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POSITIVE CONE IN p-OPERATOR PROJECTIVE TENSOR PRODUCT OF FIGÀ-TALAMANCA-HERZ ALGEBRAS

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ABSTRACT. In this paper we define an order structure on the p-operator projective tensor product of Herz algebras and we show that the canonical isometric isomorphism between $A_p(G \times H)$ and $A_p(G)\widehat{\otimes}^p A_p(H)$ is an order isomorphism for amenable groups G and H.

1. INTRODUCTION

Operator spaces were introduced in the mid 70's by E.G. Effros [6]. By Ruan's Theorem each operator space can be embedded completely isometrically in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The C^*-algebra $\mathcal{B}(\mathcal{H})$ has a natural order structure, inducing a cone on any embedded $*$-subalgebra (see [3] for more details on order structure of operator algebras). Also, the dual of a C*-algebra has a natural cone. The case is more interesting when the dual space is a Banach algebra. An example is the commutative C^*-algebra $C_0(G)$, and a less trivial example is provided by the group C^*-algebra $C^*(G)$, which is the enveloping C^*-algebra of the Banach algebra $L^1(G)$ of absolutely integrable (with respect to the Haar measure) Borel functions on G. In this case, the dual space $B(G)$, is the Fourier-Stieltjes algebra. The Fourier algebra $A(G)$ and Fourier-Stieltjes algebra $B(G)$, introduced by Eymard in the 60's [7], are commutative Banach algebras which have a natural order

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structure (given by the cone of positive-definite functions) and a quite relevant operator space structure.

Let G be a locally compact group, the Fourier algebra $A(G)$ consists of all coefficient functions of the left regular representation λ of G

$$A(G) = \{w = (\lambda\xi\,,\,\eta)\,:\;\xi\,,\,\eta \in L_2(G)\},$$

which is a Banach algebra with the norm $\|w\|_{A(G)} = \inf\{\|\xi\|_2\|\eta\|_2 : w = (\lambda\xi\,,\,\eta)\}$, introduced by Eymard in 1964 [7]. For an abelian group G, the Fourier transform yields an isometric isomorphism from $A(G)$ onto $L_1(\widehat{G})$, where $\widehat{G}$ is the Pontryagin dual group of G. In general, $A(G)$ is a two-sided closed ideal of the Fourier-Stieltjes algebra $B(G)$ [7]. This is the linear span of the set $P(G)$ of all positive definite continuous functions on G. There is a vast literature about $A(G)$ and $B(G)$. In an earlier paper, the authors studied the order structure of the Fourier algebra $A(G)$ [15].

There are reasons to believe the significance of the order structure of the Fourier algebra. For instance, it is shown by W. Arendt, J. De Cannière that for locally compact groups G_1 and G_2, the Fourier algebras $A(G_1)$ and $A(G_2)$ are order isomorphic (in either positive-definite or pointwise orders) if and only if G_1 and G_2 are isomorphic and homeomorphic, whereas a similar statement about usual Banach algebra isomorphism fails [2].

In [8], Figà-Talamanca introduced a natural generalization of the Fourier algebra, for a compact abelian group G and $p \in (1, \infty)$, by replacing $L_2(G)$ by $L_p(G)$. In [9], Herz extended the notion to an arbitrary group, leading to the commutative Banach algebra $A_p(G)$, called the Figà-Talamanca-Herz algebra. The p-analog of the Fourier-Stieltjes algebra, is defined as the multiplier algebra $B_p(G)$ of $A_p(G)$ [4],[12]. Runde in [13] defined this algebra using coefficient functions of p-representations on QSL_p-spaces.

In [16] the authors introduced and studied the order structure of $A_p(G)$, and extended the result of Arendt and De Cannière. In this paper, using the p-operator space structure on $A_p(G)\widehat{\otimes}^p A_p(H)$, for locally compact groups G, H, we introduce an order structure on this space. Then we show that the isometric isomorphism between $A_p(G \times H)$ and $A_p(G)\widehat{\otimes}^p A_p(H)$, studied in [5], is an order isomorphism for amenable groups G and H. The amenability assumption is crucial and may not be removed.

2. p-Operator Spaces

In this section we give a brief introduction to the notion of p-operator spaces, defined by Daws in [5] and studied in [1]. Let $n \in \mathbb{N}$, $p \in (1, \infty)$, and let E be a vector space. We denote the vector space of $n \times m$ matrices with entries from E by $\mathbb{M}_{n,m}(E)$. We put $\mathbb{M}_{n,m} := \mathbb{M}_{n,m}(\mathbb{C})$. The space $\mathbb{M}_n := \mathbb{M}_{n,n}$ is equipped with the operator norm $|\cdot|_n$ from its canonical action on n-dimensional L_p-space , ℓ_p^n. The matrix space $\mathbb{M}_n$ acts on $\mathbb{M}_n(E)$ by matrix multiplication. For a square matrix $a = (a_{ij}) \in \mathbb{M}_n$, we have

$$\|a\|_{B(\ell_p^n)} = \sup\{(\sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}x_j|^p)^{1/p} : x_j \in \mathbb{C}, \sum_{j=1}^{n} |x_j|^p \leq 1\}.$$

Definition 2.1. Let E be a vector space. A p-matricial norm on E is a family $(\|\cdot\|_n)_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, $\|\cdot\|_n$ is a norm on $\mathbb{M}_n(E)$ satisfying

$$\|\lambda \cdot x \cdot \mu\|_m \leq |\lambda| \|x\|_n |\mu|, \qquad \|x \oplus y\|_{n+m} = max\{\|x\|_n, \|y\|_m\},$$

for each $\lambda \in \mathbb{M}_{m,n}, \mu \in \mathbb{M}_{n,m}, x \in \mathbb{M}_n(E)$, and $y \in \mathbb{M}_m(E)$. Here $\lambda \cdot x \cdot \mu$ is the obvious matrix product, and $|\lambda|$ and $|\mu|$ are the norms of λ and μ as the members of $\mathcal{B}(\ell_p^n, \ell_p^m)$ and $\mathcal{B}(\ell_p^m, \ell_p^n)$, respectively.

The vector space E equipped with a p-matricial norm $(\|\cdot\|_n)_{n=1}^{\infty}$ is called a p-matricial normed space. If moreover, each $(\mathbb{M}_n(E), \|\cdot\|_n)$ is a Banach space, E is called an (abstract) p-operator space.

Clearly 2-operator spaces are the same as the classical operator spaces. For more details about operator spaces see [6], [13] and [20].

Definition 2.2. Let E and F be p-operator spaces, and let $T \in \mathcal{B}(E, F)$, then for each $n \in \mathbb{N}$

$$T^{(n)} : \mathbb{M}_n(E) \longrightarrow \mathbb{M}_n(F), \quad T^{(n)}([x_{ij}]) = [T(x_{ij})]$$

is the n-th amplification of T. The map T is called p-completely bounded if

$$\|T\|_{pcb} := \sup \|T^{(n)}\| < \infty.$$

If $\|T\|_{pcb} \leq 1$, we say that T is a p-complete contraction, and if $T^{(n)}$ is an isometry, for each $n \in \mathbb{N}$, we call T a p-complete isometry.

We use the notation $\cong^{p-iso}$ for p-complete isometries.

By [5, Section 4], the collection $\mathcal{CB}_p(E, F)$ of all p-completely bounded maps from E to F is a Banach space under $\|\cdot\|_{pcb}$ and a p-operator space through the identification

$$\mathbb{M}_n(\mathcal{CB}_p(E, F)) = \mathcal{CB}_p(E, \mathbb{M}_n(F)) \qquad (n \in \mathbb{N}).$$

Let SQ_p be the collection of subspaces of quotients of L_p spaces (the Banach spaces of the form $L_p(X)$ for some measure space X), where we identify spaces which are isometrically isomorphic. A concrete p-operator space is a closed subspace of $B(E)$, for some $E \in SQ_p$. Now [5, Section 4.1] shows that an abstract p-operator space can be isometrically embedded in $B(E)$ for some $E \in SQ_p$.

3. Tensor product

Daws in [5] defined the p-operator space projective tensor norm on the tensor product of two p-operator spaces X and Y, that is

$$\|u\|_\wedge = \inf\left\{\|\alpha\|\|x\|\|y\|\|\beta\|, u = \alpha(x \otimes y)\beta\right\}, u \in \mathbb{M}_m(X \otimes Y)$$

where the norm of α and β is taken as members of all linear maps on suitable ℓ_p spaces. The norm defined above, gives $X \otimes Y$ an abstract p-operator space structure and the completion is denoted by $X\widehat{\otimes}^p Y$.

Theorem 3.1. [5, Proposition 4.9] *Let X, Y and Z be p-operator spaces. There is a identification*

$$\mathcal{CB}_p(X\widehat{\otimes}^p Y, Z) \cong^{p-iso} \mathcal{CB}_p(X, CB_p(Y, Z)).$$

4. Figà-Talamanca-Herz algebras

Throughout the rest of this paper G is a locally compact group, p is a real number in $(1, \infty)$ and $q \in (1, \infty)$ is the conjugate scalar of p, that is $\frac{1}{p} + \frac{1}{q} = 1$. Figà-Talamanca-Herz algebras are our main examples of p-operator spaces, studied in [5]. For any function $f : G \longrightarrow \mathbb{C}$ we define $\tilde{f} : G \to \mathbb{C}$ by $\tilde{f}(x) = \overline{f(x^{-1})}$, $x \in G$. The Figà-Talamanca-Herz algebra $A_p(G)$ consists of those functions $f : G \to \mathbb{C}$ for which there are sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $L_q(G)$ and $L_p(G)$, respectively, such that $f = \Sigma_{n=1}^\infty \xi_n * \tilde{\eta}_n$ and

$$\Sigma_{n=1}^\infty \|\xi_n\|_q \|\eta_n\|_p < \infty.$$

The norm $\|f\|_{A_p(G)}$ of $f \in A_p(G)$ is defined as the infimum of the above sums over all possible representations of f. Then $A_p(G)$ is a Banach space which is embedded contractively in $C_0(G)$. It was shown by Herz that $A_p(G)$ is a Banach algebra under pointwise multiplication. When $p = 2$, we get the Fourier algebra $A(G)$.

Let $\lambda_p : G \longrightarrow \mathcal{B}(L_p(G))$ be the left regular representation of G on $L_p(G)$, defined by $\lambda_p(s)(f)(t) = f(s^{-1}t)$. Then λ_p can be lifted to a representation of $L_1(G)$ on $L_p(G)$. The algebra of pseudomeasures

$PM_p(G)$ is defined as the w^*-closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$. There is a canonical duality $PM_p(G) \cong A_p(G)^*$ via

$$\langle \xi * \tilde{\eta}, T\rangle := \langle \xi, T(\eta)\rangle \quad (\xi \in L_p(G)\ ,\ \eta \in L_q(G)\ ,\ T \in PM_p(G)).$$

In particular $PM_2(G)$ is the group von Neumann algebra $VN(G)$. If the map Λ_p from the projective tensor product $L_q(G)\widehat{\otimes}^{\gamma} L_p(G)$ to $C_0(G)$ is defined by

$$\Lambda_p(g \otimes f)(s) = \langle g, \lambda_p(s)f\rangle,$$

for $g \in L_q(G)$, $f \in L_p(G)$, and $s \in G$, then $A_p(G)$ is isometrically isomorphic to $L_q(G)\widehat{\otimes}^{\gamma} L_p(G)/ker\Lambda_p$ and the dual space of $A_p(G)$ is

$$PM_p(G) = A_p(G)^* = ker\Lambda_p{}^{\perp} = \{T \in \mathcal{B}(L_p(G)) : T|_{ker\Lambda_p = 0}\}.$$

Since $PM_p(G) \subseteq B(L_p(G))$, it has a natural p-operator space structure. In particular $A_p(G)$ carries a natural dual p-operator space structure [5, Section 5.1]. Also there is a quotient structure on $A_p(G)$, making the map $\Lambda_p : L_q(G)\widehat{\otimes}^{\gamma} L_p(G) \to A_p(G)$ a p-complete quotient map. When G is amenable, then the two natural p-operator space structures on $A_p(G)$ agree.[5, Theorem 7.1]

One of the main result of [5] is the following, which we shall use later.

Proposition 4.1. [5, Theorem 7.3] *Let G and H be amenable locally compact groups. Then $A_p(G)\widehat{\otimes}^{p} A_p(H) \cong^{p-iso} A_p(G \times H)$.*

For $p = 2$ this result is proved in [6, Theorem 7.2.4].

Remark 4.2. We use the above identification, whenever we consider an element $u \in A_p(G)\widehat{\otimes}^{p} A_p(H)$ as an element of $A_p(G \times H)$.

In this paper we study the above isomorphism of p-operator spaces as ordered spaces. We show that with a canonical order structure on p-operator tensor product the above equality is order isomorphism.

5. Order Structure

The concept of positivity appears in the theory of ordered spaces, lattice spaces and C^*-algebras.

If E is a real vector space, then $E_{\mathbb{C}} = E + iE$ is the complexification of E [17]. Each $\mathbb{R}$-linear map $T : E \to F$ has a unique $\mathbb{C}$-linear extension $\tilde{T} : E_{\mathbb{C}} \to F_{\mathbb{C}}$. A real vector space E endowed with an order relation $\leq$ is called a *real ordered space* if, given $x, y \in E$, the relation $x \leq y$ implies $x + z \leq y + z$ and $\alpha x \leq \alpha y$, for all $z \in E$ and $\alpha \in \mathbb{R}_+$. The *positive cone* of E is $E_+ = \{x : 0 \leq x\}$. We say that E_+ is *proper* if $E_+ \cap (-E_+) = \{0\}$. The complexification of a real ordered vector space is called an *ordered space*. For given ordered spaces $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$, a $\mathbb{C}$-linear map $T : E_{\mathbb{C}} \to F_{\mathbb{C}}$ is *positive* if $T(E_+) \subseteq F_+$, and is an *order isomorphism* if it is one-one, surjective positive map with a positive inverse.

The dual space of each (normed) ordered space is an ordered space as well. Let A be a normed space which is an ordered space. $\varphi \in A^*$ is positive if for all positive element x in A, we have $\varphi(x)$ is positive.

In this section we give the definition of positivity in spaces which will be studied in this paper.

Let $1 < p < \infty$, X be a measure space and $1 < q < \infty$ be the conjugate scalar of p.

- The natural order structure on $\mathbb{C}$ and $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ come from their C^* algebra structures. An element $u \in \mathbb{C}$ is positive if it is a real and positive scalar. A matrix $[a_{ij}] \in \mathbb{M}_n$ is positive if
$$\sum_{i,j} a_{ij} c_i \overline{c_j} > 0,$$
for all chose of scalars $c_1, \cdots, c_n \in \mathbb{C}$.
- The natural positive cone of each C^*-algebra gives an order structure. Also by Ruan's Theorem [6], each operator space E isometrically embed in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and inherits a natural order structure of C^*-algebra $\mathbf{B}(\mathcal{H})$.
- We say that $T \in \mathcal{B}(L_p(X))$ is positive if $\langle Tf, f\rangle \geq 0$ for each $f \in L_p(X) \cap L_q(X)$, where the pairing is the canonical dual action of $L_q(X)$ on $L_p(X)$. For $p = 2$ this order is the natural order on the C^*-algebra $\mathcal{B}(L_2(X))$.
- We say that $T = [T_{ij}] \in \mathbb{M}_n(\mathcal{B}(L_p(X))) \cong \mathcal{B}(L_p^n(X))$, (where $L_p^n(X)$ is the direct sum of n-copies of $L_p(X)$), is positive if
$$\sum_{i,j=1}^{n} \langle T_{ij} f_i, f_j\rangle \geq 0,$$
for each $f_1, \cdots, f_n \in L_p(X) \cap L_q(X)$.
- We know that $B(L_p(X)) \cong \big(L_q(G) \widehat{\otimes}^{\gamma} L_p(G)\big)^*$. For $T = [T_{ij}]$ belongs to $\mathbb{M}_n(\mathcal{B}(L_p(X))_*)$, $(\mathcal{B}(L_p(X))_*)$ is the predual space) we say that T is positive, if for each, $m \in \mathbb{N}$ and $\phi = [\phi_{ij}] \in \mathbb{M}_m(\mathcal{B}(L_p(X)))_+$, the natural matrix action $\langle \phi, T\rangle = [\langle \phi_{ij}, T_{kl}\rangle] \in \mathbb{M}_{n\times n}$ is positive.
- For $p' \in (1, \infty)$ and measure space Y subspaces $\mathcal{M} \subseteq \mathcal{B}(L_p(X))$ or $\mathcal{M} \subseteq \mathcal{B}(L_p(X))_*$ and $\mathcal{N} \subseteq \mathcal{B}(L_{p'}(Y))$ or $\mathcal{N} \subseteq \mathcal{B}(L_{p'}(Y))_*$, we say that a linear map $T : \mathcal{M} \longrightarrow \mathcal{N}$ is (p, p')-*completely positive*, if for all $n \in \mathbb{N}$, the n-th amplification $T^{(n)} : \mathbb{M}_n(\mathcal{M}) \longrightarrow \mathbb{M}_n(\mathcal{N})$, $T^{(n)}[x_{ij}] = [T(x_{ij})]$ of T is a positive map.

For simplicity, in the case where $p = p'$ we call such maps $p-$*completely positive.*

- Let $\mathcal{M}$ be as above, then since $\mathbb{M}_n(\mathcal{M}^*) \cong \mathcal{CB}_p(\mathcal{M}, \mathbb{M}_n)$, we say that $T \in \mathbb{M}_n(\mathcal{M}^*)$ is positive, when it is a p-complete positive map from $\mathcal{M}$ to $\mathbb{M}_n$.

Lemma 5.1. *Let the matrix $[v_{ij}]$ be a positive element in $\mathbb{M}_n(\mathcal{B}(L_p(X)))$, then for each $\alpha = (\alpha_1, \cdots, \alpha_n)$, scalar row vector, we have $\alpha[v_{ij}]\alpha^*$ is positive, where α^* is the conjugate transpose of α.*

Proof. We must show that $(\alpha_1, \cdots, \alpha_n)[v_{ij}](\overline{\alpha_1}, \cdots, \overline{\alpha_n})^t = \sum_{i,j=1}^n v_{ij}\alpha_i\overline{\alpha_j}$ is positive. Suppose that $f \in L_p(X) \cap L_q(X)$, then since $[v_{ij}]$ is in $M_n(\mathcal{B}(L_p(X)))_+$ we have

$$\begin{aligned}\langle \sum_{i,j=1}^n v_{ij}\alpha_i\overline{\alpha_j} f, f\rangle &= \langle \sum_{i,j=1}^n (v_{ij}(\alpha_i f)\overline{\alpha_j}, f\rangle \\ &= \langle \sum_{i,j=1}^n v_{ij}(\alpha_i f), \alpha_j f\rangle \geq 0.\end{aligned}$$

□

Theorem 5.2. *Let A be a commutative unital C^*-algebra, then a linear map $T : \mathcal{B}(L_p(X)) \to A$ is completely positive if and only if it is positive.*

Proof. Proof is similar to that of operator spaces, we give it here for convenience.

Without loss of generality we may assume that $A = C(\Omega)$ for some compact Hausdorff space Ω, by Gelfand-Naimark Theorem. For a positive element $v = [v_{ij}] \in \mathbb{M}_n(\mathcal{B}(L_p(X)))$ we must show that $[T(v_{ij})]$ is a positive element in $\mathbb{M}_n(C(\Omega))$. By the isometrically isomorphism of C^*-algebras $\mathbb{M}_n(C(\Omega)) \cong C(\Omega, \mathbb{M}_n)$ and for $\omega \in \Omega$ and $\alpha \in \mathbb{C}^n$ we have

$$\langle [T(v_{ij})(\omega)]\alpha, \alpha\rangle = \sum \overline{\alpha} T(v_{ij})(\omega)\alpha = T\left(\sum \overline{\alpha} v_{ij}\alpha\right)(\omega) \geq 0,$$

since $\sum \overline{\alpha} v_{ij}\alpha$ is positive by Lemma 5.1. □

6. Order Structure of Herz Algebras

Fourier algebra and its natural order structure induced by its operator space structure, which is the same as the order structure induced by positive definite functions as a proper cone, has been studied in [15], where they defined complete order amenability of the Fourier algebra and compared it with operator amenability. Also in [16] authors studied the order structure of Figà-Talamanca-Herz algebras.

With the notation of Section 4, we define the positive cone of $A_p(G)$ as the closure in $A_p(G)$, of the set of all function of the form $f = \Sigma_{i=1}^n \xi_i * \tilde{\xi}_i$, for (ξ_i) in $L_p(G) \cap L_q(G)$, and denote it by $A_p(G)_+$. Since with the norm of Fourier algebra the space $C_c(G) \cap P(G)$ is dense in $A(G) \cap P(G)$, the order structure defined above, in the case where $p = 2$, is the same as the order structure of $A(G)$, induced by the set $P(G) \cap A(G)$, as a positive cone.

The Banach space $PM_p(G)$ is a subset of $\mathcal{B}(L_p(G))$, and so has a natural order structure, induced by $\mathcal{B}(L_p(G))_+$. We say that $\varphi \in A_p(G)^*$ is positive if $\langle \varphi, f \rangle$ is a positive scalar, for all positive element $f \in A_p(G)$.

It is easy to see that $T \in PM_p(G)$ is positive as an element of $\mathcal{B}(L_p(G))$ if and only if it is positive as an element of $A_p(G)^*$. Also since $A_p(G)_+$ is closed, $u \in A_p(G)_+$ if and only if for each $T \in PM_p(G)_+$, $\langle T, u \rangle \geq 0$ [17].

By the definitions of order structure in Section 5, we can give the order structure of matrix spaces of Figà-Talamanca-Herz algebra.

Let $m, n \in \mathbb{N}$, it is easy to see that $T = [T_{ij}] \in \mathbb{M}_n(A_p(G))$ is positive if and only if for every p-completely positive and p-completely bounded linear map $\phi : PM_p(G) \longrightarrow \mathbb{M}_m$ the natural action $\langle \phi, T \rangle$ is a positive scalar matrix.

For $\mathcal{M} = A_p(G)$ or $PM_p(G))$ and $\mathcal{N} = A_{p'}(G)$ or $PM_{p'}(G)$ we say that a linear map $T : \mathcal{M} \longrightarrow \mathcal{N}$ is (p, p')-*completely positive* if for each positive integer n, the n-th amplification $T^n : \mathbb{M}_n(\mathcal{M}) \longrightarrow \mathbb{M}_n(\mathcal{N})$ of T is a positive map. If this is the case for $p = p'$, we say that T is p-*completely positive*.

In the following, the positive cone of Figà-Talamanca-Herz algebra is characterized.

Theorem 6.1. [16, Theorem 2.7] *Let G be a locally compact amenable group. Then $A_p(G)_+ = P(G) \cap A(G)$.*

Let μ be a measure, and E a Banach space. Daws defined a norm on the algebraic tensor product $L_p(\mu) \otimes E$ by embedding $L_p(\mu) \otimes E$ into $L_p(\mu, E)$ and he denoted the completion by $L_p(\mu) \otimes_p E$. It is easy to see that $L_p(\mu) \otimes E$ is dense in $L_p(\mu, E)$, and so $L_p(\mu) \otimes_p E = L_p(\mu, E)$ isometrically [5].

The Banach space $PM_p(G) \overline{\otimes} PM_p(H)$ is the w^*-closure of $PM_p(G) \otimes PM_p(H)$ in $B(L_p(G) \otimes_p L_p(G)) \cong B(L_p(G \times H))$. It has a natural order structure, coming from $B(L_p(G \times H))$.

Theorem 6.2. [5, Proposition 7.2] *Let G and H be locally compact groups, then $PM_p(G) \overline{\otimes} PM_p(H) \cong PM_p(G \times H)$.*

Let λ_p^G, λ_p^H, $\lambda_p^{G\times H}$ be the left regular representation on G, H and $G\times H$, respectively, Then clearly,

$$\lambda_p^G(s)\otimes\lambda_p^H(t)=\lambda_p^{G\times H}(s,t).$$

For $s\in G$ and $t\in H$, suppose $\lambda_p^G(s)\in PM_p(G)_+$, $\lambda_p^H(t)\in PM_p(H)_+$, and $f\otimes g$ belongs to $L_p(G\times H)\cap L_q(G\times H)$, then

$$\begin{aligned}\langle\lambda_p^{G\times H}(s,t)f\otimes g,f\otimes g\rangle&=\langle\lambda_p^G(s)(f)\otimes\lambda_p^H(t)(g),f\otimes g\rangle\\&=\langle\lambda_p^G(s)(f),f\rangle\langle\lambda_p^H(t)(g),g\rangle\end{aligned}$$

it means that $\lambda_p^{G\times H}(s,t)$ belongs to $PM_p(G\times H)_+$. So

$$Conv(PM_p(G)_+\otimes PM_p(H)_+)\subseteq PM_p(G\times H)+,$$

but since $PM_p(G\times H)_+=A_p(G\times H)^*{}_+$, is w^*-closed, and so

$$\overline{Conv(PM_p(G)_+\otimes PM_p(H)_+)}^{w^*}\subseteq PM_p(G\times H)_+$$

6.1. **p-operator tensor product of Herz algebras.** In this section we discuss if the p-isomtrically isomorphism $A_p(G)\widehat{\otimes}^pA_p(H)\cong A_P(G\times H)$, for amenable groups G and H, preserves the order structure.

Consider the following p-complete isometries defined in Proposition 4.1,

$$\Phi_0: A_p(G)\widehat{\otimes}^pA_p(H)\longrightarrow A_p(G\times H)$$

$$\Phi_1:(A_p(G)\widehat{\otimes}^pA_p(H))^*\longrightarrow\mathcal{CB}_p(A_p(G),PM_p(H)),$$

$$\varphi\to u,v\to\varphi(u\otimes v)\quad \varphi\in(A_p(G)\widehat{\otimes}^pA_p(H))^*,\ u,v\in A_p(G)$$

Here we define an order structure on p-operator projective tensor product of Herz algebras: we say that $u\in A_p(G)\widehat{\otimes}^pA_p(H)$ is positive if for all p-completely bounded and p-completely positive linear map $T:A_p(G)\longrightarrow PM_p(H)$, $\langle T,u\rangle$ is a positive scalar, where the pairing is defined as $\Phi_1(T)^{-1}(u)$. In this paper $u\in A_p(G)\widehat{\otimes}^pA_p(H)$ is considered as an element of $A_p(G\times H)$, identified with $\Phi_0(u)$.

The above definition can be given for two p-operator spaces which are subspaces of some space of the form $\mathcal{B}(L_p(X))$ or $\mathcal{B}(L_p(X))_*$, for each of preduals, if any. We use the generalized definition in the following theorem.

Theorem 6.3. *Let E and F be p-operator spaces which are subspaces of some space of the form $\mathcal{B}(L_p(X))$ or $\mathcal{B}(L_p(X))_*$, then the linear map $\varphi:E\widehat{\otimes}^pF\longrightarrow\mathbb{C}$ is positive if and only if its natural induced map $T_\varphi:E\longrightarrow F^*$ is completely positive.*

Proof. Let $\varphi : E\widehat{\otimes}^p F \to \mathbb{C}$ be a positive linear map. We must show that, for each $n \geq 1$, the n-th amplification ${T_\varphi}^{(n)} : \mathbb{M}_n(E) \to \mathbb{M}_n(F^*)$ is a positive map. But $\mathbb{M}_n(F^*) \cong \mathcal{CB}_p(F, \mathbb{M}_n)$, with its natural definition mentioned before.

Let $x = [x_{ij}]$ be in $\mathbb{M}_n(E)_+$, $m \in \mathbb{N}$ and $y = [y_{kl}]$ be in $\mathbb{M}_m(F)_+$. We need to check that $T_\varphi^{(n)}(x)(y) = [T_\varphi(x_{ij})][y_{kl}] = [\varphi(x_{ij} \otimes y_{kl})]$ is a positive scalar matrix. Let $\alpha = (c_1, \ldots, c_{mn})$ be a scalar row vector and put $u = \alpha \cdot (x \otimes y) \cdot \alpha^*$. Then u belongs to $(E\widehat{\otimes}^p F)^*_+$. Therefore $\langle {T_\varphi}^{(n)}(x)(y)\alpha, \alpha\rangle = \varphi(\alpha \cdot (x \otimes y) \cdot \alpha^*) = \varphi(u) \geq 0$.

On the other hand let $T_\varphi : E \longrightarrow F^*$ be a completely positive. We must show that he linear map $\varphi : E\widehat{\otimes}^p F \longrightarrow \mathbb{C}$ is positive. Let $u \in (E\widehat{\otimes}^p F)_+$ then by definition of positivity in $E\widehat{\otimes}^p F$, the natural action $\langle T, u\rangle$ is positive. It means that $\varphi(u) = \langle T, u\rangle$ is positive. Now the statement follows from Theorem 5.2. □

Corollary 6.4. *The linear map $T : A_p(G) \to PM_p(H)$ is completely positive if and only if the natural induced map $\varphi_T : A_p(G)\widehat{\otimes}^p A_p(H) \to \mathbb{C}$ is positive functional.*

The next proposition which is proved in [16, Proposition 3.4(ii)] will be used in the Theorem 6.6, so we mentioned it here.

Proposition 6.5. *Let G be an amenable locally compact group, $p \in (1, \infty)$ and $T : A_p(G) \longrightarrow PM_p(G)$ be a linear map. Let $T_1 = T|_{A(G)}$ be the restriction of T to $A(G)$.*

(i) *If T is p-completely positive, then T_1 is a completely positive linear map from $A(G)$ to $VN(G)$.*

(ii) *If $p \geq 2$ and T is a bounded linear map, then for each $n \in \mathbb{N}$, we have $\|{T_1}^{(n)}\| \leq \|T^{(n)}\|$.*

(iii) *If $1 < p \leq 2$ and T is bounded, then for each $n \in \mathbb{N}$ we have $\|T^{(n)}\| \leq \beta_n^2 \|{T_1}^{(n)}\|$.*

(iv) *If $1 < p \leq 2$ and T is a p-completely positive map, then there are infinitely many $n \in \mathbb{N}$ such that $\|T^{(n)}\| \leq \beta_n^2 n^2$.*

Theorem 6.6. *Let G and H be amenable groups and $2 \leq p < \infty$. Then the p-complete isomrphism between $A_p(G \times H)$ and $A_p(G)\widehat{\otimes}^p A_p(H)$ is an order isomorphism (ie. a positive map with positive inverse).*

Proof. Let G and H be amenable groups and let $u \in A_p(G \times H)_+$. By Theorem 6.1, u belongs to $P(G \times H)$, so u is positive as an element of $A(G \times H)$. We must show that u is positive as an element of $A_p(G)\widehat{\otimes}^p A_p(H)$.

For this, let $T : A_p(G) \longrightarrow PM_p(H)$ be an arbitrary p-completely bounded and p-completely positive linear map, by the definition of positivity in $A_p(G)\widehat{\otimes}^p A_p(H)$, we must show that $\langle T, u\rangle$ is a positive scalar.

In the first step we will show that the map $i : A(G) \longrightarrow A_p(G)$ is $(2, p)$-completely positive, for this we must show that the map

$$i^{(n)} : \mathbb{M}_n(A(G)) \longrightarrow \mathbb{M}_n(A_p(G))$$

is a positive map, with respect to the positive cones of $\mathbb{M}_n(A(G))$ and $\mathbb{M}_n(A_p(G))$. To this aim, for $[x_{ij}]$ in $\mathbb{M}_n(A(G))_+$, we show that $[x_{ij}]$ belongs to $\mathbb{M}_n(A_p(G))_+$. But the order structure on $\mathbb{M}_n(A_p(G))$ is defined by its dual space. So we prove that the natural action

$$\langle [S_{ij}], [x_{ij}]\rangle \in \mathbb{M}_{n^2},$$

gives a positive matrix, for $n \in \mathbb{N}$, $[S_{ij}]$ in $M_n(PM_p(G))_+$ and $[x_{ij}]$ in $\mathbb{M}_n(A(G))_+$.

For $n \in \mathbb{N}$ and $[x_{ij}] \in \mathbb{M}_n(A(G))$. Since $\mathbb{M}_n(A(G))_+ = \mathcal{CP}^\sigma(VN(G), \mathbb{M}_n)$, where $\mathcal{CP}^\sigma(VN(G), \mathbb{M}_n)$ is the set of all w^*-continuous completely positive linear maps from $VN(G)$ to $\mathbb{M}_n$, it follows that $[x_{ij}] \in \mathbb{M}_n(A(G))_+$ if and only if for each $[T_{ij}] \in \mathbb{M}_n[VN(G)]_+$, we have $[T_{kl}(x_{ij})] \in \mathbb{M}_{n^2+}$. Since G is amenable, the embedding $i : A(G) \longrightarrow A_p(G)$ is norm decreasing [9]. Therefore, each element $[S_{ij}]$ in $\mathbb{M}_n[PM_p(G)]$ can be considered as an element of $\mathbb{M}_n[VN(G)]$. Moreover, $[S_{ij}] \in \mathbb{M}_n[PM_p(G)]_+$ if and only if $\langle [S_{ij}][f], [f]\rangle \geq 0$, for all $[f] = (f_1, \cdots, f_n)$ with $f_1, \cdots, f_n \in C_c(G)$. Now it is clear that if $[S_{ij}] \in \mathbb{M}_n[PM_p(G)]_+$, then it also belongs to $\mathbb{M}_n[VN(G)]_+$, and so $\langle [S_{ij}], [x_{ij}]\rangle = [S_{kl}(x_{ij})] \in \mathbb{M}_{n^2+}$. Therefore $i : A(G) \to A_p(G)$ is a $(2, p)$-completely positive map.

This fact implies that $T|_{A(G)} : A(G) \to VN(G)$ is a completely positive linear map and by Proposition 6.5(ii) it is completely bounded. But $A(G \times H)_+$ is equal to the predual cone of all completely bounded and completely positive linear map from $A(G)$ to $VN(H)$ [15, Section 2]. It means that for $u \in A_p(G \times H)_+ = A(G \times H)_+$, $\langle T, u\rangle$ is positive and so u is positive as an element $A_p(G)\widehat{\otimes}^p A_p(H)$.

On the other hand let u be a positive element in $A_p(G)\widehat{\otimes}^p A_p(H)$, by the definition of positivity in p-operator tensor product which can be seen in Section 5, we have for each p-completely bounded and p-completely positive linear map $T : A_p(G) \to PM_p(H)$, the natural action of T on u is positive. But if $T : A_p(G) \to PM_p(H)$ is a p-completely bounded and p-completely positive linear map, then by the computation above $T_1 : A(G) \to VN(G)$ is a completely positive and by Proposition 6.5(ii), for $p \geq 2$ is completely bounded linear map.

So T_1 belongs to the positive cone of $\mathcal{CB}(A(G), VN(H))$. But the isometrical isomorphisms

$$\mathcal{CB}(A(G), VN(H)) \cong (A(G)\widehat{\otimes}^{op} A(H))^* \cong VN(G \times H),$$

are order isomorphism. It means that T which is an element of $PM_p(G\times H)$, is a positive element of $VN(G \times H)$ and so is a positive element of $PM_p(G \times H)$. But u is an element of $A_p(G \times H)$ and for each positive element T of $PM_p(G \times H)$, $T(u)$ is positive, now by the closedness of the positive cone of $A_p(G \times H)$, we have $u \in A_p(G \times H)_+$. $\square$

Remark 6.7. Let G and H be locally compact amenable groups, $p \in (1, \infty)$, and $T : A_p(G)\widehat{\otimes}^p A_p(H) \longrightarrow \mathbb{C}$ be a positive linear map, then by Theorem 6.6 (with the same nomination) $T : A_p(G \times H) \longrightarrow \mathbb{C}$ is a positive linear map. Since in this case $A_p(G \times H)_+ = A(G \times H) \cap P(G \times H)$, we have $T|_{A(G\times H)} : A(G \times H) \longrightarrow \mathbb{C}$ is a positive linear map.

In the following we show that the map $T|_{A(G\times H)}$ is continuous, with the norm of $A(G \times H)$.

For $u \in A(G\times H)$, there exists $u_1, \cdots, u_4$ with $u_i \in A(G\times H)\cap P(G\times H)$ and $\|u_i\|_{A(G)} \leq \|u\|_{A(G)}$. It means that if we denote the unit ball of $A(G \times H)$ by U then, $U \subseteq U_+ - U_+ + i(U_+ - U_+)$. Suppose $T|_{A(G\times H)}$ is not continuous on $U_+ - U_+ + i(U_+ - U_+)$, then so is on U_+. It means that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in U_+ such that $\|Tx_n\| \geq n^3$, for each $n \in \mathbb{N}$. Since $A(G\times H)_+$ is closed, $z := \sum x_n/n^2$ is in $A(G\times H)_+$. Hence $Tz \geq Tx_n/n^2 > 0$, for each $n \in \mathbb{N}$. Therefore $\|Tz\| \geq n$, for each $n \in \mathbb{N}$, which is impossible. So $T|_{A(G\times H)}$ is continuous with the norm of $A(G \times H)$ so T belongs to $A(G \times H)^*$.

This shows that the restriction of T belongs to

$$\mathcal{CB}(A(G), VN(H)) \cong VN(G)\overline{\otimes}VN(H) \cong VN(G \times H).$$

But in this case we can not conclude continuity of T. This is because of decomposition of any element of Fourier algebra into positives, which may fail for Herz algebras.

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