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I-PRIME IDEALS

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Abstract. In this paper, we introduce a new generalization of weakly prime ideals called I-prime. Suppose R is a commutative ring with identity and I a fixed ideal of R. A proper ideal P of R is I-prime if for $a, b \in R$ with $ab \in P - IP$ implies either $a \in P$ or $b \in P$. We give some characterizations of I-prime ideals and study some of its properties. Moreover, we give conditions under which I-prime ideals becomes prime or weakly prime and we construct the view of I-prime ideal in decomposite rings.

1. Introduction

Throughout this article, R will be a commutative ring with identity. Prime ideals play an essential role in ring theory. A prime ideal P of R is a proper ideal P with the property that for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$. There are several ways to generalize the notion of a prime ideal. We could either restrict or enlarge where a and/or b lie or restrict or enlarge where ab lies. In this article we will interested in a generalization obtained by restricting where ab lies. A proper ideal P of R is weakly prime if for $a, b \in R$ with $ab \in P - 0$, either $a \in P$ or $b \in P$. Weakly prime ideals were studied in some detail by Anderson and Smith (2003) in [1]. Thus any prime ideal is weakly prime. Bhatwadekar and Sharma (2005) in [2] recently defined a proper ideal I of an integral domain R to be almost prime if for $a, b \in R$ with $ab \in I - I^2$, then either $a \in I$ or $b \in I$. This definition

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can obviously be made for any commutative ring R. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is almost prime. Moreover, an ideal I of R is almost prime if and only if I/I^2 is a weakly prime ideal of R/I^2. Also almost prime ideals were generalized to n-almost prime as follows; a proper ideal I is called n-almost prime ideal if for any $a, b \in R$ with $ab \in I - I^n$, then either $a \in I$ or $b \in I$. With weakly prime ideals and almost prime ideals in mind, we make the following definition. Let R be a commutative ring and I be a fixed ideal of R. Then a proper ideal P of R is called I-prime ideal if for $a, b \in R$, $ab \in P - IP$, implies $a \in P$ or $b \in P$. So every weakly prime and n-almost prime ideal is I-prime where I taken to be zero or P^{n-1} respectively. If $I = R$, then every ideal is I-prime, so we can assume I to be proper ideal of R. For more details see [3].

Example 1.1. Consider the ring $\mathbf{Z}_{12}$ and take $P = I =< 4 >$. Then P is I-prime which is not prime nor weakly prime.

2. Main Results

We begin with the following lemma.

Lemma 2.1. *Let P be a proper ideal of a ring R. Then P is an I-prime ideal if and only if P/IP is weakly prime in R/IP.*

Proof. ($\Rightarrow$) Let P be an I-prime in R. Let $a, b \in R$ with $0 \neq (a + IP)(b + IP) = ab + IP \in P/IP$. Then $ab \in P - IP$ implies $a \in P$ or $b \in P$, hence $a + IP \in P/IP$ or $b + IP \in P/IP$. So P/IP is weakly prime ideal in R/IP.

($\Leftarrow$) Suppose that P/IP is weakly prime in R/IP and take $r, s \in R$ such that $rs \in P - IP$. Then $0 \neq rs + IP = (r + IP)(s + IP) \in P/IP$ so $r + IP \in P/IP$ or $s + IP \in P/IP$. Therefore $r \in P$ or $s \in P$. Thus P is an I-prime ideal in R. □

Theorem 2.2. *(1) Let $I \subseteq J$. If P is I-prime ideal of a ring R, then it is J-prime.*
(2) Let R be commutative ring and P an I-prime ideal that is not prime, then $P^2 \subseteq IP$. Thus, an I-prime ideal P with $P^2 \nsubseteq IP$ is prime.

Proof. (1) The proof come from the fact that if $I \subseteq J$, then $P - JP \subseteq P - IP$. (2) Suppose that $P^2 \nsubseteq IP$, we show that P is prime. Let $ab \in P$ for $a, b \in R$. If $ab \notin IP$, then P I-prime gives $a \in P$ or $b \in P$. So assume that $ab \in IP$. First, suppose that $aP \nsubseteq IP$; say $ax \notin IP$ for some $x \in P$. Then $a(x + b) \in P - IP$. So $a \in P$ or $x + b \in P$ and hence $a \in P$ or $b \in P$. So we can assume that $aP \subseteq IP$ in similar way we can assume that $bP \subseteq IP$. Since $P^2 \nsubseteq IP$, there exist $y, z \in P$ with $yz \notin IP$. Then $(a+y)(b+z) \in P - IP$. So P I-prime gives $a+y \in P$ or $b+z \in P$; Hence $a \in P$ or $b \in P$. Therefore P is prime. □

In the following we give a counter example on the converse of part (1) of Theorem 2.2.

Example 2.3. In the ring $\mathbf{Z}_{12}[x]$, put $I = 0$, $J =< 4 >$ and $P =< 4x >$. Then $P - IP =< 4x > -0$ and $P - JP =< 4x > - < 4 >< 4x >= \phi$. Hence P is J-prime but not I-prime.

Corollary 2.4. *Let P be an I-prime ideal of a ring R with $IP \subseteq P^3$. Then P is $\cap_{i=1}^{\infty} P^i$-prime.*

Proof. If P is prime, then P is $\cap_{i=1}^{\infty} P^i$-prime. Assume that P is not prime. By Theorem 2.2 $P^2 \subseteq IP \subseteq P^3$. Thus $IP = P^n$ for each $n \geq 2$. So $\cap_{i=1}^{\infty} P^i = P \cap P^2 = P^2$ and $(\cap_{i=1}^{\infty} P^i)P = P^2P = P^3 = IP$. Hence P I-prime implies P is $\cap_{i=1}^{\infty} P^i$-prime. □

Remark 2.5. Let P be I-prime ideal. Then $P \subseteq \sqrt{IP}$ or $\sqrt{IP} \subseteq P$. If $P \subsetneqq \sqrt{IP}$, then P is not prime since otherwise $IP \subseteq P$ implies $\sqrt{IP} \subseteq \sqrt{P} = P$. While if $\sqrt{IP} \subsetneqq P$, then P is prime.

Corollary 2.6. *Let P be I-prime ideal of a ring R which is not prime. Then $\sqrt{P} = \sqrt{IP}$*

Proof. By Theorem 2.2, $P^2 \subseteq IP$ and hence $\sqrt{P} = \sqrt{P^2} \subseteq \sqrt{IP}$. The other containment always holds.

□

Now we give a way to construct I-prime ideals P when $\cap_{i=1}^{\infty} P^i \subseteq IP \subseteq P^3$.

Remark 2.7. Assume that P is I-prime, but not prime. Then by Theorem 2.2, if $IP \subseteq P^2$, then $P^2 = IP$. In particular, if P is weakly prime (0-prime) but not prime, then $P^2 = 0$. Suppose that $IP \subseteq P^3$. Then $P^2 \subseteq IP \subseteq P^3$; So $P^2 = P^3$ and thus P^2 is idempotent.

Theorem 2.8. *(1) Let R, S be commutative rings and P 0-prime ideal of R. Then $P \times S$ is I-prime ideal of $R \times S$ for each ideal I of $R \times S$ with $\cap_{i=1}^{\infty}(P \times S)^i \subseteq I(P \times S) \subseteq P \times S$.*
(2) Let P be finitely generated proper ideal of commutative ring R. Assume P is I-prime where $IP \subseteq P^3$. Then either P is 0-prime or $P^2 \neq 0$ is idempotent and R decomposes as $T \times S$ where $S = P^2$ and $P = J \times S$ where J is 0-prime. Thus P is I-prime for $\cap_{i=1}^{\infty} P^i \subseteq IP \subseteq P$.

Proof. (1) Let R and S be commutative rings and P be weakly prime (0-prime) ideal of R. Then $P \times S$ need not be a 0-prime ideal of $R \times S$; In fact, $P \times S$ is 0-prime if and only if $P \times S$ (or equivalently P) is prime (see [1]). However, $P \times S$ is I-prime for each I with $\cap_{i=1}^{\infty}(P \times S)^i \subseteq I(P \times S)$. If P is prime , then $P \times S$ is prime ideal and thus is I-prime for all I. Assume that P is not prime. Then $P^2 = 0$ and $(P \times S)^2 = 0 \times S$. Hence $\cap_{i=1}^{\infty}(P \times S)^i = \cap_{i=1}^{\infty} P^i \times S = 0 \times S$. Thus $P \times S - \cap_{i=1}^{\infty}(P \times S)^i = P \times S - 0 \times S = (P - 0) \times S$. Since P is weakly prime, $P \times S$ is $\cap_{i=1}^{\infty}(P \times S)^i$-prime and as $\cap_{i=1}^{\infty}(P \times S)^i \subseteq I(P \times S)$, $P \times S$ is I-prime.

(2) If P is prime, then P is 0-prime. So we can assume that P is not prime. Then $P^2 \subseteq IP$ and hence $P^2 \subseteq IP \subseteq P^3$. So $P^2 = P^3$. Hence P^2 is idempotent. Since P^2 is finitely generated, $P^2 =< e >$ for some idempotent $e \in R$. Suppose $P^2 = 0$. Then $IP \subseteq P^3 = 0$. So $IP = 0$ and hence P is 0-prime. So assume $P^2 \neq 0$. Put $S = P^2 = Re$ and $T = R(1-e)$, so R decomposesas $T \times S$ where $S = P^2$. Let $J = P(1-e)$, so $P = J \times S$ where $J^2 = (P(1-e))^2 = P^2(1-e)^2 =< e > (1-e) = 0$. We show that J is 0-prime. Let $ab \in J^2 - 0$, so $(a,1)(b,1) = (ab,1) \in J \times S - (J \times S)^2 = J \times S - 0 \times S \subseteq P - IP$ since $IP \subseteq P^3$ implies $IP \subseteq P^3 = (J \times S)^3 = 0 \times S$. Hence $(a,1) \in P$ or $(b,1) \in P$ so $a \in J$ or $b \in J$. Therefore J is weakly prime. □

Corollary 2.9. *Let R be an indecomposable commutative ring and P a finitely generated I-prime ideal of R, where $IP \subseteq P^3$. Then P is weakly prime.*

Corollary 2.10. *Let R be a Noetherian integral domain. A proper ideal P of R is prime if and only if P is P^2-prime.*

Theorem 2.11. *Let a be a non-unit element in R.*
(1) Let $(0:a) \subseteq (a)$. Then (a) is I-prime for $I(a) \subseteq (a)^2$ if and only if (a) is prime.
Let (R,m) be quasi-local ring. Then
(2) (a) is I-prime for $I(a) \subseteq (a)^3$ if and only if (a) is 0-prime.
(3) (a) is m-prime if and only if a is irreducible.

Proof. (1) Suppose that (a) is I-prime and $bc \in (a)$. If $bc \notin I(a)$, then $b \in (a)$ or $c \in (a)$. So suppose that $bc \in I(a)$. Now $(b+a)c \in (a)$. If $(b+a)c \notin I(a)$, then $b+a \in (a)$ or $c \in (a)$ and hence $b \in (a)$ or $c \in (a)$. So assume that $(b+a)c \in I(a)$. Then $ac \in I(a)$ and hence $ac \in (a)^2$. So $ac = za^2$ and hence $c - za \in (0:a)$. Thus $c \in (0:a) + (a) = (a)$. The converse part is trivial since every prime ideal is I-prime.
(2) If (a) is 0-prime, then (a) is I-prime for each I with $I(a) \subseteq (a)^3$. Conversely, let (a) be I-prime for $I(a) \subseteq (a)^3$. Since a quasi local ring

has no nontrivial idempotents, (a) is 0-prime by Theorem 2.8 part(2).
(3) If a is irreducible means that $a = xy$ implies that $x \in (a)$ or $y \in (a)$ and (a) is m-prime means that $xy \in (a) - m(a)$ which implies that $x \in (a)$ or $y \in (a)$. But $xy \in (a) - m(a)$ if and only if $xy = za$ for some unit $z \in R$ if and only if $a = z^{-1}xy$ for some unit $z^{-1} \in R$. Thus (a) is m-prime if and only if $a = xy$ implies $x \in (a)$ or $y \in (a)$. □

We now give some characterizations of I-prime ideals.

Theorem 2.12. *Let P be a proper ideal of R. Then the following are equivalent:*
(1) P is I-prime.
(2) For $x \in R - P$, $(P : x) = P \cup (IP : x)$
(3) For $x \in R - P$, $(P : x) = P$ or $(P : x) = (IP : x)$
(4) For ideals J and K of R, $JK \subseteq P$ and $JK \nsubseteq IP$ imply $J \subseteq P$ or $K \subseteq P$.

Proof. (1) ⇒ (2) Suppose $r \in R - P$. Let $s \in (P : r)$, so $rs \in P$. If $rs \in P - IP$, then $s \in P$. If $rs \in IP$, then $s \in (IP : r)$, So $(P : r) \subseteq P \cup (IP : r)$. The other containment always holds.
(2) ⇒ (3) Note that if an ideal is a union of two ideals, then it is equal to one of them.
(3) ⇒ (4) Let J and K be two ideals of R with $JK \subseteq P$. Assume that $J \nsubseteq P$ and $K \nsubseteq P$. We claim that $JK \subseteq IP$. Suppose $r \in J$. First, Let $r \notin P$. Then $rK \subseteq P$ gives $K \subseteq (P : r)$. Now $K \nsubseteq P$, so $(P : r) = (IP : r)$. Thus $rK \subseteq IP$. Next, let $r \in J \cap P$. Choose $s \in J - P$. Then $r + s \in J - P$. So by the first case $sK \subseteq IP$ and so $(r+s)K \subseteq IP$. Let $t \in K$. Then $rt = (r+s)t - st \in IP$. So $rK \subseteq IP$. Hence $JK \subseteq IP$.
(4) ⇒ (1) Let $rs \in P - IP$. Then $(r)(s) \subseteq P$ but $(r)(s) \nsubseteq IP$. So $(r) \subseteq P$ or $(s) \subseteq P$; i-e. $r \in P$ or $s \in P$. □

Corollary 2.13. *Suppose P is I-prime ideal that is not prime. Then $P\sqrt{IP} \subseteq IP$.*

Proof. Let $r \in \sqrt{IP}$. If $r \in P$, then $rP \subseteq P^2 \subseteq IP$ by Theorem 2.2. So assume that $r \notin P$ by Theorem 2.12, $(P : r) = P$ or $(P : r) = (IP : r)$. As $P \subseteq (P : r)$, the last gives $rP \subseteq IP$. So assume that $(P : r) = P$. Let $r^n \in IP$, but $r^{n-1} \notin IP$. Then $r^n \in P$, so $r^{n-1} \in (P : r) = P$. Thus $r^{n-1} \in P - IP$, so $r \in P$, a contradiction. □

It is known that if S is a multiplicatively closed subset of a commutative ring R and P as a prime ideal of R with $P \cap S = \phi$, then $S^{-1}P$ is a prime ideal of $S^{-1}R$ and $S^{-1}P \cap R = P$. The first result extended to weakly prime ideals in [1, Proposition 13] and to almost prime ideals in

[2, Lemma 2.13]. For a fixed ideal I of R we prove that if P is I-prime with $P \cap S = \phi$, then $S^{-1}P$ is $S^{-1}I$-prime. Note that for an ideal J of R with $J \subseteq P$, $I(P/J) = (IP + J)/J$. If P is prime (respectively, weakly prime, n-almost prime), then so is P/J. We generalize this result to I-prime ideals in the following proposition.

Proposition 2.14. *Let R be a ring and I be an ideal of R. Let P be I-prime ideal of R. Then the following are true.*
(1) If J is an ideal of R with $J \subseteq P$, then P/J is I-prime ideal of R/J.
(2) Assume that S is multiplicatively closed subset of R with $P \cap S = \phi$. Then $S^{-1}P$ is a $S^{-1}I$-prime ideal of $S^{-1}R$. Moreover, If $S^{-1}P \neq S^{-1}(IP)$, then $S^{-1}P \cap R = P$.

Proof. (1) Let $x, y \in R$ with $\bar{x}\bar{y} \in P/J - I(P/J) = P/J - (IP + J)/J$. Thus $xy \in P - (IP + J)$. Hence $xy \in P - IP$, so $x \in P$ or $y \in P$. Therefore $\bar{x} \in P/J$ or $\bar{y} \in P/J$; so P/J is I-prime.
(2) Let $\frac{a}{s}\frac{b}{t} \in S^{-1}P - S^{-1}I.S^{-1}P \subseteq S^{-1}P - S^{-1}(IP) = S^{-1}(P - IP)$. So $abk \in P - IP$ for some $k \in S$. Thus P as I-prime gives $a \in P$ or $bk \in P$. So $a/s \in S^{-1}P$ or $b/t \in S^{-1}P$. Hence $S^{-1}P$ is $S^{-1}I$-prime. Let $a \in S^{-1}P \cap R$, so there exists $u \in S$ with $au \in P$. If $au \notin IP$, then $au \in P - IP$, so $a \in P$. If $au \in IP$, then $a \in S^{-1}(IP) \cap R$. So $S^{-1}P \cap R \subseteq P \cup (S^{-1}(IP) \cap R)$. Hence $S^{-1}P \cap R = P$ or $S^{-1}P \cap R = S^{-1}(IP) \cap R$. But the second case gives $S^{-1}P = S^{-1}(IP)$. □

Let R_1 and R_2 be two rings. It is known that the prime ideals of $R_1 \times R_2$ have the form $P \times R_2$ or $R_1 \times Q$, where P is a prime ideal of R_1 and Q is a prime ideal of R_2. We next, generalize this result to I-primes.

Theorem 2.15. *For $i = 1, 2$ let R_i be ring and I_i ideal of R_i. Let $I = I_1 \times I_2$. Then the I-prime ideals of $R_1 \times R_2$ have exactly one of the following three types:*
(1) $P_1 \times P_2$, where P_i is a proper ideal of R_i with $I_iP_i = P_i$.
(2) $P_1 \times R_2$ where P_1 is an I_1-prime of R_1 and $I_2R_2 = R_2$.
(3) $R_1 \times P_2$, where P_2 is an I_2-prime of R_2 and $I_1R_1 = R_1$.

Proof. We first prove that an ideal of $R_1 \times R_2$ having one of these three types is I-prime. The first type is clear since $P_1 \times P_2 - I(P_1 \times P_2) = P_1 \times P_2 - (I_1P_1 \times I_2P_2) = \phi$. Suppose that P_1 is I_1-prime and $I_2R_2 = R_2$. Let $(a, b)(x, y) \in P_1 \times R_2 - (I_1P_1 \times I_2R_2) = P_1 \times R_2 - (I_1P_1 \times R_2) = (P_1 - I_1P_1) \times R_2$. Then $ax \in P_1 - I_1P_1$ implies that $a \in P_1$ or $x \in P_1$, so $(a, b) \in P_1 \times R_2$ or $(x, y) \in P_1 \times R_2$. Hence $P_1 \times R_2$ is I-prime. Similarly we can prove the last case.

Next, let $P_1 \times P_2$ be I-prime and $ab \in P_1 - I_1P_1$. Then $(a, 0)(b, 0) = (ab, 0) \in P_1 \times P_2 - I(P_1 \times P_2)$, so $(a, 0) \in P_1 \times P_2$ or $(b, 0) \in P_1 \times P_2$, i-e, $a \in P_1$ or $b \in P_1$. Hence P_1 is I_1-prime. Likewise, P_2 is I_2-prime.

Assume that $P_1 \times P_2 \neq I_1P_1 \times I_2P_2$, say $P_1 \neq I_1P_1$. Let $x \in P_1 - I_1P_1$ and $y \in P_2$. Then $(x, 1)(1, y) = (x, y) \in P_1 \times P_2$. So $(x, 1) \in P_1 \times P_2$ or $(1, y) \in P_1 \times P_2$. Thus $P_2 = R_2$ or $P_1 = R_1$. Assume that $P_2 = R_2$. Then $P_1 \times R_2$ is I-prime, where P_1 is I_1-prime. □

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