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# ON COMPONENT EXTENSIONS OF LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. Let $\pounds$ be the category of locally compact abelian groups and $A, C \in \pounds$. In this paper, we define component extensions of $A$ by $C$ and show that the set of all component extensions of $A$ by $C$ forms a subgroup of $Ext(C, A)$ whenever $A$ is a connected group. We establish conditions under which the component extensions split and determine LCA groups which are component projective. We also gives a necessary condition for an LCA group to be component injective in $\pounds$.

## 1. INTRODUCTION

Let $\pounds$ denote the category of locally compact abelian (LCA) groups (will be written additively) with continuous homomorphisms as morphisms. The identity component of a group $G \in \pounds$ is denoted by $G_0$. A morphism is called proper if it is open onto its image and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be proper exact if $\phi$ and $\psi$ are proper morphisms. In this case the sequence is called an extension of $A$ by $C$ ( in $\pounds$ ). Following [4], we let $Ext(C, A)$ denote the (discrete) group of extensions of $A$ by $C$.The group operation on $Ext(C, A)$ is as in Theorem 2.19. The splitting problem in LCA groups is finding conditions on $A$ and $C$ under which $Ext(C, A) = 0$. In [2, 3, 5, 7, 8, 9, 12, 13] the splitting problem is studied. Sometimes, the splitting problem is limited to a subgroup or a subset of $Ext(C, A)$. Some subgroups of $Ext(C, A)$ such as

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$Pext(C, A)$,${}^*Pext(C, A)$,$Tpext(C, A)$ and $Apext(C, A)$ have been studied in [2, 7, 8, 9]. In [13], we define s-pure extensions and obtained some results. In [8] the question investigated is connected with the search for condition under which the group of pure extensions , $Pext(C, A)$, is null. In this paper, we introduce a new subgroup of $Ext(C, A)$ , namely $Ext(C, A)_0$. We study the vanishing problem for this subgroup and will find a classification for the group $C$. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a component extension if $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ is an extension. Let $Ext(C, A)_0$ denote the set of all component extensions of $A$ by $C$. In Section 2, we show that $Ext(C, A)_0$ is a subgroup of $Ext(C, A)$ whenever $A$ is a connected group (Theorem 2.19). In Section 3, we introduce component injective and component projective in $\pounds$. An LCA group $G$ is a component projective group in $\pounds$ if and only if $G \cong \mathbb{R}^n \bigoplus C \bigoplus A$ where $C$ is a compact connected group having a cotorsion dual and $A$ a discrete free group (Theorem 3.6). If $G$ is a component injective group in $\pounds$, then $G \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$ where $n$ is a nonnegative integer, $\sigma$ a cardinal number and $H$ a totally disconnected, LCA group (Theorem 3.3).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals with the discrete topology and $\mathbb{Z}$ is the group of integers with the discrete topology. The Pontrjagin dual of a group $G$ is denoted by $\hat{G}$. For more on locally compact abelian groups, see [6].

## 2. Component extensions

Let $A, C \in \pounds$. In this section, we will define component extensions and will show that the set of all component extensions of $A$ by $C$ is a subgroup of $Ext(C, A)$ whenever $A$ is a connected group.

**Definition 2.1.** An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a component extension if $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ is an extension.

**Lemma 2.2.** *An extension $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is a component extension if and only if $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ is an exact sequence.*

*Proof.* Let $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ be an exact sequence. By [6, Theorem 5.29], $\phi : A_0 \to B_0$ and $\psi : B_0 \to C_0$ are proper morphisms. Hence $E$ is a component extension. □

*Remark* 2.3. Let $G \in \pounds$ and $H$ be a connected subgroup of $G$. We know that $(G/H)_0$ is the intersection of all open subgroups of $G/H$. But an open subgroup of $G/H$ has the form $K/H$ where $K$ is an open subgroup of $G$ containing $H$. Since $H$ is connected, then by [6, Theorem 7.8], $H \subseteq K$ for every open subgroup $K$ of $G$. Hence, $(G/H)_0 = G_0/H$.

**Lemma 2.4.** *Every extension of a connected LCA group by a LCA group is a component extension.*

*Proof.* Let $E : 0 \to A \overset{\phi}{\to} B \to C \to 0$ be an extension such that $A$ is connected. Since $\phi(A)$ is a connected subgroup of $B$, so by Remark 2.3, $C_0 \cong B_0/\phi(A)$. Hence $0 \to A \overset{\phi}{\to} B_0 \to C_0 \to 0$ is a component extension. □

**Lemma 2.5.** *Every extension of a totally disconnected group by a totally disconnected group is a component extension.*

*Proof.* Let $E : 0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ be an extension such that $A$ and $C$ are totally disconnected. We claim that $B$ is totally disconnected. Since $\psi(B_0) \subseteq C_0$ and $C$ is totally disconnected, it follows that $\psi(B_0) = 0$. Hence, $B_0 \subseteq Im\phi$. But, $Im\phi$ is a totally disconnected group. Therefore, $B_0 = 0$ and $B$ is a totally disconnected group. □

The extension $0 \to A \to A \bigoplus C \to C \to 0$ is called the trivial extension.

**Lemma 2.6.** *The trivial extension of $A$ by $C$ is a component extension.*

*Proof.* It is clear. □

Recall that two extensions $0 \to A \xrightarrow{\phi_1} B \xrightarrow{\psi_1} C \to 0$ and $0 \to A \xrightarrow{\phi_2} X \xrightarrow{\psi_2} C \to 0$ are said to be equivalent if there is a topological isomorphism $\beta : B \to X$ such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi_1"] \arrow[d, "1_A"] & B \arrow[r, "\psi_1"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "\phi_2"] & X \arrow[r, "\psi_2"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative.

**Lemma 2.7.** *An extension equivalent to a component extension is a component extension.*

*Proof.* Let

$$E_1 : 0 \to A \xrightarrow{\phi_1} B \xrightarrow{\psi_1} C \to 0$$

and

$$E_2 : 0 \to A \xrightarrow{\phi_2} X \xrightarrow{\psi_2} C \to 0$$

be two equivalent extensions such that $E_1$ is a component extension. Then, there is a topological isomorphism $\beta : B \to X$ such that $\beta\phi_1 =$

$\phi_2$ and $\psi_2\beta = \psi_1$. Let $c_0 \in C_0$. Since $E_1$ is a component extension, so $\psi_1(b_0) = c_0$ for some $b_0 \in B_0$. Hence, $\psi_2(\beta(b_0)) = \psi_1(b_0) = c_0$. So, $\psi_2 : X_0 \to C_0$ is surjective. Now, let $\psi_2(x_0) = 0$ for some $x_0 \in X_0$. Since $\beta(B_0) = X_0$, so there exists $b_0 \in B_0$ such that $\beta(b_0) = c_0$. Hence, $\psi_1(b_0) = \psi_2(\beta(b_0)) = 0$. Since $E_1$ is a component extension,then $\phi_1(a_0) = b_0$ for some $a_0 \in A_0$. Consequently, $\phi_2(a_0) = \beta(\phi_1(a_0)) = x_0$. $\square$

**Definition 2.8.** Let $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ be an extension and $\alpha : A \to A'$ be a proper morphism. We define the sequence $\alpha E$ as follows:

$$\alpha E : 0 \to A' \xrightarrow{\phi'} X \xrightarrow{\psi'} C \to 0$$

where

$$X = (A' \bigoplus B)/H$$

$$H = \{(-\alpha(a), \phi(a)); a \in A\}$$

$$\phi'(a') = (a', 0) + H$$

$$\psi'((a', b) + H) = \psi(b)$$

Then, $\alpha E$ is an extension which is called the standard pushout of $E$ (See [4, Proposition 2.3]).

Let $\gamma : C' \to C$ be a proper morphism. We define the sequence $E\gamma$ as follows:

$$E\gamma : 0 \to A \xrightarrow{\phi'} X \xrightarrow{\psi'} C' \to 0$$

where

$$X = \{(b, c'); b \in B, c' \in C', \psi(b) = \gamma(c')\}$$

$$\phi'(a) = (\phi(a), 0)$$

$$\psi'(b, c') = c'$$

Then, $E\gamma$ is an extension which is called the standard pullback of $E$ (See [4, Proposition 2.3]).

**Lemma 2.9.** *A pullback of a component extension is a component extension.*

*Proof.* Suppose $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is a component extension and

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi'"] \arrow[d, "1_A"] & B' \arrow[r, "\psi'"] \arrow[d] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\phi"] & B \arrow[r, "\psi"] & C \arrow[r] & 0 \end{tikzcd}$$

is the standard pullback diagram. Then,

$$B' = \{(b, c'); \psi(b) = \gamma(c')\}$$

and

$$\phi' : a \longmapsto (\phi(a), 0),\ \ \psi' : (b, c') \longmapsto c'$$

We show that $0 \to A_0 \xrightarrow{\phi'} B'_0 \xrightarrow{\psi'} C'_0 \to 0$ is exact. Let $c'_0 \in C'_0$. Then, $\gamma(c'_0) \in C_0$. Since $\psi : B_0 \to C_0$ is surjective, so there exists $b_0 \in B_0$ such that $\psi(b_0) = \gamma(c'_0)$. Hence, $(b_0, c'_0) \in B'_0$ and $\psi'(b_0, c'_0) = c'_0$. So $\psi' : B'_0 \to C'_0$ is surjective. Now, suppose that $(b, c') \in B'_0$ and $\psi'(b, c') = 0$. Then, $c' = 0$ and $b \in B_0$. Since $\psi(b) = 0$, so there exists $a_0 \in A_0$ such that $\phi(a_0) = b$. Hence, $\phi'(a_0) = (b, 0) = (b, c')$. This shows that $Ker\psi' \mid_{B'_0} \subseteq Im\phi' \mid_{A_0}$. □

*Remark* 2.10. Let $f : A \to C$ be a proper morphism and $G \in \pounds$. Then

(1) $f_* : Ext(G, A) \to Ext(G, C)$ defined by $f_*([E]) = [fE]$
and
(2) $f^* : Ext(C, G) \to Ext(A, G)$ defined by $f^*([E]) = [Ef]$
are group homomorphisms (See [10, Theorem 2.1]).

Recall that $Ext(C, A)_0$ denotes the set of all component extensions of $A$ by $C$. We say that $Ext(C, A)_0 = 0$ if every component extension of $A$ by $C$ splits.

**Lemma 2.11.** *Let $f : A \to C$ be a proper morphism. Then, $f^*(Ext(C, G)_0) \subseteq Ext(A, G)_0$ for all $G \in \pounds$.*

*Proof.* Let $E \in Ext(C, G)_0$. Then, by definition, $Ef$ is a pullback of $E$.By Remak 2.10 (2), $f^*(E) = [Ef]$ . Now, by Lemma 2.9,$f^*(E) \in Ext(A, G)_0$. □

**Theorem 2.12.** *If $C$ is a connected group and $A$ a totally disconnedted group, then $Ext(C, A)_0 = 0$.*

*Proof.* Let $E : 0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ be a component extension. Then, $0 \to A_0 \overset{\phi}{\to} B_0 \overset{\psi}{\to} C_0 \to 0$ is an extension. Since $A$ is totally disconnected and $C$ connected, it follows that $\psi : B_0 \to C$ is a topological isomorphism. We claim that $B = B_0 + \phi(A)$. Let $b \in B$. Then $\psi(b) \in C$.

So $\psi(b) = \psi(b_0)$ for some $b_0 \in B_0$. Hence $b - b_0 \in ker\psi = im\phi$. So $b = b_0 + \phi(a)$ for some $a \in A$. Now, we show that $\phi(A) \bigcap B_0 = 0$. Let $b \in \phi(A) \bigcap B_0$. Then, $\psi(b) = 0$. Since $\psi : B_0 \to C$ is injective, so $b = 0$. Hence, $\phi(A) \bigcap B_0 = 0$. Since $B_0$ is $\sigma-$compact, it follows from [4, Corollary 3.2] that $B \cong \phi(A) \bigoplus B_0$. This shows that $E$ splits. □

**Lemma 2.13.** *Let $G \in \pounds$ be a non discrete group which contains a compact open subgroup $K$. Then $Ext(G, \mathbb{Z}) \neq 0$.*

*Proof.* Consider the exact sequence $0 \to K \to G \to G/K \to 0$. By [4, Corollary 2.10], we have the exact sequence

$$\to Ext(G/K, \mathbb{Z}) \to Ext(G, \mathbb{Z}) \to Ext(K, \mathbb{Z}) \to 0$$

If $Ext(G, \mathbb{Z}) = 0$, then $Ext(K, \mathbb{Z}) = 0$. By [4, Theorem 2.12], $Ext(\hat{\mathbb{Z}}, \hat{K})$ $\cong Ext(K, \mathbb{Z}) = 0$. It follows from [4, Proposition 2.17] that $\hat{K} = 0$ which is a contradiction because $G$ is not a discrete group. □

**Lemma 2.14.** *Let $G \in \pounds$ be a non connected and non totally disconnected group such that $G_0$ is not an open subgroup of $G$. Then $Ext(G, \mathbb{Z})_0 \neq 0$.*

*Proof.* Consider the extension $0 \to G_0 \xrightarrow{i} G \longrightarrow G/G_0 \to 0$. By [5, Corollary 2.10], we have the exact sequence

$$0 \to Ext(G/G_0, \mathbb{Z}) \xrightarrow{\pi^*} Ext(G, \mathbb{Z}) \xrightarrow{i^*} Ext(G_0, \mathbb{Z}) \to 0$$

Now, suppose that $Ext(G, \mathbb{Z})_0 = 0$. We have $\pi^*(Ext(G/G_0, \mathbb{Z})_0) \subseteq Ext(G, \mathbb{Z})_0$, so $\pi^*(Ext(G/G_0, \mathbb{Z})_0 = 0$. Since $\pi^*$ is injective, then $Ext(G/G_0, \mathbb{Z})_0 = 0$. By Lemma 2.5, $Ext(G/G_0, \mathbb{Z}) = 0$ which is a contradiction because $G/G_0$ is a totally disconnected group which contains a compact open subgroup. On the other hand, $G/G_0$ is not a discrete group. So, by Lemma 2.13, $Ext(G/G_0, \mathbb{Z}) \neq 0$. □

**Lemma 2.15.** *Let $A$ be a discrete divisible group and $G$ a LCA group. Then $Ext(G, A)_0 = 0$.*

*Proof.* Consider the extension $0 \to G_0 \xrightarrow{i} G \longrightarrow G/G_0 \to 0$. By [5, Corollary 2.10], we have the exact sequence

$$0 \to Ext(G/G_0, A) \to Ext(G, A) \xrightarrow{i^*} Ext(G_0, A) \to 0$$

Since, $G/G_0$ is a totally disconnected group, so by [4, Theorem 3.4] $Ext(G/G_0, A) = 0$. Hence, $i^*$ is injective. By Lemma 2.11,$i^*(Ext(G, A)_0)$ $\subseteq Ext(G_0, A)_0$. By Theorem 2.12, $Ext(G_0, A)_0$ $= 0$. Hence, $i^*(Ext(G, A)_0) = 0$. Now $Ext(G, A)_0 = 0$ because $i^*$ is injective. □

The following remark shows that the pushout of a component extension need not be a component extension.

*Remark* 2.16. By Lemma 2.14, if $G = \hat{\mathbb{Q}} \bigoplus \widehat{(\mathbb{Q}/\mathbb{Z})}$, then $Ext(G, \mathbb{Z})_0 \neq 0$. So there exists a non splitting component extension $E : 0 \to \mathbb{Z} \xrightarrow{\phi} X \to G \to 0$. Consider the standard pushout $iE$:

$$iE : 0 \to \mathbb{Q} \to (\mathbb{Q} \bigoplus X)/H \to G \to 0$$

where $H = \{(-n, \phi(n)); n \in \mathbb{Z}\}$. Let $iE$ be a component extension. Then by Lemma 2.15, it splits. So $(\mathbb{Q} \bigoplus X)/H \cong \mathbb{Q} \bigoplus G$. Since $\mathbb{Q} \bigoplus G$ is torsion-free, so $H$ is a pure subgroup. On the other hand, $H \subseteq \mathbb{Q} \bigoplus nX$. So $H$ is a divisible group. Hence, for a positive integer $m \neq 1$, there exists $(-n, f(n)) \in H$ such that $m(-n, f(n)) = (-1, f(1))$. It follows that $mn = 1$ which is a contradiction. So $iE$ is not a component extension.

**Lemma 2.17.** *Let $A$ be a connected group. Then a pushout of a component extension of $A$ by $C$ is a component extension.*

*Proof.* Suppose $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is a component extension and

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\mu"] & B \arrow[r, "\psi"] \arrow[d] & C \arrow[r] \arrow[d, "1_C"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\phi'"] & (A' \bigoplus B)/H \arrow[r, "\psi'"] & C \arrow[r] & 0
\end{tikzcd}$$

is a standard pushout diagram. Then

$$H = \{(\mu(a), -\phi(a)), a \in A\}$$

and

$$\phi' : a' \longmapsto (a', 0) + H, \ \psi' : (a', b) + H \longmapsto \psi(b)$$

We show that $0 \to A'_0 \xrightarrow{\phi'} ((A' \bigoplus B)/H)_0 \xrightarrow{\psi'} C_0 \to 0$ is exact. Let $c_0 \in C_0$. Since $\psi : B_0 \to C_0$ is surjective, so there exists $b_0 \in B_0$ such that $\psi(b_0) = c_0$. Hence $(0, b_0) + H \in ((A' \bigoplus B)/H)_0$. Since $\psi'((0, b_0) + H) = c_0$, so $\psi' : ((A' \bigoplus B)/H)_0 \to C_0$ is surjective. Let $(a', b) + H \in ((A' \bigoplus B)/H)_0$ and $\psi'((a', b) + H) = 0$. So $\psi(b) = 0$. Hence, there is $a \in A$ such that $\phi(a) = -b$. Since $A$ is connected, then $\mu(A) \subseteq A'_0$. □

Let $E_1 : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ and $E_2 : 0 \to A \xrightarrow{\phi'} B' \xrightarrow{\psi'} C \to 0$ be two extensions of $A$ by $C$. Then the direct sum of $E_1$ and $E_2$ is denoted by $E_1 \bigoplus E_2$ and defined as follows:

$$E_1 \bigoplus E_2 : 0 \to A \bigoplus A \xrightarrow{\phi \bigoplus \phi'} B \bigoplus B' \xrightarrow{\psi \bigoplus \psi'} C \bigoplus C \to 0$$

**Lemma 2.18.** *The direct sum of two component extensions is a component extension.*

*Proof.* It is clear. □

**Theorem 2.19.** *Let $A, C \in \pounds$ such that $A$ is a connected group. Then, $Ext(C, A)_0$ is an subgroup of $Ext(C, A)$ with respect to the operation defined by*

$$[E_1] + [E_2] = [\nabla_A(E_1 \bigoplus E_2)\triangle_C]$$

*where $E_1$ and $E_2$ are component extensions of $A$ by $C$ and $\nabla_A$ and $\triangle_C$ are the diagonal and codiagonal homomorphism.*

*Proof.* Clearly, $0 \to A \to A \bigoplus C \to C \to 0$ is a component extension. Let $[E] \in Ext(C, A)_0$. The inverse of $[E]$ is $[-1_A E]$ which belongs to $Ext(C, A)_0$ (Lemma 2.17). By Lemma 2.9 and Lemma 2.17, $[E_1] + [E_2] \in Ext(C, A)_0$ for two component extensions $E_1$ and $E_2$ of $A$ by $C$. Therefore, $Ext(C, A)_0$ is a subgroup of $Ext(C, A)$. □

## 3. Component injective and projective in $\pounds$

In this section, we define the concept of component injective and component projective in $\pounds$ and classify them.

**Definition 3.1.** Let $G \in \pounds$. We call $G$ a component injective group in $\pounds$ if for every component extension

$$0 \to A \xrightarrow{\phi} B \to C \to 0$$

and a morphism $f : A \to G$, there is a morphism $\bar{f} : B \longrightarrow G$ such that $\bar{f}\phi = f$.

We call $G$ a component projective group in $\pounds$ if for every component extension

$$0 \to A \to B \xrightarrow{\psi} C \to 0$$

and a morphism $f : G \to C$, there is a morphism $\bar{f} : G \to B$ such that $\psi\bar{f} = f$ .

**Lemma 3.2.** $\mathbf{Q}$ *is not a component injective in $\pounds$.*

*Proof.* By Lemma 2.14, there is a non splitting component extension $E : 0 \to \mathbf{Z} \xrightarrow{\phi} X \xrightarrow{\psi} G \to 0$. Let $\mathbf{Q}$ be a component injective. Then, there is a morphism $f : X \to \mathbf{Q}$ such that $f\phi = i$ where $i : \mathbf{Z} \hookrightarrow \mathbf{Q}$ is an inclusion. Since $E$ is a component extension, so $\psi(X_0) = G_0$. Hence,

$X = X_0 + \phi(\mathbf{Z})$. Since $f$ is continuous and $\mathbf{Q}$ totally disconnected,then $f(X_0) = 0$. We claim that $f(x) = n$ for some $n \in \mathbf{Z}$. Let $x \in X$. Then $x = x_0 + \phi(n)$ for some $n \in \mathbf{Z}$. So $f(x) = f(\phi(n)) = n$. An easy calculation shows that $f^{-1}(0) = X_0$. So $X_0$ is open in $X$. Hence $\psi(X_0) = G_0$ is open in $G$ which is a contradiction. □

**Theorem 3.3.** *Let $G \in \pounds$. Consider the following conditions for $G$:*

1. *$G$ is component injective in $\pounds$.*
2. *$Ext(X, G)_0 = 0$ for all $X \in \pounds$.*
3. *$G \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$ where $n$ is a nonnegative integer, $\sigma$ a cardinal number and $H$ is a totally disconnected, LCA group.*

*Then:* (1) ⇒ (2) ⇒ (3).

*Proof.* (1) ⇒ (2): It is clear.

(2) ⇒ (3): Let $G \in \pounds$ and $Ext(X, G)_0 = 0$ for all $X \in \pounds$. Let $C$ be a connected group. Consider the following exact sequence

$$0 \to Ext(C, G_0) \xrightarrow{i_*} Ext(C, G) \xrightarrow{\pi_*} Ext(C, G/G_0) \to 0$$

Since $Ext(C, G)_0 = 0$ and $i_*(Ext(C, G_0)_0) \subseteq Ext(C, G)_0 = 0$, so $Ext(C, G_0)_0 = 0$. By Lemma 2.4 (2), $Ext(C, G_0) = 0$. So By [5, Theorem 3.3], $G_0 \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma$. Hence $G \cong G_0 \bigoplus G/G_0$. Set $H = G/G_0$. So $G \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$. □

*Remark* 3.4. In Theorem 3.3, (3) may not imply (1).For example, take $G = \mathbf{Q}$. Then, by Lemma 3.2, $G$ is not a component injective group. Also, if $X = \hat{\mathbb{Q}} \bigoplus \widehat{(\mathbb{Q}/\mathbb{Z})}$ then by Lemma 2.14 $Ext(X, \mathbb{Z})_0 \neq 0$. Hence, (3) may not imply (2) as well.

**Lemma 3.5.** *Let $G \in \pounds$ and $\{H_i; i \in I\}$ be a collection in $\pounds$ such that $\bigoplus_{i\in I} H_i \in \pounds$. If $Ext(H_i, G)_0 = 0$ for every $i$, then $Ext(\bigoplus_{i\in I} H_i, G)_0 = 0$.*

*Proof.* By [4, Theorem 2.13], $\sigma : Ext(\bigoplus_{i\in I} H_i, G) \to \prod_{i\in I} Ext(H_i, G)$ defined by $\sigma([E]) = ([El_i])_{i\in I}$ is an isomorphism where $l_i : H_i \to \bigoplus_{i\in I} H_i$ is an injection. By Lemma 2.9, $[El_i] \in Ext(H_i, G)_0$ for every $i$. So $\sigma(Ext(\bigoplus_{i\in I} H_i, G)_0) \subseteq \prod_{i\in I} Ext(H_i, G)_0$. Now, suppose that $Ext(H_i, G)_0 = 0$ for every $i$. Then, $\sigma(Ext(\bigoplus_{i\in I} H_i, G)_0) = 0$. Since $\sigma$ is injective, then $Ext(\bigoplus_{i\in I} H_i, G)_0 = 0$. □

**Theorem 3.6.** *Let $G \in \pounds$. The following statements are equivalent:*

1. *$G$ is component projective in $\pounds$.*
2. *$Ext(G, X)_0 = 0$ for all $X \in \pounds$.*
3. *$G \cong \mathbb{R}^n \bigoplus C \bigoplus A$ where $C$ is a compact connected group having a cotorsion dual and $A$ a discrete free group.*

*Proof.* (1) $\Rightarrow$ (2): It is clear. (2) $\Rightarrow$ (3): Let $G \in \pounds$ and $Ext(G,X)_0 = 0$ for all $X \in \pounds$. Let $X$ be a totally disconnected group. Consider the following exact sequence

$$0 \to Ext(G/G_0, X) \xrightarrow{\pi^*} Ext(G, X) \xrightarrow{i^*} Ext(G_0, X) \to 0$$

Since $Ext(G,X)_0 = 0$, so $Ext(G/G_0, X)_0 = 0$. Hence $Ext(G/G_0, X) = 0$ for all totally disconnected groups $X$. By [4, Theorem 4.1], $G/G_0$ is a free group. So $G \cong G_0 \bigoplus G/G_0$. By Lemma 2.4, $Ext(G,C) = Ext(G,C)_0 = 0$ for all connected groups $C \in \pounds$. So $Ext(G_0, C) = 0$ for all connected groups $C \in \pounds$. by [5, Theorem 3.6], $G_0 \cong \mathbb{R}^n \bigoplus C$ where $C$ is a compact group having a cotorsion dual. (3) $\Rightarrow$ (2): Let $G \cong \mathbb{R}^n \bigoplus C \bigoplus A$ where $C$ is a compact connected group having a cotorsion dual and $A$ a discrete free group. By [11, Theorem 3.3], $Ext(\mathbb{R}^n \bigoplus A, X) = 0$ for all $X \in \pounds$. Let $X \in \pounds$. Now, we show that $Ext(C,X)_0 = 0$. Consider the following exact sequence

$$0 \to Ext(C, X_0) \to Ext(C, X) \xrightarrow{\pi_*} Ext(C, X/X_0) \to 0$$

Since $\hat{C}$ is a cotorsion group, so $Ext(C, X_0) \cong Ext(\widehat{(X_0)}, \hat{C}) = 0$. By Theorem 2.12, $Ext(C, X/X_0)_0 = 0$. Hence $\pi_*(Ext(C,X)_0) = 0$. So $Ext(C,X)_0 = 0$. Hence, by Lemma 3.5, $Ext(\mathbb{R}^n \bigoplus C \bigoplus A, X)_0 = 0$. (2) $\Rightarrow$ (1): Let $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ be a component extension and $f : G \to C$ a morphism. Consider the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi'"] \arrow[d, "1_A"] & B' \arrow[r, "\psi'"] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A \arrow[r, "\phi"] & B \arrow[r, "\psi"] & C \arrow[r] & 0
\end{tikzcd}$$

By Lemma 2.9, $Ef$ is a component extension. So by assumption, $Ef$ splits. Hence, there is a morphism $h : C \to B'$ such that $\psi' h = f$. Now, $\beta h$ is a morphism of $G$ to $B$ and $\psi\beta h = f$. So $G$ is component projective in $\pounds$.

$\square$

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