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EXACT ANNIHILATING-IDEAL GRAPH OF COMMUTATIVE RINGS

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ABSTRACT. The rings considered in this article are commutative rings with identity $1 \neq 0$. The aim of this article is to define and study the exact annihilating-ideal graph of commutative rings. We discuss the interplay between the ring-theoretic properties of a ring and graph-theoretic properties of exact annihilating-ideal graph of the ring.

1. INTRODUCTION

The study of graphs associated with algebraic structures was initiated in 1878 when Arthur Cayley introduced Cayley graph of finite groups in [4]. The annihilating-ideal graph of a commutative ring was introduced by Behboodi and Rakeei in [2]. Several interesting properties of annihilating-ideal graph were studied in [2] and [3], which indicated the interplay between commutative rings and graph theory. The rings considered in this article are commutative ring with identity $1 \neq 0$. We recall that an ideal I of a commutative ring R is called an annihilating-ideal if $Ir = (0)$ for some $r \in R - \{0\}$. Recall from [2], that for a commutative ring R with identity, the annihilating-ideal graph of R denoted by $AG(R)$ is an undirected graph, whose vertex set is the set of nonzero annihilating-ideals $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$.

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We say that an ideal I of R is an exact annihilating-ideal if there exists an ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$. In this case we say that (I, J) is a pair of exact annihilating-ideals. Motivated by the study of exact zero-divisor graph of commutative rings studied in [5] and [6], we define exact annihilating-ideal graph $EAG(R)$ of a commutative ring R to be an undirected graph whose vertex set is the set of nonzero exact annihilating-ideals $EA(R)^*$ and two distinct vertices I and J are adjacent if and only if (I, J) is a pair of exact annihilating-ideals. It is clear that for any commutative ring R, $((0), R)$ is a pair of exact annihilating-ideals. Since the vertex set of $EAG(R)$ is $EA(R)^*$, in $EAG(R)$ we always have R to be an isolated vertex. So $EAG(R)$ will always be a disconnected graph. So for the shake of betterment of results, we restrict the vertex set of $EAG(R)$ to the set of proper exact annihilating-ideals of R denoted by $EA(R)^{\#}$. So $EA(R)^{\#} = EA(R) - \{(0), R\}$. We will try to study some fundamental results for exact annihilating-ideal graph for a commutative ring R with identity $1 \neq 0$ in this article.

We call a graph G is connected if there is a path between any two distinct vertices. The length of the shortest path between any two vertices x and y is denoted by $d(x, y)$, and $d(x, y) = \infty$ if no such path exists. The diameter of a graph G is denoted and defined as $diam(G) = sup\{d(x, y) \mid x \;\&\; y \textit{ are distinct vertices of } G\}$. A cycle in a graph is a path of length at least 3 through distinct vertices with same begin and end vertices. The girth of a graph G is denoted by $g(G)$ and is defined to be the length of the shortest cycle in G. $g(G) = \infty$ if G contains no cycle. A graph is said to be complete if each vertex in the graph is adjacent to every other vertex. A complete graph with n vertices is denoted by K_n. By a null graph, we mean the edgeless graph, while by an empty graph, we mean a graph with no vertices.

For a subset $A \subset R$, $A^* = A - \{0\}$. $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{F}_m$ indicates ring of integers, ring of integers modulo n and field with m elements, respectively. $Z(R)$ and $EZ(R)$ denotes the set of zero divisors and set of exact zero divisors of R, respectively. $U(R)$ is the set of units in R. By $A[X]$, we mean a polynomial ring in one variable X over A. We follow [1] for other standard notations. To avoid trivialities, we assume that R is not an integral domain unless otherwise stated.

2. Preliminaries and Examples

In this section, we give some definitions and discuss several examples of exact annihilating-ideal graphs.

Definition 2.1. Let R be a commutative ring with identity. An ideal I of R is said to be an exact annihilating-ideal if there exists an ideal J of R such that $Ann(I) = J$ and $Ann(J) = I$.

In this case we say that (I, J) is a pair of exact annihilating-ideals. The set of all proper exact annihilating-ideals is denoted by $EA(R)^{\#}$. We note that an ideal I of a commutative ring R is said to be an annihilating-ideal if $Ir = (0)$, for some $r \in R - \{0\}$.

Definition 2.2. The exact annihilating-ideal graph EAG(R) of a commutative ring R is a simple graph with the vertex set to be $EA(R)^{\#}$ and two vertices I and J are adjacent if and only if (I, J) is a pair of exact annihilating-ideals, i.e. $Ann(I) = J$ and $Ann(J) = I$.

Example 2.3. Let $R = \mathbb{Z}_2[X]/(X^3)$. We say $Im(X) = \overline{x}$. The only nonzero proper ideals of R are $(\overline{x})$ and $(\overline{x^2})$. We can observe that $Ann(\overline{x}) = (\overline{x^2})$ and $Ann(\overline{x^2}) = (\overline{x})$. Thus $EAG(R)$ of R is as shown in figure 1.

Example 2.4. Let $R = \mathbb{Z}_2[X]/(X^3 + X)$. We say $Im(X) = \overline{x}$. The only nonzero proper ideals of R are $(\overline{x})$, $(\overline{x+1})$, $(\overline{x^2+1})$ & $(\overline{x^2+x})$. We can observe that $Ann(\overline{x}) = (\overline{x^2+1})$ and $Ann(\overline{x^2+1}) = (\overline{x})$. Also $Ann(\overline{x+1}) = (\overline{x^2+x})$ and $Ann(\overline{x^2+x}) = (\overline{x+1})$. Thus $EAG(R)$ of R is as shown in figure 1.

Example 2.5. Let $R = \mathbb{Z}_2[X]/(X^3 + 1)$. We say $Im(X) = \overline{x}$. The only nonzero proper ideals of R are $(\overline{x+1})$ and $(\overline{x^2+x+1})$. Also $Ann(\overline{x+1}) = (\overline{x^2+x+1})$ and $Ann(\overline{x^2+x+1}) = (\overline{x+1})$. Thus $EAG(R)$ is a complete graph K_2 as shown in figure 1.

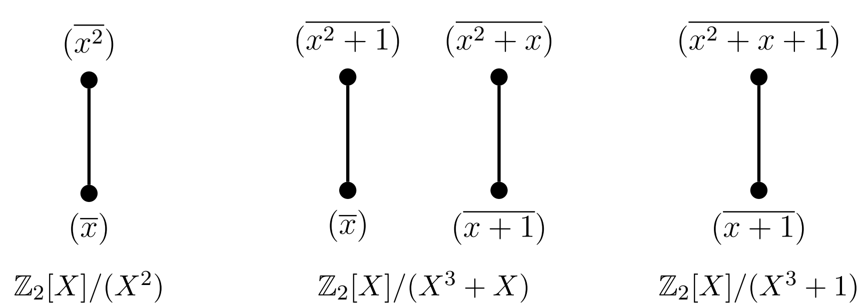

FIGURE 1

3. Properties of EAG(R)

Theorem 3.1. *For a commutative ring R, if $EAG(R)$ is connected, then $diam(EAG(R)) \leq 2$.*

Proof. Let R be a commutative ring such that the exact annihilating-ideal graph $EAG(R)$ of R is connected. Suppose that the length of the shortest path between any two vertices is bigger than two. Thus let us take the length of shortest path between two vertices I and J to be three, say $I - I_1 - I_2 - J$. By the definition of $EAG(R)$, $Ann(I) = I_1$ and $Ann(I_1) = I$. Similarly, $Ann(I_1) = I_2$ and $Ann(I_2) = I_1$; $Ann(I_2) = J$ and $Ann(J) = I_2$. But then $Ann(I) = I_1 = Ann(I_2) = J$ and $Ann(J) = I_2 = Ann(I_1) = I$. Thus $Ann(I) = J$ and $Ann(J) = I$. Hence (I, J) is a pair of exact annihilating-ideals and hence I and J are adjacent in $EAG(R)$. So the shortest length of any path between any two vertices can not exceed two. Since $EAG(R)$ is connected, $diam(EAG(R)) \leq 2$. □

Theorem 3.2. *If $EAG(R)$ contains a cycle, then $g(EAG(R)) \leq 4$.*

Proof. From above theorem, we observe that if there is a path of length three between any two vertices I and J, then $I - J$ are adjacent in $EAG(R)$. Therefore $g(EAG(R)) \leq 4$. □

Theorem 3.3. *Let $R = D_1 \times D_2$, where D_1 and D_2 are integral domains. Then $EAG(R)$ is complete graph K_2.*

Proof. Let $R = D_1 \times D_2$, where D_1 and D_2 are integral domains. Thus the vertex set of $EAG(R)$ is $\{(u, 0)R, (0, v)R | u \in U(D_1), v \in U(D_2)\}$. We note that ideals $I = (x, 0)R$ such that $x \in D_1 - U(D_1)$ and $J = (0, y)R$ such that $y \in D_2 - U(D_2)$ are not vertices in $EAG(R)$. For instance, let $I = (x, 0)R$, $x \in D_1 - U(D_1)$, then $Ann((x, 0)R) = (0, v)R$, $v \in U(D_2)$ and $Ann((0, v)R) = (u, 0)R$, $u \in U(D_1)$. But $(x, 0)R \neq (u, 0)R$. Therefore $(x, 0)R$ is not a vertex in $EAG(R)$. Similarly we can show that $(0, y)R$ is not a vertex in $EAG(R)$. Also $(u, 0)R$ and $(0, v)R$ are adjacent in $EAG(R)$. Since these are the only vertices of $EAG(R)$, $EAG(R)$ is connected and a complete graph K_2. □

Corollary 3.4. *If $R = \mathbb{Z}_{pq}$, where p and q are distinct primes. Then $EAG(R) = K_2$.*

Proof. Let $R = \mathbb{Z}_{pq}$, where p and q are distinct primes, then $R \simeq \mathbb{Z}_p \times \mathbb{Z}_q$. But $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are fields. Thus by above theorem, $EAG(R) = K_2$. □

Remark 3.5. ([2], Theorem 1.4) says that annihilating-ideal graph of a commutative ring R is finite if and only if R has only finitely many ideals. The fact is not true for exact annihilating-ideal graphs. For instance, let $R = \mathbb{Z}\times\mathbb{Z}$. Then by above theorem, $EAG(R)$ is a complete graph K_2. But R has an infinite number of proper ideals.

Remark 3.6. ([2], Theorem 1.3) says that if R is an Artinian ring, then every nonzero proper ideal is a vertex of $AG(R)$. The result fails to hold for $EAG(R)$. For instance, let $R = \mathbb{Z}_2[X,Y]/(X,Y)^2$. Then $Ann(\overline{x}) = (\overline{x,y})$. But $Ann(\overline{x,y}) = (\overline{x,y}) \neq (\overline{x})$. Thus $(\overline{x})$ is not a vertex of $EAG(R)$, even if it is a proper ideal of ring R.

Remark 3.7. ([2], Theorem 2.1) shows that $AG(R)$ is always connected for a commutative ring R. Example 2.2 shows that the fact is not true for $EAG(R)$.

Remark 3.8. We can observe that $EAG(R)$ is a subgraph of $AG(R)$. But $EAG(R)$ is not same as $AG(R)$ which can be observed by example 2.2 as we know that $AG(R)$ is always connected graph while $EAG(R)$ is not connected graph in example 2.2.

Theorem 3.9. *Let $R = \mathbb{Z}_{p^n}$, where p is a prime and $n \geq 2$ is a positive integer. Then $EAG(R)$ is disjoint union of $[n/2]$ number of complete graphs, where $[n/2]$ is integer part of $n/2$.*

Proof. Let $R = \mathbb{Z}_{p^n}$, where p is a prime and $n \geq 2$ is a natural number. Thus only proper ideals of R are $(\overline{p})$, $(\overline{p^2})$, $\cdots$, $(\overline{p^{n-1}})$. Also $Ann(\overline{p}) = (\overline{p^{n-1}})$ and $Ann(\overline{p^{n-1}}) = (\overline{p})$. $Ann(\overline{p^2}) = (\overline{p^{n-2}})$ and $Ann(\overline{p^{n-2}}) = (\overline{p^2})$. This process (say process *) will continue up to $n/2$ or $(n-1)/2$ steps, depending upon whether n is even or odd.
Case I: n is even.
If n is an even integer, then the process * stops after $n/2 = [n/2]$ steps, where $[n/2]$ denotes the integer part of $n/2$. Also each $(\overline{p^i})$ is adjacent with $(\overline{p^{n-i}})$ only, which gives a either a complete graph K_2 if $i \neq n/2$ or a complete graph K_1 if $i = n/2$. Thus in this case $EAG(R)$ is disjoint union of $[n/2]$ number of complete graphs.
Case II: n is odd integer.
If n is an odd integer, then the process * stops after $(n-1)/2 = [n/2]$ steps. Also each (p^i) is adjacent with $(\overline{p^{n-i}})$ only, which gives a complete graph K_2. Thus in this case $EAG(R)$ is disjoint union of $[n/2]$ number of complete graphs. □

Corollary 3.10. *If $R = \mathbb{Z}_{p^2}$, where p is a prime, then $EAG(R)$ is a complete graph K_2.*

Proof. This can be seen by taking $n = 2$ in above theorem. □

Remark 3.11. From the proof of above theorem, we can observe that for $R = \mathbb{Z}_{p^n}$, where p is a prime and $n \geq 2$ is a positive integer, $EAG(R) = K_1 \cup \bigcup_{i=1}^{[n/2]-1} K_2$, if n is an even integer and $EAG(R) = \bigcup_{i=1}^{[n/2]} K_2$, if n is an odd integer.

Theorem 3.12. *If $EAG(R)$ is a star graph, then $EAG(R) = K_2$.*

Proof. Let $EAG(R)$ be a star graph. Therefore there is a vertex I of $EAG(R)$ which is adjacent to every vertex of the graph, say $(I_\alpha)_{\alpha\in\Lambda}$. Thus by the definition of $EAG(R)$, $Ann(I) = (I_\alpha)$ and $Ann(I_\alpha) = I$, for each $\alpha \in \Lambda$. Hence $\Lambda = \{\alpha\}$, which gives $EAG(R) = K_2$. □

Remark 3.13. Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where each $\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3$ are fields. We will discuss about the the structure of $EAG(R)$. Let $\alpha_1, \alpha_2, \alpha_3$ be arbitrary elements from $\mathbb{F}_1^*, \mathbb{F}_2^*, \mathbb{F}_3^*$, respectively. Then $EAG(R)$ is a disconnected graph as in figure 2.

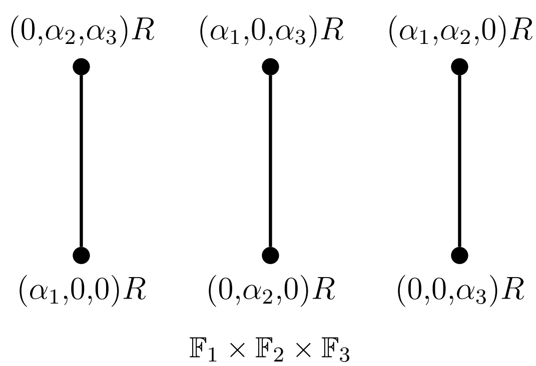

FIGURE 2

Remark 3.14. From above remark we can observe that if $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, then $EAG(R)$ is disconnected graph. Thus for $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, if $EAG(R)$ is connected, then $n = 2$.

We generalize the fact of remark 3.13 in next theorem and discuss the structure of $EAG(R)$ if $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$.

Theorem 3.15. *Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$, $(1 \leq i \leq n)$ is a field. Then the exact annihilating-ideal graph $EAG(R)$ of R is a disjoint union of $2^{n-1} - 1$ number of complete graphs, if n is an odd integer and is a disjoint union of $2^{n-1} - 1 + \binom{n}{2}/2$ number of complete graphs if n is an even integer.*

Proof. Let $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$, $(1 \leq i \leq n)$ is a field. Then we can observe that for each $1 \leq i \leq n$, the vertex of the form $(0, 0, \cdots, 0, \alpha_i, 0, \cdots, 0)R$ with $\alpha_i(\neq 0) \in \mathbb{F}_i$ is adjacent with $(\alpha_1, \cdots, \alpha_{i-1}, 0, \alpha_{i+1}, \cdots, \alpha_n)R$, which gives $\binom{n}{1}$ number of disjoint complete components of $EAG(R)$. Similarly, the vertices with exactly two nonzero $\alpha_i's$ gives $\binom{n}{2}$ number of disjoint complete components of $EAG(R)$. If n is odd, the total number of components of $EAG(R)$ is $\sum_{i=1}^{(n-1)/2} \binom{n}{i} = 2^{n-1} - 1$. Thus $EAG(R)$ is disjoint union of $2^{n-1} - 1$ number of complete graphs. Similarly, if n is even, then the number of components are $\sum_{i=1}^{\frac{n}{2}} \binom{n}{i} = 2^{n-1} - 1 + \binom{n}{2}/2$. Thus in this case $EAG(R)$ is disjoint union of $2^{n-1} - 1 + \binom{n}{2}/2$ number of complete graphs. □

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