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G-WEIGHTS AND p-LOCAL RANK

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Abstract. Let k be field of characteristic p, and let G be any finite group with splitting field k. Assume that B is a p-block of G. In this paper, we introduce the notion of radical B-chain C_B, and we show that the p-local rank of B is equals the length of C_B. Moreover, we prove that the vertex of a simple kG-module S is radical if and only if it has the same vertex of the unique direct summand, up to isomorphism, of the Sylow permutation module whose radical quotient is isomorphic to S. Finally, we prove the vertices of certain direct summands of the Sylow permutation module are bounds for the vertices of simple kG-modules.

1. Introduction

Let k be field of characteristic p, and G a finite group with splitting field k. The p-subgroup Q of G is a radical p-subgroup if $O_p(N_G(Q)) = Q$. Assume that B is a p- block of G.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n$$

of G, define $|C| = n$, the i-th subchain $C_i : P_0 < P_1 < \cdots < P_i$, and

$$N(C) = N_G(C) = N_C(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n).$$

The chain C is radical if it satisfies the following two conditions:

(a): $P_0 = O_p(G)$,
(b): $P_i = O_p(N(C_i))$ for $1 \leq i \leq n$.

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Denote by $\mathfrak{R} = \mathfrak{R}(G)$ the set of all radical p-chains of G. Let B be a p-block and let D be a defect group of B. The p-local rank of B is the number

$$plr(B) = \max\{|C| : C \in \mathfrak{R}, C : P_0 < P_1 < \cdots < P_n \leq D\}.$$

In this paper, we introduce the notion of radical B-chain C_B, and we show that the p-local rank of B is equals the length of C_B. Other question that we study in this paper correspond to the radical vertices of simple modules. In [4] it is proved that vertices of simple kG-modules for p-solvable groups G are radical. However, when the group is not p-solvable this is not necessarily the case. There are at present very few examples known of simple modules whose vertices are not radical, and our purpose here is to restrict the places to search for further examples, whilst generalizing the results of [4]. We prove that the vertex of a simple kG-module S is radical if and only if it has the same vertex of the unique G-weight, up to isomorphism, whose radical quotient is isomorphic to S. In [9] it is proved that if Q is the vertex of a simple kG-module S which belongs to a block B, then we can write

$$Z(D) \leq Q \leq D,$$

where D is the defect group of B. This result can be interpreted as giving " bounds" for the vertices of simple kG-modules. Finally, in the present paper, we show that the vertices of the G-weights are better bounds for the vertices of the simple modules.

2. Preliminary Results

In this section, we study the connection between the isomorphism type of indecomposable projective kG-modules of kG and the isomorphism type of indecomposable kG-modules direct summands of $Ind_Q^G(k)$, being Q a p-subgroup of G.

Let G be a finite group with splitting field k, and let Q be a p-subgroup of G. Assume that $n = |G : Q|$ and let $X^+ = \{x_1, \ldots, x_n\}$ be a full set of representatives in G of the cosets in G/Q. Then $Ind_Q^G(k)$ is isomorphic to kGQ^+ as left kG-module, where $Q^+ = \{\sum_{x \in X^+} \alpha x \in kG\}$.

Set $X = \{x_i - x_i y, y \in Q - 1\}$. Then $I_Q(G)$ denotes the left ideal generated by X in kG. We claim that

$$rank_k(I_Q(G)) = |G : Q|(|Q| - 1).$$

Thus, we have

$$kG/I_Q(G) \cong kGQ^+, \tag{2.1}$$

as k-modules.

It is well known, that

$$kG = \bigoplus_{j=1}^{r} P_{S_j}^{\dim S_j}, \tag{2.2}$$

where P_{S_j} is the projective cover of the simple kG-module S_j and r is the number of conjugacy classes of p-regular elements of G.

From (2.2), the following holds

$$kGQ^+ = \bigoplus_{j=1}^{r} M_j^Q, \tag{2.3}$$

where $M_j^Q = P_{S_j}^{\dim S_j} / P_{S_j}^{\dim S_j} I_Q(G)$.

The following lemmas are easy but useful to our results.

Lemma 2.1. *Let G be a finite group with splitting field k of characteristic p, and let S be a simple kG-module with projective cover P_S. Assume that U is an indecomposable kG-module. Then $Soc(U)$ is a simple kG-module.*

Proof. Since U is an indecomposable kG-module $Rad(U)$ is the unique maximal proper submodule of U, therefore $U/Rad(U)$ is simple. By assumption, we also may assert that $U/Rad(U) \cong Soc(U)$. So we are done. □

Lemma 2.2. *Let G be a finite group with splitting field k of characteristic p, and let U be an indecomposable kG-module finitely generated. If $Rad(U) \neq 0$, then $Soc(U) \subseteq Rad(U)$.*

Proof. By assumption, we may write $U/Rad(U) \cong Soc(U)$. Moreover, from Nakayamas lemma, we may assert that $U \longrightarrow U/Rad(U)$ is an essential epimorphism. Since U is an indecomposable kG-module $Rad(U)$ is the unique maximal proper submodule of U, hence, we may assert that $U/Rad(U)$ is simple. So we are done. □

Lemma 2.3. *Let G be a finite group with splitting field k of characteristic p. Fixed $P \in Syl_p(G)$. We denote the Jacobson radical of kG by $J(G)$. Then $J(G) \subseteq I_P(G)$ if and only if $kG/I_P(G)$ is semisimple.*

Proof. Since every direct summand of $kG/I_P(G)$ is annihilated by $I_P(G)$, the result follows.

The converse implication is trivial. □

The following result is very useful for our main result.

Lemma 2.4. *Let G be a finite group with splitting field k of characteristic p, and let $O_p(G)$ be the largest normal p-subgroup of G. If*

$O_p(G) \neq 1$ then there exists at least one simple projective $k[G/O_p(G)]$-module with vertex $O_p(G)$.

Proof. Let us write Q for $O_p(G)$. It is well-known that $I_Q(G) \neq 0$ is a nilpotent ideal of kG. Furthermore, we have

$$kGQ^+ \cong k[G/Q], \tag{2.4}$$

as kG-modules. We claim that kG is the projective cover of $k[G/Q]$. Assume that $\alpha : kG \longrightarrow k[G/Q]$ is an essential epimorphism. Let $S_1, \ldots, S_r$ be a complete list of simple kG-modules with projective covers $P_{S_1}, \ldots, P_{S_r}$, respectively. From Lemma 2.2, we may assert that $SocP_{S_i} \subseteq RadP_{S_i}$ for all $i \in \{1, \ldots, r\}$. Therefore $kG/J(G) \cong SockG \cong Sock[G/Q]$ with $Soc(kG) \cong Soc(J(G))$. Since α is an essential epimorphism and $\alpha(Soc(J(G)$
$)) \cong Soc(J(G/Q))$, we deduce that $Soc(k[G/Q]) \not\subset Soc(J(G/Q))$. This imply that there is at last a $\alpha(P_{S_i})$ such that $Rad(\alpha(P_{S_i})) = 0$. Thus, from (2.4), the result follows. □

Many of the properties of the kG-modules with trivial source was studied by several authors. In particular, Okuyama's obtained the following result (See [14]).

Lemma 2.5. *Let S be a simple kG-module with vertex Q and trivial source. Then the Green correspondent $f(S)$ of S is a simple projective $k[N_G(Q)/Q]$-module.*

Other extremely important result, in such sense, was achieved by Alperin's (See [2]).

Lemma 2.6. *Let P be a Sylow p-subgroup of G. If W is a weight of G and U its Green correspondent, then U is a direct summand of kGP^+.*

3. G-Weights and its Main Properties

We will now give a description of the direct summands of a Sylow permutation module. Firstly, we will show the conditions under which M_j^P is a projective kG-module.

Theorem 3.1. *Let G be a finite group with splitting field k of characteristic p, and let S be a simple kG-module. Set $P \in Syl_p(G)$ fixed. Then M_j^P is a projective kG-module if and only if P_{S_j} is a block of defect zero.*

Proof. Let $J(G)$ be the Jacobson radical of kG. We to check two cases.
Case (1): $J(G) \subseteq I_P(G)$.
In this case, by Lemma 2.3, the assertion follows.

Case (2): $J(G) \not\subseteq I_P(G)$.

Assume that $M_j^P \cong P_{S_j}^l$ is a projective kG-module, where l is the multiplicity of P_{S_j} as direct summand of M_j^P. We show that P_{S_j} is a simple kG-module.

Since $I_P(G)$ is left ideal of kG, from (2.2), it follows that

$$I_P(G) = P_{S_1}^{\dim S_1} I_P(G) \oplus \cdots \oplus P_{S_r}^{\dim S_r} I_P(G). \tag{3.1}$$

By assumption, we have $P_{S_j}^l I_P(G) = 0$. Therefore, we deduce that $P_{S_j}^{\dim S_j} I_P(G)$ is a projective kG-module, where the multiplicity of P_{S_j} is equal to $\dim(S_j) - l$, i.e., we have

$$P_{S_j}^{\dim S_j} I_P(G) = P_{S_j}^{\dim(S_j)-l}.$$

Thus, we may assert that $P_{S_j} I_P(G)$ is a right indecomposable $I_P(G)$-module such that

$$(P_{S_j} I_P(G))^{\dim S_j} = P_{S_j}^{\dim(S_j)-l}. \tag{3.2}$$

We assume that $\alpha = \dim(P_{S_j} I_P(G))$ and $\beta = \dim(P_{S_j})$. According to (3.2), we way write the following equality:

$$\alpha \dim(S_j) = \beta(\dim(S_j) - l). \tag{3.3}$$

From (3.3), it follows that

$$\frac{\alpha}{\dim(S_j) - l} = \frac{\beta}{\dim S_j}. \tag{3.4}$$

We now claim that the equality (3.4) is true if and only if $\dfrac{\alpha}{\dim(S_j) - l} = \dfrac{\beta}{\dim S_j} = 1$. Hence, the following holds $\dim S_j = \dim P_{S_j}$. This completes the proof of this implication.

Conversely, combining the Lemmas 2.5 and 2.6, we deduce that P_{S_j} is a direct summand of M_j^P. Since the radical quotient of all direct summand of M_j^P is P_{S_j} the result follows. □

Remark 3.2. We observe that, according to last theorem, the kG-module M_j^P can be decompose as a direct sum of copy of a block defect zero or a direct sum where at least a direct summand is an indecomposable not projective kG-module.

Let Q be a p-subgroup of G such that $Q = O_p(N_G(Q))$. Such subgroups are called p-radical subgroups. We want to remark that the p-radical subgroups play an important role in the study of the global and local properties of finite groups. In this context the following result is very interesting.

Theorem 3.3. *Let G be a finite group with splitting field k of characteristic p, and let $P \in Syl_p(G)$ fixed. Then every indecomposable kG-module direct summand of kGP^+ has a radical vertex.*

Proof. Let $N_G(P)$ be the normalizer of P. According to the Green correspondence every indecomposable kG-module direct summand of $kGP^+ \cong Ind^G_{N_G(P)} Ind_P^{N_G(P)}(k)$ has vertex P or a vertex of the form $P \cap P^g, g \in G - N_G(P)$. Observe that if P is a normal subgroup of G then kGP^+ is semisimple, so every indecomposable kG-module direct summand of kGP^+ is a simple kG-module with vertex P. Therefore, we now consider the case where P is not a normal subgroup of G. Assume that U is an indecomposable kG-module with vertex $Q \leq P$, being $U \mid kGP^+$. We to check two cases.

- Case (1): $Q = 1$.
  If $Q = P \cap P^g = 1$ then $O_p(G) = 1$. Since $O_p(G)$ is a radical p-subgroup, the result follows.
- Case (2): $1 < Q < P$.
  In this case $Q = P \bigcap P^g$, for some $g \in G - N_G(P)$. Let $N_P(Q)$ be the normalizer of Q in the Sylow p-subgroup P. We claim that $P \bigcap N_G(Q) = N_P(Q)$ and $P^g \bigcap N_G(Q) = N_P^g(Q)$ are Sylow p-subgroups of $N_G(Q)$. Since $N_P(Q)$ and $N_P^g(Q)$ are conjugated in $N_G(Q)$, we deduce that Q is a tame intersection of $N_G(Q)$-conjugate Sylow p-subgroups of G. Thus, we can write $Q = N_P(Q) \cap N_P^g(Q)$ with $g \in N_G(Q)$. Hence, we have
  $$O_p(N_G(Q)) \leq Q. \tag{3.5}$$
  Since Q is a normal p-subgroup, we can write
  $$Q \leq O_p(N_G(Q)). \tag{3.6}$$
  Combining (3.5) and (3.6), it follows that $Q = O_p(N_G(Q))$.

□

In the rest of this paper, we will assume the notations and terminologies used in the last section.

Theorem 3.4. *Let G be a finite group, k be a splitting field for G and $P \in Syl_p(G)$. If M_j^P has an indecomposable non-projective kG-module as direct summand, then it is unique, up to isomorphism.*

Proof. By the Krull-Schmidt theorem, each left kG-module M_j^P can be decomposed, of unique manner, as a direct sum of indecomposable

kG-modules,i.e., we may write

$$M_j^P = \bigoplus_{\gamma=1}^{\mu} U_\gamma, \tag{3.7}$$

where U_γ is an indecomposable kG-module.

We now assume that U_γ is an indecomposable non-projective kG-module such that $U_\gamma \mid M_j^P$. Let P_{S_j} be the projective cover of the simple kG-module S_j, which is isomorphic to the radical quotient $U_\gamma/Rad(U_\gamma)$. We show that P_{S_j} is the projective cover of U_γ.

By Nakayama's lemma, we may assert that there exists two essential epimorphisms $h_1 : P_{S_j} \to P_{S_j}/Rad(P_{S_j})$ and $h_2 : U \to U/Rad(U)$. Since $P_{S_j}/Rad(P_{S_j}) \cong U_\gamma/Rad(U_\gamma) \cong S_j$ and P_{S_j} is projective, we deduce that there is an essential epimorphism $\theta_1 : P_{S_j} \to U_\gamma$ such that $h_1 = h_2 \circ \theta_1$. We now show that U_γ is unique, up to isomorphism.

Suppose that $U_{\gamma'}$ is other indecomposable non-projective kG-module in the decomposition (3.7). Since P_{S_j} is projective cover of U_γ and $U_{\gamma'}$, we deduce that there are two essential epimorphisms $\theta_1 : P_{S_j} \to U_\gamma$ and $\theta_2 : P_{S_j} \to U_{\gamma'}$. We define the homomorphism $\sigma : U_\gamma \to U_{\gamma'}$ given by $\sigma(\theta_1(a)) = \theta_2(a), a \in P_{S_j}$. Let $\Omega(U_\gamma)$ and $\Omega(U_{\gamma'})$ be the Heller operators of U_γ and $U_{\gamma'}$, respectively. Then, we can write

$$\Omega(U_\gamma) \cap \Omega(U_{\gamma'}) = 0. \tag{3.8}$$

Since θ_1 and θ_2 are essential epimorphisms, we have

$$\sigma(U_\gamma) = U_{\gamma'}.$$

We show that σ is injective. Clearly, we may see that

$$\ker \sigma = \{\theta_1(a) \in U_\gamma : a \in \Omega(U_{\gamma'})\}.$$

Therefore, from (3.8), we may assert that $\ker \sigma \cong \Omega(U_{\gamma'})$. Thus, we have

$$P_{S_j}/\Omega(U_{\gamma'}) \cong U_\gamma/\ker \sigma \cong U_{\gamma'}. \tag{3.9}$$

From (3.9), we deduce that $P_{S_j} \cong U_\gamma$ or $\ker \sigma = 0$. By assumption, we have $P_{S_j} \ncong U_\gamma$, therefore $\ker \sigma = 0$. Hence $U_\gamma \cong U_{\gamma'}$, which is what we need to prove. □

We now introduce the notion of G-weight.

Definition 3.5. A G-weight for G is a pair (Q, U), where U is an indecomposable kG-module direct summand of kGP^+ with vertex Q, which is simple or non-projective kG-module.

Remark 3.6. Strictly speaking, we can say that the pair (Q, U_S) is the G-weight, but we may also refer to the kG-module U_S as a G-weight.

In terms of G-weights, the following result is a consequence immediate of Theorem 3.4.

Corollary 3.7. *Let G be a finite group with splitting field k of characteristic p. Then the direct summands of kGP^+ are G-weights or non-simple projective kG-modules.*

Proof. If M_j^P is projective then M_j^P can be write as a direct sum of copies of a simple projective kG-module by Theorem 3.1. If M_j^P is not projective then, by Theorem 3.4, we may assert that M_j^P can be decomposed as a direct sum, whose direct summands are copies of a G-weight U or copies of the projective cover of U. So we are done. □

Remark 3.8. We observe that if P_S is the projective cover of the simple kG-module S, then $P_S I_P(G)$ is equal to $0, P_S$ or $\Omega(U_S)$, where U_S is a G-weight such that $U_S/Rad(U_S) \cong S$. If $J(G) \subseteq I_P(G)$ then $P_S I_P(G)$ is equal to 0 or $\Omega(S)$.

It is well known that the number of non-isomorphic simple kG-modules equals the number of conjugacy classes of p-regular elements of G. The following result establish that the G-weights also satisfy this condition.

Theorem 3.9. *Let G be a finite group with splitting field k of characteristic p. Then the number of non-isomorphic G-weights equals the number of conjugacy classes of p-regular elements of G.*

Proof. From (2.3), we have

$$kGP^+ = \bigoplus_{j=1}^{r} M_j^P,$$

where r is the number of conjugacy classes of p-regular elements of G and M_j^P is a left kG-module such that

$$M_j^P \cong P_{S_j}^{\dim S_j} / P_{S_j}^{\dim S_j} I_P(G),$$

for some simple kG-module S_j.

We check two cases.

Case (1): M_j^P is projective.

According to Theorem 3.1, we have $M_j^P = \bigoplus P_{S_j}$, being P_{S_j} a block of defect zero. Therefore, by assumption, we may assert that $(P_{S_j}, 1)$ is a G-weight.

Case (2): M_j^P is non-projective.

By Theorem 3.4, we may assert that M_j^P has a unique direct summand U (up to isomorphism), which is an indecomposable non-projective

kG-module with vertex Q. Thus, (Q, U) is a unique G-weight, which proves what we want. □

Finally, in this section, we give some conditions that can be used in the classification of the p-radical subgroups of any finite group.

Theorem 3.10. *Let G be a finite group with splitting field k of characteristic p, and let $Q \neq 1$ be a p-subgroup of G. Then Q is p-radical if and only if it is a vertex of a G-weight.*

Proof. By assumption, we can write $Q = O_p(N_G(Q))$. Thus, according to Lemma 2.4, we may assert that there exist at least one simple projective $kN_G(Q)/Q$-module W with vertex Q. We denote the Green correspondent of W by U. Combining Lemma 2.6 and Corollary 3.7, we deduce that U is a G-weight, which is our assertion.

The converse implication follows by Theorem 3.3. □

4. G-weight and p-Local Rank

Let G be a finite group with splitting field k of characteristic p, and let B be a p-block of G with defect group D. We denote the set of all the vertices of the G-weights that belongs to B by $\mathbb{V}(B)$. Set $V(D) = \{O_p(G)\} \cup \mathbb{V}(B)\}$. We define the radical p-chain

$$C_B : P_0 < P_1 < \cdots < P_n,$$

where $P_i \in V(D)$(for all $i(0 < i < n)$), $P_n = D$ and $|C_B| = |\mathbb{V}(B)|$. In this case, we say that C_B is a GB-weight chain of B. We claim that the GB-weight chain of B is unique up to conjugacy.

The next following result is an extremely important toll for handling the radical vertex of indecomposable modules.

Theorem 4.1. *Let G be a finite group with splitting field k of characteristic p, and let B be a a p-block of G with defect group D. Then $plr(B) = |C_B|$.*

Proof. We denote the Brauer correspondent of B by b. We distinguish two cases.

Case I: $D = 1$.

In such case B is block of defect zero, so the result follows by assumption.

Case II: $D \neq 1$.

Suppose that $\tilde{Q} \leq D$ is a radical p-subgroup of G. Firstly, we show that there is a weight with vertex $\tilde{Q}$ which belongs to b. By assumption, we can write

$$N_G(\tilde{Q}) \geq N_G(D) \tag{4.1}$$

Since $\tilde{Q}$ is radical, we have $\tilde{Q} = O_p(N_G(\tilde{Q}))$. Thus, applying the Lemma 2.4, we may assert that there is at least one simple $N_G(\tilde{Q})$-module W with vertex $\tilde{Q}$ and trivial source. Therefore $(\tilde{Q}, W)$ is a weight of G. From (4.1), we deduce that $(\tilde{Q}, W)$ belongs to b. We now prove that the Green correspondent of W is a G-weight. We denote the weight Green correspondent of W by U. According to Lemma 2.6, we may assert that U is a direct summand of kGP^+, being $P \in Syl_p(G)$. Since U is not a projective kG-module, we deduce that U is a G-weight by Corollary 3.7. We have $bW = W$, which implies that $BU = U$ by first main theorem. So we are done.

□

In this context we obtained the following result for simple modules.

Theorem 4.2. *Let S be a simple kG-module with vertex Q_S and G-weight (Q, U_S). Then Q_S is radical if and only if $Q_S = Q$.*

Proof. By assumption, we can write

$$k[N_G(Q_S)]Q_S^+ \cong k[N_G(Q_S)]/Q_S, \tag{4.2}$$

as $kN_G(Q_S)$-modules. Let S^{Q_S} be the fixed point module under the action of Q_S on S. We claim that $I_{Q_S}(N_G(Q_S))$ is an annihilator of S^{Q_S}. This, assets that S^{Q_S} is a direct summand of $k[N_G(Q_S)]Q_S^+$. Hence, from (4.2), we may deduce that S^{Q_S} is an indecomposable projective $k[N_G(Q_S)]/Q_S$-module which has vertex Q_S and trivial source. We denote the Green correspondent of U_S by $f(U_S)$. By Theorem 3.3, we may asset that Q is a radical p-subgroup. Therefore, since $k[N_G(Q)]Q^+ \cong k[N_G(Q)]/Q$, we can assert that $f(U_S)$ is an indecomposable projective $k[N_G(Q)]/Q$-module with vertex Q and trivial source. Let S^Q be the fixed point module under the action of Q on S. Since $I_Q(N_G(Q))$ is an annihilator of S^Q, we can deduce that S^Q is an indecomposable $k[N_G(Q)]$-module with vertex Q and trivial source. From Lemma 2.1, it follows that S^Q is a direct summand of $Res^G_{N_G(Q)}(U_S)$. Therefore

$$S^Q = f(U_S) \tag{4.3}$$

holds.

By assumption, the last theorem implies that it suffice to check only two cases.

Case I: $Q \leq Q_S$.

By assumption, we can write

$$S^{Q_S} \leq S^Q. \tag{4.4}$$

From Lemma 2.1, we may assert that S^{Q_S} is the unique direct summand of $Res^G_{N_G(Q_S)}(U_S)$ with vertex Q_S and trivial source. Hence, combining (4.3) and (4.4), we can see immediately that $S^Q = S^{Q_S}$. This implies that the equality $Q = Q_S$ holds.

Case II: $Q_S \leq Q$.

In this case we have

$$S^Q \leq S^{Q_S}. \tag{4.5}$$

Since S^{Q_S} is the unique direct summand of $Res^G_{N_G(Q_S)}(U_S)$, with vertex Q_S and trivial source, and $N_G(Q) \leq N_{Q_S}$, we conclude that the result follows combining (4.3) and (4.5).

The converse implication follows applying again the Theorem 3.3. □

In the following result, we show that the vertices of the G-weights are bounds for the vertices of simple kG-modules.

Theorem 4.3. *Let S be a simple kG-module with vertex Q_S and G-weight (Q, U_S). Then we have*

$$Z(Q) \leq Q_S \leq Q.$$

Proof. Suppose that Q_S is not a radical p-subgroup. We show that $Q_S < Q$. Assume that U_S belongs to the p-block B of G. We write b for the Brauer correspondent of B. Moreover, we denote the Green correspondent of S by $f(S)$. Since S is the radical quotient of U_S, we deduce that $f(S)$ lies in b by Brauer morphism. Let D be a defect group of B. It is well known that

$$N_G(D) \leq N_G(Q). \tag{4.6}$$

Thus, we claim that b is a p-block of $N_G(Q)$. By Theorem 3.3, we can write $O_p(N_G(Q)) = Q$. This implies that b lies in $k[C_{N_G(Q)}(Q)]$ and is the sum of $N_G(Q)$-orbit of blocks of $C_{N_G(Q)}(Q)$. Hence, we may assert that $f(S)$ is a $kN_G(Q)$-module. Moreover, we also have $N_G(Q_S) \leq N_G(Q)$. Thus, by assumption, we can write

$$Q_S < Q. \tag{4.7}$$

According to Theorem 4.2, we may assert that if Q_S is radical then we have

$$Q_S = Q. \tag{4.8}$$

Therefore, combining (4.7) and (4.8) we obtain the inequality

$$Q_S \leq Q, \tag{4.9}$$

as general case. Now, from (4.9), it follows that $Z(Q) \leq Z(Q_S)$. Hence $Z(Q) \leq Q_S$. So we are done. □

Remark 4.4. Observe that if Q is an abelian group then Q_S is radical.

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