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LEFT I-QUOTIENTS OF BAND OF RIGHT CANCELLATIVE MONOIDS

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ABSTRACT. Let Q be an inverse semigroup. A subsemigroup S of Q is a left I-order in Q and Q is a semigroup of left I-quotients of S if every element $q \in Q$ can be written as $q = a^{-1}b$ for some $a, b \in S$. If we insist on a and b being $\mathcal{R}$-related in Q, then we say that S is straight in Q. We characterize semigroups which are left I-quotients of left regular bands of right cancellative monoids with certain conditions.

1. **Introduction and summary**

Many authors are interested in the theory of semigroups of quotients. We recall that a subsemigroup S of a group G is a *left order* in G or G is a *group of left quotients* of S if any element in G can be written as $a^{-1}b$ where $a, b \in S$. Ore and Dubreil [1] showed that a semigroup S has a group of left quotients if and only if S is right reversible and cancellative. By saying that a semigroup S is *right reversible* we mean for any $a, b \in S, Sa \cap Sb \neq \emptyset$. A different definition proposed by Fountain and Petrich in 1985 [4] was restricted to completely 0-simple semigroups of left quotients and then shortly after to that of semigroup of left quotients by Gould [8]. The idea is that a subsemigroup S of a semigroup Q is a *left order* in Q if every element in Q can be written as $a^{\sharp}b$ where $a, b \in S$ and $a^{\sharp}$ is an inverse of a in a subgroup of Q and if, in addition, every square-cancellable element of S lies in a subgroup of Q. In this case we say that Q is a semigroup of *left quotients* of S. *Right orders* and *semigroup of right quotients* are defined dually. If S

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is both a left and right order in Q, then S is an *order* in Q and Q is a *semigroup of quotients* of S.

In the case where Q is an inverse semigroup, McAlister introduced two concepts of quotients for an inverse semigroup Q with subsemigroup S in [12]. The first one says that Q is a semigroup of *strong quotients* of S if every element in Q can be written as $ab^{-1}c$ where $a, c, b \in S$ and $b \in Sa \cap cS$. The second one is that Q is a *semigroup of quotients* of S if every element in Q can be written as $ab^{-1}c$ where $a, c, b \in S$.

The author and Gould in [6] have extended the classical notion of left orders in inverse semigroups. They introduced the following definition of left I-orders in inverse semigroups: A subsemigroup S of an inverse semigroup Q is a *left I-order* in Q and Q is a semigroup of *left I-quotients* of S if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and a^{-1} is the inverse of a in the sense of an inverse semigroup theory. *Right I-orders* and semigroups of *right I-quotients* are defined dually. If S is a left and right I-order in an inverse semigroup Q, we say that S is an *I-order* in Q and Q is a semigroup of *I-quotients* of S. Let S be a left I-order in Q. Then S is *straight* in Q if every $q \in Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a \,\mathcal{R}\, b$ in Q.

It is shown in [9] that a semigroup has a semilattice of groups (or a Clifford semigroup) as its semigroup of left quotients if and only if it is a semilattice of right reversible, cancellative semigroups. El-Qallali [3] has been generalized this result to a left regular band of groups. He proved that a semigroup has a left regular band of groups as its semigroup of left quotients if and only if it is a left regular band of right reversible, cancellative semigroups.

We remark that if a semigroup S is a left order in Q in the sense of [8, 9] (in particular Q is inverse), then it is certainly a left I-order.

Clifford [1] showed that any right cancellative monoid S with the (LC) condition is the $\mathcal{R}$-class of the identity of its inverse hull $\Sigma(S)$. Moreover, (in our terminology) S is a left I-order in $\Sigma(S)$. By saying that a semigroup S has the (LC) condition we mean for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. Clifford established that precisely bisimple inverse monoids can be regarded as inverse hulls of right cancellative monoids S satisfying the (LC) condition. It is clear that (LC) is a rather stronger condition than right reversiblity.

Gantos [10] extended the work of Clifford to semilattices of right cancellative monoids. He considered semigroups which are semilattices of right cancellative monoids with certain conditions. He developed a structure for semigroups Q which are semilattices Y of bisimple inverse monoids Q_α, $\alpha \in Y$ such that the set of identities elements forms a subsemigroup.

The author and Gould in [6] have extended Cliford's work to a left ample semigroup with Condition (LC). To be more precise, they proved the following theorem.

Theorem 1.1. [6] *Let S be a left ample semigroup. Then $S\theta_S$ is a left I-order in its inverse hull if and only if S has Condition (LC).*
If Condition (LC) holds, then $S\theta_S$ is a union of $R^{\Sigma(S)}$-classes.

The purpose of this article is to study left I-quotients of left regular band of right cancellative monoids. Then we get a generalization of Gantos's result.

By sections, this work is divided as follows. Section 2 contains notations and terminology. In Section 3 we characterize semigroups S which have a semigroup Q of left I-quotients, where Q is a left regular bands of bisimple inverse monoids.

2. **Notation and terminology**

In this section we introduce necessary notions and notations that are needed in our further considerations. For the undefined notions and notation the reader is referred to [11] and [13].

We begin by giving some elementary facts about $\mathcal{R}^*$; dual for $\mathcal{L}^*$. The relation $\mathcal{R}^*$ is defined on a semigroup S by the rule that $a\,\mathcal{R}^*\,b$ in S if $a\,\mathcal{R}\,b$ in some oversemigroup T of S this equivalent to $a\mathcal{R}^*\,b$ if and only if

$$xa = ya \quad \text{if and only if} \quad xb = yb$$

for all $x, y \in S^1$. The relation $\mathcal{L}^*$ is defined dually and $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$. It is clear that $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{L} \subseteq \mathcal{L}^*$ where $\mathcal{R}$ and $\mathcal{L}$ are the usual Green's relations.

A semigroup S is *left ample* if $E(S)$ is a semilattice, every $\mathcal{R}^*$-class contains a (necessary unique) idempotent a^+ and S satisfies the left ample identity which is:

$$(ae)^+a = ae \text{ for all } a \in S \text{ and } e \in E(S).$$

We can note easily that, any right cancellative monoid is left ample. By a right cancellative semigroup we mean, a semigroup S such that for all $x, y, z \in S$

$$xz = yz \text{ implies } x = y.$$

Following [7], for any left ample semigroup S we can construct an embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ as follows. For each $a \in S$ we let $\rho_a \in \mathcal{I}_S$ be given by

$$\operatorname{dom}\rho_a = Sa^+ \text{ and } \operatorname{im}\rho_a = Sa$$

and for any $x \in \operatorname{dom}\rho_a$.

$$x\rho_a = xa.$$

Then the map $\theta_S : S \to \mathcal{I}_S$ is a $(2,1)$-embedding.

The inverse hull of a left ample semigroup S is the inverse subsemigroup $\Sigma(S)$ of $\mathcal{I}_S$ generated by $\operatorname{im}\theta_S$. If S is a right cancellative monoid, then for any $a \in S$ we have $a^+ = 1$. Then $\rho_a : S \longrightarrow Sa$ is defined by

$$x\rho_a = xa \quad \text{for each } x \text{ in } S.$$

Hence $\operatorname{dom}\rho_a = S = \operatorname{dom} I_S$, giving that $\operatorname{im}\theta_S \subseteq R_1$ where R_1 is the $\mathcal{R}$-class of I_S in $\mathcal{I}_S$.

As in [6] we say that a $(2,1)$-morphism $\phi : S \to T$, where S and T are left ample semigroups with Condition (LC), is *(LC)-preserving* if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that

$$T(b\phi) \cap T(c\phi) = T(w\phi).$$

Recall [1] that a semigroup B is called a *band* if every element in B is an idempotent. A commutative band is called a *semilattice*. A band B is said to be *left regular* if it satisfies the identity $iji = ij$ for any $i, j \in B$. By $\leq$ we will denote the natural partial order on B, i.e. a relation on B defined by $j \leq i \Longleftrightarrow ij = ji = j$ $(i, j \in B)$. We shall define the quasiorders (that is, reflexive and transitive) $\preceq_r$, $\preceq_l$ on a band B by: for $i, j \in B$

$$j \preceq_r i \Longleftrightarrow ij = j, \qquad j \preceq_l i \Longleftrightarrow ji = j.$$

It is clear that

$$j \preceq_r i \Longleftrightarrow (\exists k \in B)ik = j, \text{ and } j \preceq_l i \Longleftrightarrow (\exists k \in B)ki = j.$$

Clearly, $\preceq_r = \preceq_l = \leq$ concide if B is a semilattice. In a left regular band, if $j \preceq_r i$ and $i \preceq_r j$, then $i = iji = ij = j$. It follows that a left regular band B is partially ordered with respect to $\preceq_r$. We call this partial order the *$\mathcal{R}$-order* on B. It is easy to see that the relation $\preceq_l$ is reflexive and transitive, but not necessarily antisymmetric.

A semigroup S is a band of right cancellative monoids if $S = \bigcup(S_\alpha, \alpha \in B)$ where S_α is a right cancellative monoid, $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$, B is a band and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

3. Left I-orders

El-Qallai [3] starts with a left regular band B of right reversible cancellative semigroups S_α, for every $\alpha \in B$, lets G_α be the group of left quotients of S_α, and on $Q = \bigcup_{\alpha\in B} G_\alpha$ defines a multiplication which makes it a semigroup of left quotients of S. In this section we are concerned with a semigroup of left I-quotients of a semigroup S which is a left regular band B of right cancellative monoids S_α, $\alpha \in B$ such that each S_α has the (LC) condition. We start with the following useful lemma.

Lemma 3.1. *Let S be a left regular band Y of right cancellative monoids $S_\alpha, \alpha \in Y$. Let e_α denote the identity of $S_\alpha, \alpha \in Y$. Then*
(i) $e_{\beta\alpha}e_\alpha = e_{\beta\alpha}$
(ii) $e_\alpha e_{\beta\alpha} = e_{\alpha\beta}$;
(iii) $E(S) = \{e_\alpha : \alpha \in Y\}$ *is a left regular band;*
(iv) $a_\alpha e_{\beta\alpha} = e_{\alpha\beta}a_\alpha$;
(v) for any $a, b \in S$, $a\,\mathcal{R}^*\,b$ in S if and only if $a, b \in S_\alpha$ for some α in Y;
(vi) S satisfies the left ample identity.

Proof. (i) Let $e_\alpha \in S_\alpha$ be the identitie of S_α. Since $e_{\beta\alpha}e_\alpha \in S_{\beta\alpha}$ we have

$$(e_{\beta\alpha}e_\alpha)(e_{\beta\alpha}e_\alpha) = [(e_{\beta\alpha}e_\alpha)e_{\beta\alpha}]e_\alpha = (e_{\beta\alpha}e_\alpha)e_\alpha = e_{\beta\alpha}(e_\alpha e_\alpha) = e_{\beta\alpha}e_\alpha.$$

Hence $e_{\beta\alpha}e_\alpha$ is an idempotent in $S_{\beta\alpha}$. But there is only one idempotent in $S_{\beta\alpha}$, so that $e_{\beta\alpha}e_\alpha = e_{\beta\alpha}$.

(ii) Let $e_\alpha \in S_\alpha$ be the identitie of S_α. From (i) we have

$$(e_\alpha e_{\beta\alpha})(e_\alpha e_{\beta\alpha}) = e_\alpha(e_{\beta\alpha}e_\alpha)e_{\beta\alpha} = e_\alpha e_{\beta\alpha}e_\alpha = e_\alpha(e_{\beta\alpha}e_\alpha) = e_\alpha e_{\beta\alpha}.$$

Hence $(e_\alpha e_{\beta\alpha})^2 = e_\alpha e_{\beta\alpha} \in S_{\alpha\beta}$, that is, $e_\alpha e_{\beta\alpha}$ is an idempotent in $S_{\alpha\beta}$. Since $e_{\alpha\beta}$ is only one idempotent of $S_{\alpha\beta}$, we conclude that $e_\alpha e_{\beta\alpha} = e_{\alpha\beta}$.

(iii) Let $e_\alpha \in S_\alpha$ and $e_\beta \in S_\beta$ for some $\alpha, \beta \in Y$. Then $e_\alpha e_\beta \in S_{\alpha\beta}$ and from (ii) we have that

$$e_\alpha e_\beta = (e_\alpha e_\beta)e_{\alpha\beta} = e_\alpha(e_\beta e_{\alpha\beta}) = e_\alpha e_{\beta\alpha} = e_{\alpha\beta}.$$

Similarly, $e_\beta e_\alpha = e_{\beta\alpha}$ so that $E(S)$ is a band. Using (ii) we have

$$e_\beta e_\alpha e_\beta = e_\beta(e_\alpha e_\beta) = e_\beta e_{\alpha\beta} = e_{\beta\alpha\beta} = e_{\beta\alpha} = e_\beta e_\alpha.$$

(iv) Let $e_{\beta\alpha} \in S_{\beta\alpha}$ and $a_\alpha \in S_\alpha$ for some $\alpha, \beta \in Y$. Then

$$\begin{aligned} a_\alpha e_{\beta\alpha} &= (a_\alpha e_\alpha) e_{\beta\alpha} \\ &= a_\alpha (e_\alpha e_{\beta\alpha}) \\ &= a_\alpha e_{\alpha\beta} \\ &= e_{\alpha\beta}(a_\alpha e_{\alpha\beta}) \\ &= (e_{\alpha\beta} a_\alpha) e_{\alpha\beta} = e_{\alpha\beta} a_\alpha \end{aligned}$$

(v) Suppose that $a\,\mathcal{R}^*\,b$ in S where $a \in S_\alpha$ and $b \in S_\beta$. Then $e_\alpha a = a$ and so $e_\alpha b = b$. Thus $\alpha\beta = \beta$, that is $\beta \preceq_r \alpha$. Dually, $\alpha \preceq_r \beta$ and hence $\alpha = \beta$; clearly, $a\,\mathcal{R}^*\,b$ in S_α.

Conversely, suppose that $b \in S_\alpha$ and $xb = yb$ for some $x, y \in S$ where $x \in S_\beta$ and $y \in S_\gamma$. Then $\beta\alpha = \gamma\alpha$ and so $\alpha\beta = \alpha\beta\alpha = \alpha\gamma\alpha = \alpha\gamma$. Thus $xbe_{\beta\alpha} = ybe_{\beta\alpha}$ so that from (iv) we get $xe_{\alpha\beta}b = ye_{\alpha\beta}b$, and so $e_\alpha x e_{\alpha\beta} b = e_\alpha y e_{\alpha\beta} b$. Now $e_\alpha x, e_\alpha y, e_{\alpha\beta} b$ all lie in $S_{\alpha\beta} = S_{\alpha\gamma}$ which is right cancellative, so that $e_\alpha x = e_\alpha y$. Hence $e_{\beta\alpha} x = e_\beta e_\alpha x = e_\beta e_\alpha y = e_{\beta\alpha} y$, by (iii). From (iv) again we have $xe_{\alpha\beta} = ye_{\alpha\beta}$. From (iii) we have $xe_\alpha e_\beta = ye_\alpha e_\beta$ and so $(xe_\alpha)(e_\beta e_{\beta\alpha}) = (ye_\alpha)(e_\beta e_{\beta\alpha})$. As $xe_\alpha, e_\beta e_{\beta\alpha}, ye_\alpha \in S_{\beta\alpha}$ which is right cancellative, so that $xe_\alpha = ye_\alpha$. Also, if $xb = b$, that is, $xb = e_\alpha b$, then $xe_\alpha = e_\alpha e_\alpha = e_\alpha$. Thus $b\,\mathcal{R}^*\,e_\alpha$ in S. Hence for any $a \in S_\alpha$ we have that $a\,\mathcal{R}^*\,b$ in S as required.

(vi) From (v) we deduce that each $\mathcal{R}^*$-class contains an idempotent which must be unique. Notice that if $a \in S_\alpha$, then $a^+ = e_\alpha$. To see that S satisfies the left ample identity, let $a \in S_\alpha$ and $e_\beta \in S_\beta$ so that $ae_\beta \in S_{\alpha\beta}$. Using (iii) and (iv) we have

$$\begin{aligned} (ae_\beta)^+ a &= e_{\alpha\beta} a \\ &= ae_{\beta\alpha} \\ &= (ae_\alpha)e_{\beta\alpha} \\ &= a(e_\alpha e_{\beta\alpha}) \\ &= ae_{\alpha\beta} \\ &= (ae_\alpha)e_\beta \\ &= ae_\beta \end{aligned}$$

as required. □

In the case where Y is a semilattice we have

Lemma 3.2. *(cf. [5]) Let S be a semilattice Y of right cancellative monoids $S_\alpha, \alpha \in Y$. Let e_α denote the identity of $S_\alpha, \alpha \in Y$. Then*

(i) *$e_\beta a_\alpha = a_\alpha e_\beta$ if $\alpha \geq \beta$;*

(ii) *$e_\alpha e_{\alpha\beta} = e_{\alpha\beta}$ where $e_\alpha, e_{\alpha\beta}$ are the identities of S_α and $S_{\alpha\beta}$ respectively;*

(iii) *$E(S)$ is a semilattice;*

(iv) *the idempotents are central;*

(v) for any $a, b \in S$, $a\,\mathcal{R}^*\,b$ in S if and only if $a, b \in S_\alpha$ for some α in Y;

(vi) S is a left ample semigroup.

Lemma 3.3. *Let S be a left regular band Y of right cancellative monoids S_α with the (LC) condition. Suppose that (C) holds, where (C): whenever $\beta \preceq_l \alpha$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ ($a_\alpha, b_\alpha, c_\alpha \in S_\alpha$), then $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$. Then S has (LC).*

Proof. Suppose that $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ implies $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$ for all $\beta \preceq_l \alpha$.
Let $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. Then $e_{\alpha\beta}a, e_{\alpha\beta}b \in S_{\alpha\beta}$ so that as $S_{\alpha\beta}$ has (LC) we know that

$$S_{\alpha\beta}(e_{\alpha\beta}a) \cap S_{\alpha\beta}(e_{\alpha\beta}b) = S_{\alpha\beta}c$$

for some $c \in S_{\alpha\beta}$. Now, let $d \in Sa \cap Sb$, say $d \in S_\gamma$ so that $\gamma \preceq_l \alpha\beta$ and $d = ua = vb$ for some $u, v \in S$ where $u \in S_\eta$ and $v \in S_\zeta$. Then

$$S_\gamma(e_{\alpha\beta}a) \cap S_\gamma(e_{\alpha\beta}b) = S_\gamma c,$$

by (C). Hence

$$S_\gamma e_\gamma(e_{\alpha\beta}a) \cap S_\gamma e_\gamma(e_{\alpha\beta}b) = S_\gamma c.$$

Since $\gamma\alpha\beta = \gamma$ we have $e_\gamma e_{\alpha\beta} = e_{\gamma\alpha\beta} = e_\gamma$, by Lemma 3.1. Thus

$$S_\gamma e_\gamma a \cap S_\gamma e_\gamma b = S_\gamma a \cap S_\gamma b = S_\gamma c.$$

Since Y is a left regular band and $\gamma = \eta\alpha = \zeta\beta$ we have $\gamma = \gamma\eta = \gamma\zeta$. Then

$$d = ua = vb = (e_\gamma u)a = (e_\gamma v)b \in S_\gamma a \cap S_\gamma b = S_\gamma c$$

as $e_\gamma u, e_\gamma v \in S_\gamma$. Then $d \in S_\gamma c \subseteq Sc$. Thus $Sa \cap Sb \subseteq Sc$. Also, $c \in S_{\alpha\beta}a \subseteq Sa$ and $c \in S_{\alpha\beta}b \subseteq Sb$. Thus $c \in Sa \cap Sb$. Hence $Sc \subseteq Sa \cap Sb$ and we get $Sc = Sa \cap Sb$. □

Corollary 3.4. *Let S be a semilattice Y of right cancellative monoids S_α with the (LC) condition and S has (C). Then S has (LC).*

Now, we state the following two lemmas, which will be used to prove the main result of this section.

Lemma 3.5. *(cf. [2, Lemma 4.1]) Let T be a right cancellative monoid. Then for $a, b \in T$ we have*

$$a\,\mathcal{L}\,b \text{ if and only if } a = ub,$$

for some unit u of T.

The proof of the following is entirely routine, but we provide it for completeness.

Lemma 3.6. *Let Q be an inverse monoid with identity 1 and let R_1, the $\mathcal{R}$-class of 1. Let $a, b, c, d \in R_1$. Then*

$$a^{-1}b = c^{-1}d \quad \textit{if and only if} \quad a = uc \quad \textit{and} \quad b = ud$$

for some unit u.

Proof. Suppose that $a^{-1}b = c^{-1}d$ where $a, b, c, d \in R_1$. It is clear that $a^{-1} \mathcal{R}\, a^{-1}b$ and $c^{-1} \mathcal{R}\, c^{-1}d$. Then

$$a^{-1} \mathcal{R}\, a^{-1}b = c^{-1}d \,\mathcal{R}\, c^{-1} \text{ in Q.}$$

Then $a \,\mathcal{L}\, c$ in Q. Since $a \,\mathcal{R}\, b$, it follows that $b = aa^{-1}b = ac^{-1}d$. We claim that ac^{-1} is a unit. As $a \,\mathcal{L}\, c$, it follows that $ac^{-1} \,\mathcal{L}\, cc^{-1} = 1$. Since $a^{-1} \mathcal{R}\, c^{-1}$ we have that $1 = aa^{-1} \mathcal{R}\, ac^{-1}$ and hence $u = ac^{-1}$ is a unit, and we obtain $b = ud$. Since $u = ac^{-1}$ and $a \,\mathcal{L}\, c$ we have that $uc = ac^{-1}c = a$. The converse is clear. □

Let $\Sigma(S)$ be the inverse hull of left I-quotents of a right cancellative monoid S with (LC). In the rest of this section we identify S with $S\theta_S$, where θ_S is the embedding of S into $\mathcal{I}_S$. We write $a^{-1}b$ short for the element $\rho_a^{-1}\rho_b$ of $\Sigma(S)$ where $a, b \in S$.

We thus are able to prove our main result.

Theorem 3.7. *Let S be a left regular band Y of right cancellative monoids S_α with identity e_α, such that each S_α, has (LC). Suppose in addition that (C) holds. For each $\alpha \in Y$, let Σ_α be the inverse hull of S_α. Then S is a left I-order in Q where Q is a left regular band Y of bisimple inverse monoids Σ_α with identity e_α, such that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q.*

Proof. By Corollary 3.11 of [6], for each $\alpha \in Y$, S_α is a left I-order in Σ_α where S_α is the $\mathcal{R}$-class of the identity of Σ_α. We may assume that $\Sigma_\alpha \cap \Sigma_\beta = \emptyset$ for any $\alpha, \beta \in Y$, $\alpha \neq \beta$. Put $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$. Notice that if $b \in S_\alpha$ and $c \in S_\beta$, then $bc \in S_{\alpha\beta}$. As Y is a left regular band we have $bcb \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ has (LC), there exist $\acute{x}, \acute{y} \in S_{\alpha\beta}$ with $\acute{y}bc = \acute{x}bcb = \acute{z}$. Putting $x = \acute{x}bc, y = \acute{y}b$ one sees that $yc = xb$ and $x, y \in S_{\alpha\beta}$. Clearly, for any $a \in S_\alpha, d \in S_\beta$ we have $xa \in S_{\alpha\beta\alpha} = S_{\alpha\beta}, yd \in S_{\alpha\beta\beta} = S_{\alpha\beta}$ and so $(xa)^{-1}(yd)$ exists in $\Sigma_{\alpha\beta}$.

Define a product $\cdot$ on Q by

$$a^{-1}b \cdot c^{-1}d = (xa)^{-1}(yd)$$

where $S_{\alpha\beta}b \cap S_{\alpha\beta}c = S_{\alpha\beta}w$ and $xb = yc = w$ for some $x, y \in S_{\alpha\beta}$.

Lemma 3.8. *This product is well-defined.*

Proof. Suppose that we have elements a_1, b_1, a_2, b_2 of S_α, c_1, d_1, c_2, d_2 of S_β such that

$$a_1^{-1}b_1 = {a_2}^{-1}b_2 \text{ in } \Sigma_\alpha \text{ and } c_1^{-1}d_1 = {c_2}^{-1}d_2 \text{ in } \Sigma_\beta.$$

By Lemma 3.6,

$$a_1 = u_1a_2, b_1 = u_1b_2$$

for some unit $u_1 \in S_\alpha$ and

$$c_1 = v_1c_2, d_1 = v_1d_2$$

for some unit $v_1 \in S_\beta$. By definition,

$$a_1^{-1}b_1 \cdot {c_1}^{-1}d_1 = (t_1a_1)^{-1}(r_1d_1)$$

where

$$S_{\alpha\beta}b_1 \cap S_{\alpha\beta}c_1 = S_{\alpha\beta}w_1 \text{ and } t_1b_1 = r_1c_1 = w_1$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta}$. Also,

$$a_2^{-1}b_2 \cdot {c_2}^{-1}d_2 = (t_2a_2)^{-1}(r_2d_2)$$

where

$$S_{\alpha\beta}b_2 \cap S_{\alpha\beta}c_2 = S_{\alpha\beta}w_2 \text{ and } t_2b_2 = r_2c_2 = w_2$$

for some $t_2, r_2, w_2 \in S_{\alpha\beta}$.
We have to show that $a_1^{-1}b_1 \cdot {c_1}^{-1}d_1 = a_2^{-1}b_2 \cdot {c_2}^{-1}d_2$, that is,

$$(t_1a_1)^{-1}(r_1d_1) = (t_2a_2)^{-1}(r_2d_2)$$

and to do this we need to prove that

$$t_1a_1 = ut_2a_2 \text{ and } r_1d_1 = ur_2d_2$$

for some unit u in $S_{\alpha\beta}$, using Lemma 3.6. We aim to prove that $S_{\alpha\beta}w_1 = S_{\alpha\beta}w_2$. We get this if we prove that $S_{\alpha\beta}b_1 = S_{\alpha\beta}b_2$ and $S_{\alpha\beta}c_1 = S_{\alpha\beta}c_2$. Since $b_1 = u_1b_2$, we have that

$$e_{\alpha\beta}b_1 = b_1e_{\beta\alpha} = u_1b_2e_{\beta\alpha} = u_1e_{\alpha\beta}b_2 = (u_1e_{\alpha\beta})(e_{\alpha\beta}b_2),$$

by Lemma 3.1 and as $b_2 = u_1^{-1}b_1$, we have $e_{\alpha\beta}b_2 = (e_{\alpha\beta}u_1^{-1})(e_{\alpha\beta}b_1)$. Then

$$S_{\alpha\beta}b_1 = S_{\alpha\beta}e_{\alpha\beta}b_1 = S_{\alpha\beta}e_{\alpha\beta}b_2 = S_{\alpha\beta}b_2.$$

Similarly, $S_{\alpha\beta}c_1 = S_{\alpha\beta}c_2$. Hence $S_{\alpha\beta}w_1 = S_{\alpha\beta}w_2$ so that $w_1 \,\mathcal{L}\, w_2$ in $S_{\alpha\beta}$. By Lemma 3.5, $w_1 = lw_2$ for some unit l in $S_{\alpha\beta}$. Then

$$w_1 = t_1b_1 = lw_2 = l(t_2b_2) = lt_2(u_1^{-1}b_1).$$

But, by Lemma 3.1 $a_1 \,\mathcal{R}^*\, b_1$ in S, it follows that $t_1a_1 = lt_2u_1^{-1}a_1 = lt_2a_2$. Since

$$w_1 = r_1c_1 = lw_2 = lr_2c_2 = lr_2v_1^{-1}c_1$$

and $c_1\,\mathcal{R}^*\,d_1$ in S, again using Lemma 3.1 we have

$$r_1d_1 = lr_2v_1^{-1}d_1 = lr_2v_1^{-1}v_1d_2 = lr_2d_2$$

as required. □

In order to prove the associative law we need to introduce subsidiary lemmas. The proof of the next lemma is depends only on the fact that S_α is right cancellative and the proof can be found in [10].

Lemma 3.9. $(S_\alpha a_\alpha \cap S_\alpha b_\alpha)c_\alpha = S_\alpha a_\alpha c_\alpha \cap S_\alpha b_\alpha c_\alpha$ *for all* $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$.

Lemma 3.10. *Let* $a^{-1}b, a^{-1}e_\alpha \in \Sigma_\alpha$ *and* $c^{-1}d, e_\beta d \in \Sigma_\beta$ *where* $a, b \in S_\alpha, c, d \in S_\beta$ *and* e_α, e_β *are the identities elements in* S_α *and* S_β *respectively. Then*
(i) $a^{-1}b \cdot e_\beta d = (e_{\alpha\beta}a)^{-1}(bd)$;
(ii) $a^{-1}e_\alpha \cdot c^{-1}d = (e_{\alpha\beta}ca)^{-1}(e_{\alpha\beta}d)$.

Proof. (i) We have that $S_{\alpha\beta}e_\beta \cap S_{\alpha\beta}b = S_{\alpha\beta} \cap S_{\alpha\beta}b = S_{\alpha\beta}b$ and

$$e_{\alpha\beta}b = be_{\beta\alpha} = b(e_{\beta\alpha}e_\beta) = (be_{\beta\alpha})e_\beta = (e_{\alpha\beta}b)e_\beta,$$

using Lemma 3.1. We have

$$\begin{aligned}(a^{-1}b)\cdot(e_\beta d) &= (a^{-1}b)\cdot(e_\beta^{-1}d)\\ &= (e_{\alpha\beta}a)^{-1}(e_{\alpha\beta}bd)\\ &= (e_{\alpha\beta}a)^{-1}(bd).\end{aligned}$$

(ii) Let $a^{-1}e_\alpha$ and $c^{-1}d$ be as in the hypothesis. Then

$$a^{-1}e_\alpha \cdot c^{-1}d = (xa)^{-1}(yd)$$

where $S_{\alpha\beta}w = S_{\alpha\beta}e_\alpha \cap S_{\alpha\beta}c$ and $xe_\alpha = yc = w$ for some $x, y \in S_{\alpha\beta}$. Usiny Lemma 3.1,

$$S_{\alpha\beta}w = S_{\alpha\beta}e_{\alpha\beta}e_\alpha \cap S_{\alpha\beta}e_{\alpha\beta}c = S_{\alpha\beta}e_{\alpha\beta} \cap S_{\alpha\beta}e_{\alpha\beta}c = S_{\alpha\beta} \cap S_{\alpha\beta}e_{\alpha\beta}c = S_{\alpha\beta}e_{\alpha\beta}c$$

and so $e_{\alpha\beta}c\,\mathcal{L}\,w = yc$ in $S_{\alpha\beta}$. Thus, $ye_{\alpha\beta}c = yc = ue_{\alpha\beta}c$ for some unit u in $S_{\alpha\beta}$, by Lemma 3.5. Since $S_{\alpha\beta}$ is right cancellative we have $y = u$, that is, y is a unit. Then

$$\begin{aligned}a^{-1}e_\alpha \cdot c^{-1}d &= (xa)^{-1}(yd) && \text{since}\, xa = x(e_\alpha a) = (xe_\alpha)a = yca\\ &= (yca)^{-1}(yd)\\ &= (ye_{\alpha\beta}ca)^{-1}(yd) && \text{as}\, y = ye_{\alpha\beta}\\ &= (e_{\alpha\beta}ca)^{-1}y^{-1}(yd) && \text{as}\, y, e_{\alpha\beta}ca, yd \in S_{\alpha\beta} \subseteq \Sigma_{\alpha\beta}\\ &= (e_{\alpha\beta}ca)^{-1}(y^{-1}yd)\\ &= (e_{\alpha\beta}ca)^{-1}(e_{\alpha\beta}d).\end{aligned}$$

□

Lemma 3.11. *Let* $a^{-1}b \in \Sigma_\alpha, e_\beta d, d^{-1}e_\beta \in \Sigma_\beta$ *and* $x^{-1}y \in \Sigma_\gamma$ *where* e_β *is the identity element in* S_β *where* $a, b \in S_\alpha, e_\beta, d \in S_\beta$ *and* $x, y \in S_\gamma$. *Then*

(*i*) $(a^{-1}b \cdot e_\beta d) \cdot x^{-1}y = a^{-1}b \cdot (e_\beta d \cdot x^{-1}y)$;
(*ii*) $(a^{-1}b \cdot d^{-1}e_\beta) \cdot x^{-1}y = a^{-1}b \cdot (d^{-1}e_\beta \cdot x^{-1}y)$.

Proof. (*i*) Let $a^{-1}b, e_\beta d, x^{-1}y$ be as in the hypothesis. Then

$$\begin{aligned}(a^{-1}b \cdot e_\beta d) \cdot x^{-1}y &= (e_{\alpha\beta}a)^{-1}(bd) \cdot x^{-1}y && \text{by Lemma 3.10,}\\ &= (t_1 e_{\alpha\beta}a)^{-1}(r_1 y)\\ &= (t_1 e_{\alpha\beta\gamma}e_{\alpha\beta}a)^{-1}(r_1 y) && \text{as } t_1 = t_1 e_{\alpha\beta\gamma}\\ &= (t_1 e_{\alpha\beta\gamma}a)^{-1}(r_1 y) && \text{by Lemma 3.1}\\ &= (t_1 a)^{-1}(r_1 y).\end{aligned}$$

where $t_1 bd = r_1 x = w_1$ and

$$S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}x = S_{\alpha\beta\gamma}w_1$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta\gamma}$.

On the other hand, by definition of multiplication,

$$\begin{aligned}a^{-1}b \cdot (e_\beta d \cdot x^{-1}y) &= a^{-1}b \cdot (t_2 e_\beta)^{-1}(r_2 y)\\ &= a^{-1}b \cdot (t_2 e_{\beta\gamma}e_\beta)^{-1}(r_2 y) && \text{as } t_2 = t_2 e_{\beta\gamma}\\ &= a^{-1}b \cdot (t_2(e_{\beta\gamma}e_\beta))^{-1}(r_2 y)\\ &= a^{-1}b \cdot (t_2 e_{\beta\gamma})^{-1}(r_2 y) && \text{by Lemma 3.1}\\ &= a^{-1}b \cdot t_2^{-1}(r_2 y) && \text{as } t_2 e_{\beta\gamma} = t_2\\ &= (t_3 a)^{-1}(r_3 r_2 y)\end{aligned}$$

where $t_2 d = r_2 x = w_2$ with

$$S_{\beta\gamma}d \cap S_{\beta\gamma}x = S_{\beta\gamma}w_2 \tag{3.1}$$

for some $t_2, r_2, w_2 \in S_{\beta\gamma}$ and $t_3 b = r_3 t_2 = w_3$ with

$$S_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}t_2 = S_{\alpha\beta\gamma}w_3 \tag{3.2}$$

for some $t_3, r_3, w_3 \in S_{\alpha\beta\gamma}$. Since $\alpha\beta\gamma \preceq_l \beta\gamma$ we have

$$S_{\alpha\beta\gamma}d \cap S_{\alpha\beta\gamma}x = S_{\alpha\beta\gamma}w_2, \tag{3.3}$$

by (C) and (3.1).

We must show that $(t_1 a)^{-1}(r_1 y) = (t_3 a)^{-1}(r_3 r_2 y)$. By using Lemma 3.6, we have to show that $t_1 a = u t_3 a$ and $r_1 y = u r_3 r_2 y$ for some unit u in $S_{\alpha\beta\gamma}$.

Once we show that $w_1 \,\mathcal{L}\, w_3 d$ in $S_{\alpha\beta\gamma}$, we have that $w_1 = h w_3 d$ for some unit h in $S_{\alpha\beta\gamma}$, by Lemma 3.5. Hence $t_1 bd = h t_3 bd$ so that $t_1 e_{\alpha\beta\gamma} bd = h t_3 e_{\alpha\beta\gamma} bd$. Since t_1, ht_3 and $e_{\alpha\beta\gamma}bd$ are in $S_{\alpha\beta\gamma}$, which is right cancellative we obtain $t_1 = h t_3$ so that $t_1 a = h t_3 a$.

Now,

$$w_1 = r_1 x = t_1 bd = h t_3 bd = h r_3 t_2 d = h r_3 r_2 x.$$

As r_1, hr_3r_2 and $e_{\alpha\beta\gamma}x$ are in $S_{\alpha\beta\gamma}$ again by right cancellativity in $S_{\alpha\beta\gamma}$ we have that $r_1 = hr_3r_2$ and so $r_1y = hr_3r_2y$.

We show that $w_1 \,\mathcal{L}\, w_3d$ in $S_{\alpha\beta\gamma}$. We have

$$\begin{array}{rcll} S_{\alpha\beta\gamma}w_1 & = & S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}x & \\ & = & S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}d \cap S_{\alpha\beta\gamma}x & \text{as } S_{\alpha\beta\gamma}bd \subseteq S_{\alpha\beta\gamma}d \\ & = & S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}w_2 & \text{by (3.3)} \\ & = & S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}t_2d & \\ & = & S_{\alpha\beta\gamma}e_{\alpha\beta\gamma}(bd) \cap S_{\alpha\beta\gamma}e_{\alpha\beta\gamma}(t_2d) & \\ & = & S_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}b)d \cap S_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}t_2)d & \\ & = & S_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}b)e_{\alpha\beta\gamma}d \cap S_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}t_2)e_{\alpha\beta\gamma}d & \\ & = & (S_{\alpha\beta\gamma}e_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}e_{\alpha\beta\gamma}t_2)e_{\alpha\beta\gamma}d & \text{by Lemma 3.9} \\ & = & (S_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}t_2)e_{\alpha\beta\gamma}d & \\ & = & S_{\alpha\beta\gamma}w_3e_{\alpha\beta\gamma}d & \text{by (3.2)} \\ & = & S_{\alpha\beta\gamma}w_3d. & \end{array}$$

Hence $w_1 \,\mathcal{L}\, w_3d$ in $S_{\alpha\beta\gamma}$.
The proof of (ii) is similar.

□

Lemma 3.12. *The multiplication introduced above is associative.*

Proof. Suppose that $a^{-1}b \in \Sigma_\alpha, c^{-1}d \in \Sigma_\beta$ and $s^{-1}t \in \Sigma_\gamma$ where $a, b \in S_\alpha, c, d \in S_\beta$ and $s, t \in S_\gamma$. From Lemma 3.11, we have that

$$\begin{array}{rcl} a^{-1}b \cdot [c^{-1}d \cdot s^{-1}t] & = & a^{-1}b \cdot [(c^{-1}e_\beta e_\beta d).s^{-1}t] \\ & = & a^{-1}b \cdot [(c^{-1}e_\beta.e_\beta d) \cdot s^{-1}t] \\ & = & a^{-1}b \cdot [c^{-1}e_\beta \cdot (e_\beta d.s^{-1}t)] \\ & = & (a^{-1}b \cdot c^{-1}e_\beta) \cdot (e_\beta d.s^{-1}t) \\ & = & [(a^{-1}b \cdot c^{-1}e_\beta).e_\beta d] \cdot s^{-1}t \\ & = & [a^{-1}b \cdot (c^{-1}e_\beta.e_\beta d)] \cdot s^{-1}t \\ & = & [a^{-1}b \cdot c^{-1}d] \cdot s^{-1}t. \end{array}$$

□

To see that S is a subsemigroup of Q we need only to show that the multiplication on Q extends the multiplication on S. Let $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. By Lemmas 3.1 and 3.10 ,

$$\begin{array}{rcl} e_\alpha a e_\beta b & = & e_\alpha^{-1}a \cdot e_\beta^{-1}b \\ & = & (e_{\alpha\beta}e_\alpha)(ab) \\ & = & e_{\alpha\beta}(ab) = ab. \end{array}$$

and we get the following lemma;

Lemma 3.13. *The multiplication on Q extends the multiplication on S.*

It is clear that Q is an inverse semigroup. Hence S is a left I-order in $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$. By Lemma 3.1, $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of S. It is easy to see that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. This concludes the proof of Theorem 3.7.

□

The following corollary is immediate.

Corollary 3.14. *With the notation in Theorem 3.7 S is a straight left I-order in Q.*

We aim now to prove the converse of Theorem 3.7. Let Q be a left regular band Y of bisimple inverse monoids Q_α, (with identity e_α) such that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. Let S_α be the $\mathcal{R}$-class of the identity e_α in Q_α. By Theorem 3.1 of [2], S_α is a right cancellative monoid with (LC). Put $S = \bigcup_{\alpha \in Y} S_\alpha$. It is clear that S is a left regular band Y of right cancellative monoids S_α, $\alpha \in Y$. We show that whenever $\beta \preceq_l \alpha$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, then $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$.

The following easy lemmas will be useful.

Lemma 3.15. *Let $\alpha, \beta \in Y$. Then $e_\alpha e_\beta = e_{\alpha\beta}$ and $e_{\alpha\beta} q_\alpha = q_\alpha e_\beta$ for any $q_\alpha \in Q_\alpha$.*

Proof. Since $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q and $e_\alpha e_\beta \in Q_{\alpha\beta}$, it follows that $e_\alpha e_\beta$ is the identity element of $Q_{\alpha\beta}$. Thus $e_\alpha e_\beta = e_{\alpha\beta}$.

Since $e_{\alpha\beta} q_\alpha \in Q_{\alpha\beta}$ and $e_{\alpha\beta} = e_\alpha e_\beta$ is the identity of $Q_{\alpha\beta}$ we have

$$\begin{aligned} e_{\alpha\beta} q_\alpha = (e_{\alpha\beta} q_\alpha) e_{\alpha\beta} = e_{\alpha\beta}(q_\alpha e_{\alpha\beta}) &= q_\alpha e_{\alpha\beta} \\ &= q_\alpha (e_\alpha e_\beta) = (q_\alpha e_\alpha) e_\beta = q_\alpha e_\beta. \end{aligned}$$

□

Lemma 3.16. *(cf. [13, Lemma X.1.3]) Let Q be a bisimple inverse monoid and let R be the $\mathcal{R}$-class of the identity. For any $a, b, c \in R$,*

$$Ra \cap Rb = Rc \text{ if and only if } a^{-1}ab^{-1}b = c^{-1}c.$$

Returning to our argument before Lemma 3.15. Let $S_\alpha a \cap S_\alpha b = S_\alpha c$ where $a, b, c \in S_\alpha$. Then, we have that $a^{-1}ab^{-1}b = c^{-1}c$, by Lemma 3.16. We claim that $(e_\beta a)^{-1}(e_\beta a)(e_\beta b)^{-1}(e_\beta b) = (e_\beta c)$ where $\beta \preceq_l \alpha$. It is clear that $e_\beta a, e_\beta b, e_\beta c \in S_\beta$. Using Lemmas 3.15 and 3.16

we have

$$\begin{aligned}(e_\beta a)^{-1}(e_\beta a)(e_\beta b)^{-1}(e_\beta b) &= a^{-1}e_\beta e_\beta ab^{-1}e_\beta e_\beta b\\ &= a^{-1}e_\beta ab^{-1}e_\beta b\\ &= e_{\alpha\beta}a^{-1}ae_{\alpha\beta}b^{-1}b\\ &= e_{\alpha\beta}a^{-1}ab^{-1}b\\ &= e_{\alpha\beta}c^{-1}c\\ &= c^{-1}e_\beta c = (e_\beta c)^{-1}(e_\beta c).\end{aligned}$$

Hence our claim is established. By the above lemma $S_\beta e_\beta a \cap S_\beta e_\beta b = S_\beta e_\beta c$ where $\beta \preceq_l \alpha$, so that $S_\beta a \cap S_\beta b = S_\beta c$, as required.

Theorem 3.17. *Let Q be a left regular band Y of bisimple inverse monoids Q_α, (with identity e_α) such that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. Then there is a subsemigroup S of Q which is a left regular band of right cancellative monoids S_α where S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α and for any $\beta \preceq_l \alpha$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, then $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$. Moreover, S is a left I-order in Q.*

Combining Theorems 3.7, 3.17 and Lemma 3.3 , we get the following corollary.

Corollary 3.18. *Let S be a left regular band Y of right cancellative monoids S_α with identity e_α, such that each S_α has the (LC) condition, satisfying (C). For each $\alpha \in Y$, let Q_α be the inverse hull of S_α, so that Q_α is a bisimple inverse monoid, and S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α. Then S is a left I-order in Q where Q is a left regular band Y of bisimple inverse semigroups.*

Conversely, let Q be a left regular band Y of bisimple inverse monoids Q_α, with identity e_α, such that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup. Then there is a subsemigroup S of Q with the (LC) condition which is a left regular band of right cancellative monoids, such that S is a left I-order in Q.

Corollary 3.19. [6] *(cf.* [10, Main Theorem]*) Let $S = \mathcal{S}(Y; S_\alpha)$ be a semilattice Y of right cancellative monoids S_α with identity e_α, such that each S_α has (LC). Suppose in addition that for any $\alpha \geq \beta$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, then $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$. For each $\alpha \in Y$, let Q_α be the inverse hull of S_α, so that Q_α is a bisimple inverse monoid, and S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α. Then $Q = \mathcal{S}(Y; Q_\alpha)$ is a semigroup of left I-quotients of S, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup.*

Conversely, let $Q = \mathcal{S}(Y; Q_\alpha)$ be a semilattice Y of bisimple inverse monoids Q_α, with identity e_α, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup. Then $S = \mathcal{S}(Y; R_{e_\alpha})$ is a semilattice of right cancellative monoids R_{e_α}, such that each R_{e_α} has (LC) and for any $\alpha \geq \beta$, if $R_{e_\alpha} a_\alpha \cap R_{e_\alpha} b_\alpha = R_{e_\alpha} c_\alpha$, then $R_{e_\beta} a_\alpha \cap R_{e_\beta} b_\alpha = R_{e_\beta} c_\alpha$.

As mentioned in section 2, any right cancellative monoid is a left ample semigroup. Hence, using Lemma 4.1 and Theorem 4.3 of [6] and Corollary 3.4 we have

Corollary 3.20. *With the notation in Corollary 3.19, S has the (LC) condition and Q is isomorphic to the inverse hull of S .*

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