that $N \subseteq M_3$. If $K \cap M_3 \not\ll M$, then $N - M_3 - K$ is a path. If $K \cap M_3 \ll M$, then $K \cap M_3 \subseteq Rad(M) \subseteq M_1$. So we have $N - M_1 - K$. Therefore, $d(N, K) = 2$.

□

Theorem 2.12. *Let M be an R-module and $G(M)$ contains a cycle. Then $g(G(M)) = 3$.*

Proof. Let $|Max(M)| = 2$. Then $G(M)$ is union of two disjoint subgraphs by Theorem 2.8. So if $G(M)$ contains a cycle, then $g(G(M)) = 3$. Now let $|Max(M)| \geq 3$ and choose M_1, M_2, and $M_3 \in Max(M)$. Then by Lemma 2.4, $M_1 - M_2 - M_3 - M_1$ is a cycle in $G(M)$. Hence $g(G(M)) = 3$. □

A vertex a in a connected graph G is a cut vertex if $G - \{a\}$ is disconnected.

Theorem 2.13. *Let M be an R-module and $G(M)$ be a connected graph. Then $G(M)$ has no cut vertex.*

Proof. Let L be a cut vertex of $G(M)$. Then $G(M) \setminus L$ is not connected. So there exist at least two vertices N, K of $G(M)$ such that L lies between every path from N to K. By Theorem 2.11, we see the shortest path between N, K is length of 2. Hence $N - L - K$ is a path. So $N \cap K \ll M$, $N \cap L \not\ll M$ and $K \cap L \not\ll M$. We claim that L is a maximal submodule of M. Otherwise, by Lemma 2.2 (a), there exists a maximal submodule H of M such that $L \subseteq H$. Since $L \cap N \subseteq H \cap N$ and $L \cap N \not\ll M$, we have $H \cap N$ is a non-small submodule of M. By similar arguments we have $H \cap K \not\ll M$. Hence $N - H - K$ is a path in $G(M) \setminus L$ which is a contradiction. Thus L is a maximal submodule. Now we show that there exists a maximal submodule $M_i \neq L$ of M such that $N \nsubseteq M_i$. Otherwise, if $N \subseteq M_i$ for each $M_i \in Max(M)$, then $N \subseteq \bigcap_{M_i \neq L} M_i$. Hence $N \cap L \subseteq \bigcap_{M_i \in Max(M)} M_i = Rad(M)$. So by Lemma 2.2 (b), $N \ll M$, a contradiction. Similarly, there exists $M_j \neq L$ such that $K \nsubseteq M_j$. We claim that for each $M_t \in Max(M)$, $N \subseteq M_t$ or $K \subseteq M_t$. Since $N \cap K \ll M$, by Lemma 2.2 (b), $N \cap K \subseteq Rad(M) \subseteq M_t$ for each $M_t \in Max(M)$. Hence $N \subseteq M_t$ or $K \subseteq M_t$ by Lemma 2.2 (a). Since $G(M)$ is connected, $|Max(M)| \geq 3$ by Theorem 2.8. Now Assume that $L \neq M_i, M_j \in Max(M)$ such that $N \nsubseteq M_i$ and $K \nsubseteq M_j$. Hence we have $N \subseteq M_j$ and $K \subseteq M_i$. Thus $N - M_j - M_i - K$ is a path in $G(M) \setminus L$, a contradiction. Therefore, $G(M)$ has no cut vertex. □

Theorem 2.14. *Let M be an R-module. Then $G(M)$ can not be a complete n-partite graph.*

Proof. Let $G(M)$ be a complete n-partite graph with parts $V_1, ..., V_n$. Then by Lemma 2.4, M_i and M_j are adjacent for every $M_i, M_j \in Max(M)$. So each V_i contains at most one maximal submodule of M. By Pigeon hole principal, $|Max(M)| \leq n$. Now we claim that $|Max(M)| = n$. Otherwise, let $|Max(M)| = t$, where $t < n$. Suppose that $M_i \in V_i$, $1 \leq i \leq t$. Then V_{t+1} contains no maximal submodule of M. By Lemma 2.4, we see that $\cap_{j \neq i} M_j$ is a non-small submodule of M. Since $\cap_{j \neq i} M_j \cap M_i = Rad(M)$, we have $\cap_{j \neq i} M_j$ and M_i are non-adjacent by Lemma 2.2 (b). Thus $\cap_{j \neq i} M_j \in V_i$. Let N be a vertex in V_{t+1}. Then there exists a maximal submodule M_k of M such that $N \subseteq M_k$. Thus N is adjacent to M_k. Since $G(M)$ is a complete n-partite graph and $M_k \in V_k$, N is adjacent to all vertices of V_k. Hence N is adjacent to $\cap_{j \neq k} M_j$. But this is a contradiction because $N \cap (\cap_{j \neq k} M_j) \subseteq M_k \cap (\cap_{j \neq k} Mj) = Rad(M) \ll M$. Thus $|Max(M)| = n$. Now we assume that $H = \cap_{i=3}^{n} M_i$. By Lemma 2.4, H is a non-small submodule of M. Since $H \cap M_1 = \cap_{i \neq 2} M_i \not\ll M$, we have H is adjacent to M_1. By similar arguments H is adjacent to M_2. Hence $H \notin V_1, V_2$. Further for each i $(3 \leq i \leq n)$, $H \cap M_i = H \not\ll M$. Thus H is adjacent to all maximal submodules M_i of M. Hence for each i $(1 \leq i \leq n)$, $H \notin V_i$, a contradiction. □

Theorem 2.15. *Let M be an R-module with $|Max(M)| < \infty$. Then we have the following.*

(a) *There is no vertex in $G(M)$ which is adjacent to every other vertex.*
(b) *$G(M)$ can not be a complete graph.*

Proof. (a) Let $|Max(M)| = t$. Suppose on the contrary that there exists a non-small submodule $N \in V(G(M))$ such that N is adjacent to every vertex. By Lemma 2.2 (a), there exists a maximal submodule M_i of M such that $N \subseteq M_i$. Now $K := \cap_{j \neq i} M_j$ is a non-small submodule of M by Lemma 2.4. Since N is adjacent to all other vertices, $N \cap K \not\ll M$. But $N \cap K \subseteq M_i \cap (\cap_{j \neq i} M_j) = Rad(M)$. Thus $N \cap K$ is a small submodule of M by Lemma 2.2 (b), a contradiction.

(b) This is an immediate consequence of part (a). □

The next example shows that the condition "$Max(M)$ is a finite set" can not be omitted in Theorem 2.15.

Example 2.16. Let $M = \mathbb{Z}$ be as a $\mathbb{Z}$-module. One can see that $|Max(M)| = \infty$ and 0 is the only small submodule of M. So every submodule of M is non-small and they are adjacent to each other. Thus $G(M)$ is a complete graph.

A vertex of a graph G is said to be *pendent* if its neighbourhood contains exactly one vertex.

Theorem 2.17. *Let M be an R-module.*

(a) *$G(M)$ contains a pendent vertex if and only if $|Max(M)| = 2$ and $G(M) = G_1 \cup G_2$, where G_1, G_2 are two disjoint complete subgraphs and $|V(G_i)| = 2$ for some $i = 1, 2$.*

(b) *$G(M)$ is not a star graph.*

Proof. (a) Let N be a pendent vertex of $G(M)$. Suppose on the contrary that $|Max(M)| \geq 3$. By Lemma 2.4, for each $M_i \in Max(M)$, M_i is adjacent to other maximal submodules of M. Thus $deg(M_i) \geq 2$ and hence N is not a maximal submodule. By Lemma 2.2 (a), there exists a maximal submodule M_i of M such that $N \subseteq M_i$. Without loss of generality, we may assume that $N \subseteq M_1$. Then N and M_1 are adjacent. Since $deg(N) = 1$, we have the only vertex of $G(M)$ which is adjacent to N is M_1 in other word there is no maximal submodule $M_i \neq M_1$ such that $N \subseteq M_i$. Thus $N \cap M_2 \ll M$. Hence by Lemma 2.2 (b), $N \cap M_2 \subseteq Rad(M) \subseteq M_j$ for each $M_j \neq M_1, M_2$. Thus $N \subseteq M_j$ $(j \neq 1, 2)$ by Lemma 2.2 (c), a contradiction. Therefore, $|Max(M)| = 2$. By Theorem 2.8, $G(M) = G_1 \cup G_2$ where G_1, G_2 are complete subgraphs of $G(M)$. Let $N \in G_i$ for i $(1 \leq i \leq 2)$. Then $|V(G_i)| = 2$ because G_i is a complete subgraph and $deg(N) = 1$. The converse is straightforward.

(b) Let $G(M)$ be a star graph. Then $G(M)$ contains a pendent vertex and hence $|Max(M)| = 2$ by part (a). Therefore, $G(M)$ is not connected by Theorem 2.8, a contradiction. □

A *regular* graph is a graph where each vertex has the same number of neighbours (i.e. every vertex has the same degree). A regular graph is *r-regular* (or regular of degree r) if the degree of each vertex is r.

Theorem 2.18. *Let M be an R module.*

(a) *If N and K are two vertex of $G(M)$ such that $N \subseteq K$, then $deg(N) \leq deg(K)$;*

(b) *If $G(M)$ is an r-regular graph, then $|Max(M)| = 2$ and $|V(G(M))| = 2r + 2$.*

Proof. (a) Suppose that N and K are two vertex of $G(M)$ such that $N \subseteq K$. Let L be a vertex adjacent to N. Thus $L \cap N \not\ll M$ and hence $L \cap K \not\ll M$. This implies that K is adjacent to L so that $deg(N) \leq deg(K)$.

(b) Assume on the contrary that $|Max(M)| \geq 3$. Then for each $M_i \in Max(M)$, since $deg(M_i) = r$ and M_i is adjacent to all maximal submodules by Lemma 2.4, we have $Max(M)$ is a finite set. Now for

$M_1, M_2 \in Max(M)$, $deg(M_1 \cap M_2) \leq deg(M_1)$ by part (a). Clearly, $deg(M_1 \cap M_2) \neq deg(M_1)$ because if $N = \cap_{j \neq 2} M_j$, then N is adjacent to M_1 but N is not adjacent to $M_1 \cap M_2$ by Lemma 2.2 (b). Hence $deg(M_1 \cap M_2) < r$, a contradiction. Therefore, $|Max(M)| \leq 2$. Clearly, $|Max(M)| \neq 1$. Thus $|Max(M)| = 2$ and $G(M)$ is a union of two disjoint complete subgraphs by Theorem 2.8. Let $Max(M) = \{M_1, M_2\}$ and assume that $M_i \in G_i$. Since for each $i = 1, 2$, $deg(M_i) = r$, we have $|G_i| = r + 1$. It follows that $|V(G(M))| = 2r + 2$. □

3. CLIQUE NUMBER, INDEPENDENCE NUMBER, AND DOMINATION NUMBER

In this section, we will study the clique number, independence number, and domination number of the small intersection graph. We recall that M is a multiplication R-module.

Proposition 3.1. *Let M be an R-module. Then we have the following.*

(a) *If $G(M)$ is a non-empty graph, then $\omega(G(M)) \geq |Max(M)|$.*
(b) *If $G(M)$ is an empty graph, then $\omega(G(M)) = 1$ if and only if $Max(M) = \{M_1, M_2\}$, where M_1 and M_2 are finitely generated hollow R-modules.*
(c) *If $\omega(G(M)) < \infty$, then $|Max(M)| < \infty$.*
(d) *If $\omega(G(M)) < \infty$, then $\omega(G(M)) \geq 2^{|Max(M)|-1} - 1$.*

Proof. (a) If $|Max(M)| = 2$, then $\omega(G(M)) \geq 2$ by Theorem 2.8. If $|Max(M)| \geq 3$, then the subgraph of $G(M)$ with the vertex set of $\{M_i\}_{M_i \in Max(M)}$ is a complete subgraph of $G(M)$ by Lemma 2.4. Hence $\omega(G(M)) \geq |Max(M)|$.

(b) This follows directly from Theorem 2.6.
(c) This is clear by part (a) and (b).
(d) Let $Max(M) = \{M_1, ..., M_t\}$. Also for each $1 \leq i \leq t$, set
$$A_i = \{M_1, ..., M_{i-1}, M_{i+1}, M_t\}.$$
Now let $P(A_i)$ be the power set of A_i and for each $X \in P(A_i)$, set $M_X = \bigcap_{M_j \in X} M_j$ for $1 \leq j \leq t$. The subgraph of $G(M)$ with the vertex set $\{M_X\}_{X \in P(A_i) \setminus \{\emptyset\}}$ is a complete subgraph of $G(M)$ by Lemma 2.4. Clearly, $|\{M_X\}_{X \in P(A_i) \setminus \{\emptyset\}}| = 2^{|Max(M)|-1} - 1$. Thus $\omega(G(M)) \geq 2^{|Max(M)|-1} - 1$.

□

The following remarks show that the condition " M is a multiplication module" can not be omitted in Proposition 3.1.

Remark 3.2. Let $M := \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be as Example 2.7 which is an empty graph. Then we see that $|Max(M)| = 3$; but $\omega(G(M)) = 1$.

Remark 3.3. Let $M := \mathbb{Z}_2 \oplus \mathbb{Z}_4$ be as Example 2.10. Then $G(M)$ is a non-empty graph with $\omega(G(M)) < |Max(M)|$.

Corollary 3.4. *Let M be a finitely generated R-module. If $\omega(G(M)) < \infty$, then $M/Rad(M)$ is a cyclic R-module.*

Proof. This follows from Proposition 3.1 (c) and [8, Theorem 2.8]. □

Theorem 3.5. *Let R be a ring and M be an R-module. Then $\gamma(G(M)) \leq 2$. Moreover, if $Max(M)$ is a finite set, then $\gamma(G(M)) = 2$.*

Proof. Since $G(M)$ is a non-null graph, we have $|Max(M)| \geq 2$. Set $S := \{M_1, M_2\}$, where $M_1, M_2 \in Max(M)$. Let $N \in V(G(M))$. We claim that N is adjacent to M_1 or M_2. Clearly, when $N \subseteq M_1$ or $N \subseteq M_2$, the claim is true. So we assume that $N \nsubseteq M_1$ and $N \nsubseteq M_2$. Without loss of generality, we may assume that N is not adjacent to M_1. Then $N \cap M_1 \subseteq Rad(M)$ by Lemma 2.2 (b). It follows that $N \subseteq M_2$, a contradiction. Similarly, N is adjacent to M_2. Hence $\gamma(G(M)) \leq 2$. The last assertion follows from Theorem 2.15. □

Example 3.6. Let $M := \mathbb{Z}_{36}$ be as Example 2.9. Then we see that $|Max(M)| < \infty$ and $\gamma(G(M)) = 2$.

Remark 3.7. The condition "M is a multiplication module" can not be omitted in Theorem 3.5. For example, let $M = \mathbb{Z}_2 \oplus Z_6$ be a $\mathbb{Z}$-module. Then $V(G(M)) = \{(0,1)\mathbb{Z}, (0,2)\mathbb{Z}, (0,3)\mathbb{Z}, (1,0)\mathbb{Z}, (1,1)\mathbb{Z}, (1,2)\mathbb{Z}, (1,3)\mathbb{Z}, (1,0)\mathbb{Z} + (0,3)\mathbb{Z}\}$ and $Max(M) = \{(0,1)\mathbb{Z}, (1,2)\mathbb{Z}, (1,1)\mathbb{Z}, (1,0)\mathbb{Z} + (0,3)\mathbb{Z}\}$. We see that $|Max(M)| < \infty$; but $\gamma(G(M)) = 3$.

Theorem 3.8. *Let M be an R-module and $|Max(M)| < \infty$. Then $\alpha(G(M)) = |Max(M)|$.*

Proof. Let $Max(M) = \{M_1, ..., M_n\}$. Since $T := \{\bigcap_{j=1, i\neq j}^{n} M_j\}_{i=1}^{n}$ is an independent set in $G(M)$, we have $n \leq \alpha(G(M))$ (Note that if $\alpha, \beta \in T$, then $\alpha \cap \beta = Rad(M)$, so α is not adjacent to β by Lemma 2.2 (c)). Now let $\alpha(G(M)) = m$ and let $S = \{N_1, N_2, ..., N_m\}$ be a maximal independent set in $G(M)$. Then for each $N \in S$, $N \not\ll M$. By Lemma 2.2 (b), $N \nsubseteq M_t$ for some $M_t \in Max(M)$. If $m > n$, then by Pigeon hole principal, there exists $1 \leq i, j \leq n$ such that $N_i \nsubseteq M_t$ and $N_j \nsubseteq M_t$. Since S is an independent set, N_i and N_j are not adjacent and $N_i \cap N_j \ll M$. So $N_i \cap N_j \subseteq M_t$ by Lemma 2.2 (b). Hence $N_i \subseteq M_t$ or $N_j \subseteq M_t$ by Lemma 2.2 (c), a contradiction. We have similar arguments when $\alpha(G(M)) = \infty$. Thus $\alpha(G(M)) = |Max(M)|$ and the proof is complete. □

Acknowledgments

We would like to thank the referee for careful reading and valuable comments.

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