**Journal of Algebra and Related Topics**
**Vol. 5, No 1, (2017), pp 1-10**

# GOOD STRONGLY REGULAR RELATIONS ON WEAK $\Gamma$-(SEMI)HYPERGROUPS

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ABSTRACT. In this paper first we introduce the notion of weak $\Gamma$-(semi)hypergroups, next some classes of equivalence relations which are called good regular and strongly good regular relations are defined. Then we investigate some properties of this kind of relations on weak $\Gamma$-(semi)hypergroups.

## 1. Introduction

The algebraic hyperstructure notion was introduced in 1934 by a French mathematician F. Marty [10], at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non-commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Around the 40's, the general aspects of the theory, the connections with groups and various applications in geometry were studied. The theory knew an important progress starting with the 70's, when its research area enlarged. A book on hyperstructures [4] points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Many authors studied different aspects of semihypergroups, for instance, P. Bonansinga and P. Corsini [1, 3], Davvaz [5], Freni [6], Leoreanu [9]

MSC(2010): 20N20
Keywords: (semi)hypergroup, weak $\Gamma$-(semi)hypergroup, good regular relation.
Received: 6 January 2017, Accepted: 1 March 2017.
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and Onipchuk [12]. In 1986, Sen and Saha [17] defined the notion of a $\Gamma$-semigroup as a generalization of a semigroup. Many classical notions of semigroups have been extended to $\Gamma$- semigroups and a lot of results on $\Gamma$-semigroups are published by a lot of mathematicians, for instance [8, 13, 14, 15, 16, 17, 18, 19].

In this research first we introduce the notion of weak $\Gamma$-(semi) hypergroup as a generalization of $\Gamma$-(semi)hypergroups. Using the class of weak $\Gamma$-(semi)hypergroups we introduce the associated semihypergroup $(S, \circ_\Gamma)$ of a weak $\Gamma$-(semi)hypergroup $(S, \Gamma)$. We introduce good regular and strongly good regular relations in weak $\Gamma$-(semi)hypergroups and we obtain some properties of these kind of relations in weak $\Gamma$-(semi)hypergroups. Finally, we generalize the notion of left and right $\Gamma$-hyperideal. For a weak $\Gamma$-semihypergroup $S$ we introduce a weak left(right) $\Gamma$-hyperideal of $S$.

## 2. Preliminaries

We recall here some basic notions of hypergroup theory and we fix the notations used in this note. We encourage the readers to the following fundamental book Corsini [2].

Let $H$ be a non-empty set and $P^*(H)$ denote the set of all non-empty subsets of $H$. Let $\circ$ be a *hyperoperation* (or *join operation*) on $H$, that is, a function from the cartesian product $H \times H$ into $P^*(H)$. The image of the pair $(a, b) \in H \times H$ under the hyperoperation $\circ$ in $P^*(H)$ is denoted by $a \circ b$. The join operation can be extended in a natural way to subsets of $H$ as follows: for non-empty subsets $A, B$ of $H$, define $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$. The notation $a \circ A$ is used for $\{a\} \circ A$ and $A \circ a$ for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its element $a$. The hyperstructure $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ for every $a, b, c \in H$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A semihypergroup $(H, \circ)$ is called *complete* if, for every natural numbers $n, m \geq 2$ and all tuples $(x_1, x_2, \ldots, x_n) \in H^n$ and $(y_1, y_2, \ldots, y_m) \in H^m$, we have the following implication:

$$\prod_{i=1}^{n} x_i \cap \prod_{j=1}^{m} y_j \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j,$$

where $\prod\limits_{i=1}^{n} x_i = x_1 \circ x_2 \circ \cdots \circ x_n$ and $\prod\limits_{j=1}^{m} y_j = y_1 \circ y_2 \circ \cdots \circ y_m$. In practice, the next characterization is more useful.

**Theorem 2.1.** *([2]) A (semi)hypergroup $(H, \circ)$ is complete if it can be written as the union $H = \bigcup_{s \in S} A_s$ of its subsets, where $S$ and $A_s$ satisfy the conditions:*

1. *$(S, \cdot)$ is a (semi)group;*
2. *for every $(s, t) \in S^2$, where $s \neq t$, we have $A_s \bigcap A_t = \emptyset$;*
3. *if $(a, b) \in A_s \times A_t$, then $a \circ b = A_{s \cdot t}$.*

A semihypergroup $(H, \circ)$ is called a *hypergroup* if the reproduction law holds: $a \circ H = H \circ a = H$, for every $a \in H$.

## 3. Weak Γ-semihypergroups

In this section, we introduce a generalization of the notion of Γ-(semi) hypergroup by the notion of weak Γ-(semi)hypergroup. Using the class of weak Γ-(semi)hypergroups we introduce the associated semihypergroup $(S, \circ_\Gamma)$ of a weak Γ-(semi)hypergroup $(S, \Gamma)$.

**Definition 3.1.** (see [11]) Let $S$ and $\Gamma$ be two non-empty sets. S is called a Γ-semihypergroup if for every $\gamma \in \Gamma$ is a hyperoperation on $S$, i.e. $x\gamma y \subseteq S$, for every $x, y \in S$, and for every $(\alpha, \beta) \in \Gamma^2$ and $(x, y, z) \in S^3$, we have $(x\alpha y)\beta z = x\alpha(y\beta z)$.

The Γ-semihypergroup $S_\Gamma$ is called *non-cover* if for every $x, y \in S$ and $\gamma \in \Gamma$ we have $x\gamma y \neq S$.
From now on we show the (semi)hypergroup $(S, \gamma)$ with $S_\gamma$. If every $\gamma \in \Gamma$ is an operation, then S is a Γ-semigroup. If $S_\gamma$ is a hypergroup for every $\gamma \in \Gamma$, then $S$ is called a Γ-hypergroup.

**Definition 3.2.** Let $S$ and $\Gamma$ be two non-empty sets. S is called a weak Γ-semihypergroup if for every $\gamma \in \Gamma$ is a hyperoperation on $S$, i.e. $x\gamma y \subseteq S$, for every $x, y \in S$, and for every $(\alpha, \beta) \in \Gamma^2$ and $(x, y, z) \in S^3$, we have

$$[(x\alpha y)\beta z] \cup [(x\beta y)\alpha z] = [x\alpha(y\beta z)] \cup [x\beta(y\alpha z)].$$

Moreover, S is called a weak Γ-hypergroup if for every $\gamma \in \Gamma$, we have $a\gamma S_\gamma = S_\gamma \gamma a = S_\gamma$, for every $a \in S_\gamma$.

Let $(S, \Gamma)$ be a weak Γ-hypergroup. We define the hyperoperation $\circ_\Gamma$ on $S$ as follows:

$$x \circ_\Gamma y = \bigcup_{\gamma \in \Gamma} x\gamma y$$

for every $(x, y) \in S^2$.

**Example 3.3.** *Let $n, t \in \mathbb{N}$, $2 \leq n$ and $(Z_n, +)$ be the cyclic group of order $n$. Consider the operation $\bar{a} +_t \bar{b} = \overline{a + b + t}$, for every $\bar{a}, \bar{b} \in Z_n$. Now let $\Gamma = \{+_k \mid k \in K \subseteq \mathbb{N}\}$ then $(Z_n, \Gamma)$ is a weak Γ-group.*

**Proposition 3.4.** *If $(S,\Gamma)$ be a weak $\Gamma$-semihypergroup then $(S,\circ_\Gamma)$ is a semihypergroup.*

*Proof.* Let $(x,y,z)\in S^3$ and $a\in(x\circ_\Gamma y)\circ_\Gamma z$. Then there exists $(\alpha,\beta)\in\Gamma^2$ such that $a\in(x\alpha y)\beta z$ so $a\in x\alpha(y\beta z)$ or $a\in x\beta(y\alpha z)$. Hence $a\in x\circ_\Gamma(y\circ_\Gamma z)$. Therefore $(x\circ_\Gamma y)\circ_\Gamma z\subseteq x\circ_\Gamma(y\circ_\Gamma z)$. Similarly we have $x\circ_\Gamma(y\circ_\Gamma z)\subseteq(x\circ_\Gamma y)\circ_\Gamma z$. Thus $(S,\circ_\Gamma)$ is a semihypergroup. □

From now on we call $(S,\circ_\Gamma)$ is the associated (semi)hypergroup of $(S,\Gamma)$.

**Example 3.5.** *Suppose that in Example 3.3, $n=3$ and $K=\{1,3\}$. Then the associated hypergroup is as follow:*

| $\circ_\Gamma$ | $\bar 0$ | $\bar 1$ | $\bar 2$ |
|---|---|---|---|
| $\bar 0$ | $\bar 0,\bar 1$ | $\bar 1,\bar 2$ | $\bar 0,\bar 2$ |
| $\bar 1$ | $\bar 1,\bar 2$ | $\bar 0,\bar 2$ | $\bar 0,\bar 1$ |
| $\bar 2$ | $\bar 0,\bar 2$ | $\bar 0,\bar 1$ | $\bar 1,\bar 2$ |

## 4. Good relations on weak $\Gamma$-(semi)hypergroups

In this section we introduce good regular and strongly good regular relations in weak $\Gamma$-(semi)hypergroups. Also we obtain some properties of these kind of relations in weak $\Gamma$-(semi)hypergroups.

**Definition 4.1.** Let $S$ be weak $\Gamma$ -(semi)hypergroup and $R$ be an equivalence relation on $S$, we set

$$A\overline{R}B\Leftrightarrow aRb,(\forall a\in A,\exists b\in B)\&(\forall b\in B,\exists a\in A),$$

$$A\overline{\overline{R}}B\Leftrightarrow aRb,\forall a\in A,\forall b\in B,$$

for every pairs $(A,B)$ of non-empty subsets of $S^2$.

(1) The relation $R$ is called *regular on the left* (*on the right, respectively*) if $xRy\Rightarrow a\gamma x\overline{R}a\gamma y$ ( $xRy\Rightarrow x\gamma a\overline{R}y\gamma a$, respectively), for every $(x,y,a)\in S^3$ and $\gamma\in\Gamma$. Moreover, $R$ is called *regular* if it is regular on the right and on the left.

(2) The relation $R$ is called *strongly regular on the left* (*on the right, respectively*) if $xRy\Rightarrow a\gamma x\overline{\overline{R}}a\gamma y$ ( $xRy\Rightarrow x\gamma a\overline{\overline{R}}y\gamma a$, respectively), for every $(x,y,a)\in S^3$ and $\gamma\in\Gamma$. Moreover, $R$ is called *strongly regular* if it is strongly regular on the right and on the left.

**Definition 4.2.** A regular relation $R$ is called good regular ( strongly regular relation $R$ is called good strongly regular, resp.) if it satisfies

(1) $(x\alpha y)\beta z\overline{R}(x\beta y)\alpha z,((x\alpha y)\beta z\overline{\overline{R}}(x\beta y)\alpha z);$

(2) $x\alpha(y\beta z)\overline{R}x\beta(y\alpha z), (x\alpha(y\beta z) \overset{=}{R} x\beta(y\alpha z))$;
for every $(x, y, z) \in S^3, (\alpha, \beta) \in \Gamma^2$.

**Definition 4.3.** Let $S$ be weak $\Gamma$ -(semi)hypergroup and $R$ be an equivalence relation on $S$. Let $R(a)$ be the equivalence class of $a$ with respect to $R$ and let $\frac{S}{R} = \{R(a) \mid a \in S\}$. For every $\gamma \in \Gamma$ we define $\frac{\gamma}{R}$ on $\frac{S}{R}$ as follows:

$$R(a)\frac{\gamma}{R}R(b) = \{R(x) \mid x \in a\gamma b\}.$$

Now we set $\frac{\Gamma}{R} = \{\frac{\gamma}{R} \mid \gamma \in \Gamma\}$.

**Theorem 4.4.** *Let $S$ be a weak $\Gamma$ -(semi)hypergroup and $R$ be an equivalence relation on $S$. The following conditions are equivalent :*

(1) *$R$ is regular relation,*
(2) *$\frac{S}{R}$ is a weak $\frac{\Gamma}{R}$ -(semi)hypergroup.*

*Proof.* ($1 \Rightarrow 2$). We need to prove that for every $\gamma \in \Gamma$ the hyperoperation $\frac{\gamma}{R}$ is well defined. Let $\gamma \in \Gamma$ and $x, y \in S$. If $x^{'} \in R(x)$ and $y^{'} \in R(y)$, we prove that $R(x)\frac{\gamma}{R}R(y) = R(x^{'})\frac{\gamma}{R}R(y^{'})$. We have $x^{'}Rx$, $y^{'}Ry$, and $R$ is regular so $x\gamma y\overline{R}x\gamma y^{'}$ and $x\gamma y^{'}\overline{R}x^{'}\gamma y^{'}$. Hence, for every $w \in x\gamma y$, there exists $u \in x\gamma y^{'}$ such that $wRu$ and for every $v \in x\gamma y^{'}$, there exists $z \in x^{'}\gamma y^{'}$ such that $zRv$. Thus, for every $w \in x\gamma y$ there exists $z \in x^{'}\gamma y^{'}$ such that $wRz$. Therefore, for every $R(w) \in R(x)\frac{\gamma}{R}R(y)$ there exists $z \in x'\gamma y'$ such that $R(w) = R(z) \in R(x')\frac{\gamma}{R}R(y')$, that is

$$R(x)\frac{\gamma}{R}R(y) \subseteq R(x')\frac{\gamma}{R}R(y').$$

Also we have

$$R(x')\frac{\gamma}{R}R(y') \subseteq R(x)\frac{\gamma}{R}R(y).$$

Now we want to prove that, for every $\frac{\alpha}{R}, \frac{\beta}{R} \in \frac{\Gamma}{R}$ and $R(x), R(y), R(z) \in \frac{S}{R}$,

$$(R(x)\frac{\alpha}{R}[R(y)\frac{\beta}{R}R(z)]) \cup (R(x)\frac{\beta}{R}[R(y)\frac{\alpha}{R}R(z)]) =$$

$$([R(x)\frac{\alpha}{R}R(y)]\frac{\beta}{R}R(z))\cup([R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z)).$$

Let $R(w)\in(R(x)\frac{\alpha}{R}[R(y)\frac{\beta}{R}R(z)])\cup(R(x)\frac{\beta}{R}[R(y)\frac{\alpha}{R}R(z)])$ then $R(w)\in(R(x)\frac{\alpha}{R}[R(y)\frac{\beta}{R}R(z)])$ or $R(w)\in(R(x)\frac{\beta}{R}[R(y)\frac{\alpha}{R}R(z)])$. Without loss of generality suppose that $R(w)\in(R(x)\frac{\alpha}{R}[R(y)\frac{\beta}{R}R(z)])$. Then there exists $R(v)\in R(y)\frac{\beta}{R}R(z)$ such that $R(w)\in R(x)\frac{\alpha}{R}R(v)$. Hence we can suppose that $v\in y\beta z$ and $w\in x\alpha(y\beta z)\cup x\beta(y\alpha z)=(x\alpha y)\beta z\cup(x\beta y)\alpha z$ then $w\in(x\alpha y)\beta z$ or $w\in(x\beta y)\alpha z$. Hence $R(w)\in[R(x)\frac{\alpha}{R}R(y)]\frac{\beta}{R}R(z))$ or $R(w)\in[R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z))$. Hence

$$(R(x)\frac{\alpha}{R}[R(y)\frac{\beta}{R}R(z)])\cup(R(x)\frac{\beta}{R}[R(y)\frac{\alpha}{R}R(z)])\subseteq$$

$$([R(x)\frac{\alpha}{R}R(y)]\frac{\beta}{R}R(z))\cup([R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z)).$$

In a similar way we obtain

$$([R(x)\frac{\alpha}{R}R(y)]\frac{\beta}{R}R(z))\cup([R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z))\subseteq$$

$$(R(x)\frac{\alpha}{R}[R(y)\frac{\beta}{R}R(z)])\cup(R(x)\frac{\beta}{R}[R(y)\frac{\alpha}{R}R(z)]).$$

$(2\Rightarrow 1)$. Let $xRx^{'}$ then $R(x)=R(x^{'})$ and so for every $y'\in S$, we have $R(x)\frac{\gamma}{R}R(y')=R(x')\frac{\gamma}{R}R(y')$. Now let $u\in x\gamma y'$. Then $R(u)\in R(x)\frac{\gamma}{R}R(y')=R(x')\frac{\gamma}{R}R(y')$ and so there exists $v\in x'\gamma y'$ such that $R(v)=R(u)$. In the same way, for every $z\in x'\gamma y'$ there exists $w\in x\gamma y'$ such that $R(w)=R(z)$. Therefore, $x\gamma y'\overline{R}x'\gamma y'$ and so $R$ is regular on the right. Similarly it is regular on the left and so it is regular. □

**Proposition 4.5.** *If $(S,\Gamma)$ is a weak $\Gamma$-(semi)hypergroup and $R$ is a good relation on $S$, then $\frac{S}{R}$ is a $\frac{\Gamma}{R}$-(semi)hypergroup*

*Proof.* We have $R$ is a good regular relation on $S$, so $(x\alpha y)\beta z\overline{R}(x\beta y)\alpha z$ and $x\alpha(y\beta z)\overline{R}x\beta(y\alpha z)$, for every $(x,y,z)\in s^3,(\alpha,\beta)\in\Gamma^2$.

Let $R(w) \in (R(x)\frac{\alpha}{R}R(y))\frac{\beta}{R}R(z)$. Then there exists $R(v) \in R(x)\frac{\alpha}{R}R(y)$ such that $R(w) \in R(v)\frac{\beta}{R}R(z)$. Hence we can suppose that $v \in x\alpha y$ and $w \in v\beta z$ then $w \in (x\alpha y)\beta z$, so there exists $w' \in (x\beta y)\alpha z$ such that $R(w) = R(w')$, therefor $R(w) \in ([R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z))$. Then

$$([R(x)\frac{\alpha}{R}R(y)]\frac{\beta}{R}R(z)) \subseteq ([R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z)).$$

In a similar way, we obtain

$$([R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z)) \subseteq ([R(x)\frac{\alpha}{R}R(y)]\frac{\beta}{R}R(z)).$$

Therefore

$$[R(x)\frac{\beta}{R}R(y)]\frac{\alpha}{R}R(z) = R(x)\frac{\beta}{R}[R(y)\frac{\alpha}{R}R(z)],$$

for every $\frac{\alpha}{R}, \frac{\beta}{R} \in \frac{\Gamma}{R}$ and $R(x), R(y), R(z) \in \frac{S}{R}$. □

**Example 4.6.** *Suppose that $S = \{e, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$. Consider the semihypergroups $S_\alpha$ and $S_\beta$, where the hyperoperations $\alpha$ and $\beta$ are defined on $S$ by the following tables:*

| $\alpha$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $b$ | $c$ | $e$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $e$ | $a$ | $b$ |

| $\beta$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

*Let $R = \{(e, e), (a, a), (b, b), (c, c), (e, b), (b, e), (a, c), (c, a)\}$. It is easy to see that $R$ is a good relation and $\frac{\alpha}{R} = \frac{\beta}{R}$, where*

| $\frac{\alpha}{R}$ | $R(e)$ | $R(a)$ |
|---|---|---|
| $R(e)$ | $R(e)$ | $R(a)$ |
| $R(a)$ | $R(a)$ | $R(e)$ |

**Theorem 4.7.** *Let $S$ be a weak $\Gamma$-(semi)hypergroup and $R$ be an equivalence relation on $S$. The following conditions are equivalent:*

1. *$R$ is a strongly regular relation,*
2. *if $x_1Ry_1$ and $x_2Ry_2$ then $x_1\gamma x_2 \overline{\overline{R}}\, y_1\gamma y_2$*
3. *$\frac{S}{R}$ is a weak $\frac{\Gamma}{R}$-(semi)group.*

*Proof.* $(1 \Leftrightarrow 2)$ is clear.
$(1 \Rightarrow 3)$. Let $x, y \in S$ and $\gamma \in \Gamma$. Since $R$ is strongly regular, then for every $u, v \in x\gamma y$ we have $R(u) = R(v)$ and so $R(x)\frac{\gamma}{R}R(y)$ is singleton. Now, by Theorem 4.5, the proof is completed.
$(3 \Rightarrow 1)$. Let $xRy, a \in S$ and $\gamma \in \Gamma$, since $R(x)\frac{\gamma}{R}R(a) = \{R(z) \mid z \in x\gamma a\}$, $R(y)\frac{\gamma}{R}R(a) = \{R(t) \mid t \in y\gamma a\}$ are singleton, then for every $z \in x\gamma a$ and $t \in y\gamma a$. we have $zRt$ and so $R$ is strongly regular to the right and the same way implies that $R$ is strongly regular to the left. Therefore, $R$ is strongly regular. □

**Theorem 4.8.** *Let $S$ be a weak $\Gamma$ -(semi)hypergroup and $R$ be an equivalence good relation on $S$. The following conditions are equivalent:*

(1) *$R$ is a strongly good regular relation,*
(2) *$\frac{S}{R}$ is a $\frac{\Gamma}{R}$ -(semi)group*

*Proof.* The proof follows Theorem 4.7 and Proposition 4.5. □

## 5. Γ-hyperideals of weak Γ-semihypergroups

In this section, we generalize the notion of left and right Γ-hyperideal. For a weak Γ-semihypergroup $S$ we introduce a weak left(right) Γ-hyperideal of $S$.

**Definition 5.1.** (see [7]) A non-empty subset $I$ of a Γ-semihypergroup $S$ is called a left(right) Γ-hyperideal of $S$, if $S\Gamma I \subseteq I(I\Gamma S \subseteq I)$.

**Definition 5.2.** A non-empty subset $I$ of a weak Γ-semihypergroup( or a weak Γ-semigroup) $S$ is called a weak left(right) Γ-hyperideal(or a weak left(right) Γ-ideal) of $S$, if $S\Gamma I \cap I \neq \emptyset(I\Gamma S \cap I \neq \emptyset)$.

**Example 5.3.** *Suppose that $S = \{e, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$. Consider the semihypergroups $S_\alpha$ and $S_\beta$, where the hyperoperations $\alpha$ and $\beta$ are defined on $S$ by the following tables:*

| $\alpha$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $S$ | $c$ |
| $b$ | $b$ | $e, a, b$ | $b$ | $b, c$ |
| $c$ | $c$ | $a, c$ | $c$ | $S$ |

| $\beta$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $S$ | $S$ | $S$ | $S$ |
| $a$ | $S$ | $S$ | $S$ | $S$ |
| $b$ | $S$ | $S$ | $S$ | $S$ |
| $c$ | $S$ | $S$ | $S$ | $S$ |

*In this case $(S, \Gamma)$ is a weak $\Gamma$-hypergroup and $I = \{e, a\}$ is weak left $\Gamma$-hyperideal of $S$ which is not a weak right $\Gamma$-hyperideal, indeed, $a\alpha c \cap I = \emptyset$.*

**Example 5.4.** *Consider $I = \{a, b, c\}$ in Example 5.3, $I$ is a weak $\Gamma$-hyperideal which is not an $\Gamma$-hyperideal of $S$.*

**Definition 5.5.** Let $(S, \Gamma)$ be a $\Gamma$-semihypergroup and $\alpha \in \Gamma$. Then subset $A$ of $S_\alpha$ is called

(i) $\alpha$-complete if the following implication valid:

$$\prod_{i=1}^{n} a_i \cap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A,$$

where $n > 1$, $(a_1, a_2, ..., a_n) \in S^n$ and $\prod_{i=1}^{n} a_i = a_1 \alpha a_2 \alpha \cdots \alpha a_n$;

(ii) $\Gamma$-complete if it is $\gamma$-complete, for every $\gamma \in \Gamma$.

**Proposition 5.6.** *Let $(S, \Gamma)$ be a $\Gamma$-complete semihypergroup and $I$ a non- empty subset of $S$. Then $I$ is a weak left(right) $\Gamma$-hyperideal of $S$ if and only if it is an left(right) $\Gamma$-hyperideal of $S$.*

*Proof.* The proof is straightforward. □

**Proposition 5.7.** *Let $(S, \Gamma)$ be a weak $\Gamma$-semihypergroup and $I$ be a weak left(right) $\Gamma$-hyperideal of $S$. If $R$ be a regular relation on $S$. Then $\frac{I}{R}$ is a weak left(right) $\frac{\Gamma}{R}$-hyperideal of the $\frac{\Gamma}{R}$-(semi)hypergroup $\frac{S}{R}$.*

*Proof.* First we prove that $\frac{S}{R}\frac{\Gamma}{R}\frac{I}{R} \cap \frac{I}{R} \neq \emptyset$. From $S\Gamma I \cap I \neq \emptyset$ we conclude that there exist $x \in S, a \in I$ and $\gamma \in \Gamma$, such that $x\gamma a \cap I \neq \emptyset$. Thus $R(x)\frac{\gamma}{R}R(a) \cap \frac{I}{R} \neq \emptyset$ and so $\frac{S}{R}\frac{\Gamma}{R}\frac{I}{R} \cap \frac{I}{R} \neq \emptyset$. Similarly we have $\frac{I}{R}\frac{\Gamma}{R}\frac{S}{R} \cap \frac{I}{R} \neq \emptyset$. □

**Proposition 5.8.** *Let $(S, \Gamma)$ be a weak $\Gamma$-semihypergroup and $I$ be a weak left(right) $\Gamma$-hyperideal of $S$. If $R$ be a strongly regular relation on $S$. Then $\frac{I}{R}$ is a weak left(right) $\frac{\Gamma}{R}$-ideal of the $\frac{\Gamma}{R}$-(semi)group $\frac{S}{R}$.*

*Proof.* The proof is straightforward. □

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