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σ-SPORADIC PRIME IDEALS AND SUPERFICIAL ELEMENTS

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Abstract. Let A be a Noetherian ring, I be an ideal of A and σ be a semi-prime operation, different from the identity map on the set of all ideals of A. Results of Essan proved that the sets of associated prime ideals of $\sigma(I^n)$, which denoted by $Ass(A/\sigma(I^n)$, stabilize to $A_\sigma(I)$. We give some properties of the sets $S_n^\sigma(I) = Ass(A/\sigma(I^n)\backslash A_\sigma(I)$, with n small, which are the sets of σ-sporadic prime divisors of I. We also give some relationships between $\sigma(f_I)$-superficial elements and asymptotic prime σ-divisors, where $\sigma(f_I)$ is the σ-closure of the I-adic filtration $f_I = (I^n)_{n\in\mathbb{N}}$.

1. Introduction

Let A be a commutative Noetherian ring and I be a regular ideal of A. A prime ideal $P \subset A$ is an associated prime of I if there exists an element x in A such that $P = (I :_A x)$. The set of associated primes of I, denoted $Ass(A/I)$, is the set of all prime ideals associated to I. A well-known result of Brodmann [2] proved that the sets of associated prime ideals of I^n, which denote by $Ass(A/I^n)$, stabilize to $A^*(I)$, that is, there exists a positive integer n_0 such that $Ass(A/I^n) = Ass(A/I^{n_0})$ for all $n \geq n_0$. For small n it may happen that there are prime ideals P with $P \in Ass(A/I^n) \setminus A^*(I)$. Such a prime is called a sporadic prime divisor of I. In [7], MacAdam gave some properties of sporadic prime of regular ideals.

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Now let us assume that I is an ideal of A, which is not necessarily regular. Let σ be a semi-prime operation on the set $\mathcal{I}(A)$ of all ideals of A, with $\sigma \neq id_{\mathcal{I}(A)}$. A result of Essan [3] proves that the sequence $(Ass(A/\sigma(I^n))_{n\in\mathbb{N}^*}$ stabilize to a set denoted $A_\sigma(I)$, that is $Ass(A/\sigma(I^n) = A_\sigma(I)$ for all large n. For small n it may happen that there are prime ideals P with $P \in Ass\big(A/\sigma(I^n)\big) - A_\sigma(I)$. Such a prime is called a σ-sporadic prime divisor of I. For all integer $n \geq 1$, we put $\mathcal{S}_n^\sigma(I) = Ass\big(A/\sigma(I^n)\big) - A_\sigma(I)$ and $\mathcal{S}^\sigma(I) = \cup_{n\in\mathbb{N}^*}\mathcal{S}_n^\sigma(I)$, that is $\mathcal{S}^\sigma(I)$ is the set of all σ-sporadic prime of I. Moreover, Essan [4] proves that the sequence $\left(Ass(A/(I^n)_\sigma)\right)_{n\in\mathbb{N}^*}$, with $(I^n)_\sigma = \sigma(I^{k+n}) :_A \sigma(I^k), \quad k \gg 0$ is an increasing sequence.
In section 3, we are interested in the σ-sporadic prime of an ideal I of a ring A. We prove that for all integer $n \geq 1$, $\mathcal{S}_n^\sigma(I) \subseteq Ass\big((I^n)_\sigma/\sigma(I^n)\big)$ (cf. Theorem 3.4). We will also prove a generalization of [9], Lemma 2.5. and a generalization of [9], 4.15.
In section 4, we suppose that $(A, \mathcal{M})$ is a Noetherian local ring with infinite residue field. We put $\sigma(f_I) = (\sigma(I^n))_{n\in\mathbb{N}}$, which is the σ-closure of the I-adic filtration $f_I = (I^n)_{n\in\mathbb{N}}$. An element $x \in I$ is said to be $\sigma(f_I)$-superficial if there exists an integer n_0 such that $\big(\sigma(I^{n+1}) :_A x\big) \cap \sigma(I^{n_0}) = \sigma(I^n)$, for all $n \geq n_0$. Let I be an $\mathcal{M}$-primary ideal of the ring A. We prove that if $x \in I$ is a $\sigma(f_I)$-superficial element, then for all $n \geq 1$ we have (i) $\big((I^{n+1})_\sigma : x\big) = (I^n)_\sigma$, (ii) $(x) \cap (I^{n+1})_\sigma = x(I^n)_\sigma$ (Proposition 4.2). It follows that $\sigma(I^{k+1}) : x = \sigma(I^k)$ and $\sigma(I^{n+1}) : I = \sigma(I^n)$, for all $k \geq \rho_\sigma^I(A)$, with $\rho_\sigma^I(A) = min\{n \,|\, (I^i)_\sigma = \sigma(I^i) \text{ for all } i \geq n\}$ (Corollary 4.3 and Theorem 4.6).

2. Preliminary

Throughout this paper the letter A will denote a commutative ring with identity.
(1) A filtration on the ring A is a sequence $f = (I_n)_{n\in\mathbb{N}}$ of ideals of A such that $I_0 = A$, $I_{n+1} \subseteq I_n$ and $I_nI_m \subseteq I_{n+m}$ for all n, $m \in \mathbb{N}$.

Definition 2.1. [5]
Let $\mathcal{I}(A)$ be the set of all ideals of a ring A. We consider the following properties of a map $\sigma : \mathcal{I}(A) \longrightarrow \mathcal{I}(A)$:

(a) $I \subseteq \sigma(I)$ for all $I \in \mathcal{I}(A)$

(b) if $I \subseteq J$ then $\sigma(I) \subseteq \sigma(J)$ for all $I, J \in \mathcal{I}(A)$

(c) $\sigma(\sigma(I) = \sigma(I)$

(d) $\sigma(I)\sigma(J) \subseteq \sigma(IJ)$,

(e) $\sigma(b\,I) = b\,\sigma(I)$ for all regular element $b \in I$

Then σ is a semi-prime operation on $\mathcal{I}(A)$ if $(a) - (d)$ hold for all $I, J \in \mathcal{I}(A)$; it is a prime operation if $(a) - (e)$ hold for all $I, J \in \mathcal{I}(A)$ and any regular element b of A.

It follows from (d) of Definition 2.1 that $\sigma(\sigma(I)\sigma(J)) = \sigma(IJ)$ for all $I, J \in \mathcal{I}(A)$.
(2) If $f = (I_n)_{n\in\mathbb{N}}$ is a filtration on the ring A and σ is a semi-prime operation on $\mathcal{I}(A)$ then $\sigma(f) = \big(\sigma(I_n)\big)_{n\in\mathbb{N}}$ is a filtration on A.
(3) Let I be an ideal of A. A filtration $f = (I_n)_{n\in\mathbb{N}}$ on A is said to be I-good if $I.I_n \subseteq I_{n+1}$ for all $n \geq 0$ and there exists $n_0 \in \mathbb{N}$ such that $\forall\, n \geq n_0$, $I.I_n = I_{n+1}$. It follows that $I^n I_{n_0} = I_{n_0+n}$, $\forall\, n \geq 1$.
(4) Let $(A, \mathcal{M})$ be a Noetherian local ring with infinite residue field $A/\mathcal{M}$ and $f = (I_n)_{n\in\mathbb{N}}$ be an I-good filtration on A. An element $x \in I$ is said to be f-superficial if there is an integer n_0 such that $\big(I_{n+1} :_A x\big) \cap I_{n_0} = I_n$ for all $n \geq n_0$.

3. σ-sporadic prime of an ideal

Throughout this section A is a Noetherian ring, I is a nonzero ideal in A and σ is a semi-prime operation on $\mathcal{I}(A)$.

Let $S \subset A$ be a multiplicative set, that is, suppose that $1_A \in S$ and $xy \in S$ for all $x, y \in S$. An ideal I of A is said to be *satured* with respect to S (or S-*satured*) in A if for all $(a, s) \in A \times S$ such that $as \in I$ we have $a \in I$. Let us put $I_{sat} = \{a \in A/\, ab \in I \text{ for some } b \in S\}$. Then I_{sat} is a S-satured ideal of A. It is the intersection of all S-satured ideal of A containing I. It is obvious that $I_{sat} = \cup_{s\in S}(I : s)$ and I is a S-satured ideal in A if and only if $I = I_{sat}$.
Let $S^{-1}A$ be the ring of fractions of A with respect to S. We put

$$I^e = \{\frac{a}{s} \in S^{-1}A \,/\, a \in I,\, s \in S\},$$

which is called the *extension* of the ideal I to $S^{-1}A$. For any ideal J of $S^{-1}A$ we put

$$J^c = \{a \in A \,/\, \frac{a}{1} \in J\}.$$

This is called the *contracted* ideal of J.
In these notations, the inclusions $I \subseteq I^{ec}$ and $J^{ce} \subseteq J$ follows immediately from the definitions. From the first inclusion we get $I^e \subseteq I^{ece}$, but substituting $J = I^e$ in the second gives $I^{ece} \subseteq I^e$, and hence

$$I^{ece} = I^e, \quad \text{and similary} \quad J^{cec} = J^c$$

Remark 3.1. Let I be an ideal of the ring A. Then we have $I_{sat} = I^{ec}$.

Indeed, let $a \in I^{ec}$. We have $\frac{a}{1} \in I^e$. There exist $b \in I$ and $s \in S$ such that $\frac{a}{1} = \frac{b}{s}$, that is, there exists $u \in S$ such that $u(as - b) = 0$, hence $usa = ub$ with $ub \in I$ and $us \in S$. It follows that $a \in I_{sat}$ and $I^{ec} \subseteq I_{sat}$. Conversely let $a \in I_{sat}$. There exists $s \in S$ such that $as \in I$, hence $\frac{as}{1} \in I^e$. Since $\frac{1}{s} \in S^{-1}A$, we have $\frac{a}{1} = \frac{1}{s}\frac{as}{1} \in I^e$, thus $a \in I^{ec}$ and $I_{sat} \subseteq I^{ec}$, therefore $I_{sat} = I^{ec}$.

Proposition 3.2. *The map $\sigma : \mathcal{I}(A) \longrightarrow \mathcal{I}(A)$, $I \longmapsto \sigma(I) = I_{sat}$ is a semi-prime operation on $\mathcal{I}(A)$.*

Proof. (i) (a), (b), (c) of Definition 2.1 follow immediately from the definition of S-satured ideal.
(ii) Let $I, J \in \mathcal{I}(A)$ such that $I \subseteq J$. For all $a \in I_{sat}$, there exists $s \in S$ such that $as \in I$. Since $I \subseteq J$, $as \in J$, hence $a \in J_{sat}$. This proves that $I_{sat} \subseteq J_{sat}$.
(iii) Let $I, J \in \mathcal{I}(A)$. For all $a \in I_{sat}$ and $b \in J_{sat}$ there exist $s, u \in S$ such that $as \in I$ and $bu \in J$. It follows that $absu \in IJ$, with $su \in S$, hence $ab \in (IJ)_{sat}$ and $I_{sat}J_{sat} \subseteq (IJ)_{sat}$. □

Lemma 3.3. *Let P be a prime ideal of the ring A and $A_P = S^{-1}A$ with $S = A \backslash P$. Then the map $\sigma_P : \mathcal{I}(A_P) \longrightarrow \mathcal{I}(A_P)$, $IA_P \longmapsto I_{sat}A_P$ (where $I \in \mathcal{I}(A)$) is a semi-prime operation on $\mathcal{I}(A_P)$.*

Proof. We put $\sigma(I) = I_{sat}$ for all ideal I of A. Let us first prove that σ_P is well-defined. Indeed, let $I, J \in \mathcal{I}(A)$ such that $IA_P = JA_P$, that is $I^e = J^e$. Then we have $I^{ec} = J^{ec}$, so that $I_{sat} = J_{sat}$, hence $\sigma(I) = \sigma(J)$ and we have $\sigma(I)A_P = \sigma(J)(A_P)$, thus $\sigma_P(IA_P) = \sigma_P(JA_P)$.
We now prove that σ_P is a semi-prime operation on $\mathcal{I}(A_P)$.
(a) Let $IA_P \in \mathcal{I}(A_P)$. Since $I \subseteq \sigma(I)$, we have $IA_P \subseteq \sigma(I)A_P$.
(b) Let $IA_P \in \mathcal{I}(A_P)$. Since σ is a semi-prime operation, we have $\sigma_P[\sigma_P(IA_P)] = \sigma_P[\sigma(I)A_P] = \sigma(\sigma(I))A_P = \sigma(I)A_P$.
(c) Let $IA_P, JA_P \in \mathcal{I}(A_P)$ such that $IA_P \subseteq JA_P$, that is $I^e \subseteq J^e$. Then $I^{ec} \subseteq J^{ec}$. By remark 3.1, $I_{sat} \subseteq J_{sat}$, that is $\sigma(I) \subseteq \sigma(J)$. We have $\sigma(I)A_P \subseteq \sigma(J)A_P$, therefore $\sigma_P(IA_P) \subseteq \sigma_P(JA_P)$.
(d) $\sigma_P(IA_P)\sigma_P(JA_P) = \sigma(I)A_P\sigma(J)A_P = \sigma(I)\sigma(J)A_P \subseteq \sigma(IJ)A_P = \sigma_P(IJA_P) = \sigma_P(IA_PJA_P)$. □

Theorem 3.4. *Let A be a Noetherian ring and σ be a semi-prime operation on $\mathcal{I}(A)$. Suppose that for $P \in Spec(A)$, there is a semi-prime operation $\hat{\sigma}_P$ on $\mathcal{I}(A_P)$ such that $\hat{\sigma}_P(IA_P) = \sigma(I)A_P, \forall\, I \in \mathcal{I}(A)$. Then*

(i) *I_σ is σ-closed.*
(ii) *Let n and q be large enough integers such that for a nonzero ideal I in A, we have $(I^k)_\sigma = \sigma(I^{n+k}) : \sigma(I^n)$ and $\sigma(I^{q+1}) : I = \sigma(I^q)$ for all $k \geq 1$. Then $(I^{nq})_\sigma = \sigma(I^{nq})$ and $Ass\big(A/(I^{nq})_\sigma\big) = A_\sigma(I)$.*
(iii) *For every integer $n \geq 1$, $\mathcal{S}_n^\sigma(I) \subseteq Ass\big((I^n)_\sigma/\sigma(I^n)\big)$.*

Proof. (i) It is sufficient to prove that $\sigma(I_\sigma) \subseteq I_\sigma$. We have $I_\sigma = \sigma(I^{n+1}) : \sigma(I^n)$, since n is large enough. It follows that

$$\sigma(I_\sigma) = \sigma[\sigma(I^{n+1}) : \sigma(I^n)] \subseteq \sigma(\sigma(I^{n+1})) : \sigma(\sigma(I^n)) = \sigma(I^{n+1}) : \sigma(I^n)$$

and $\sigma(I^{n+1}) : \sigma(I^n) = I_\sigma$, (cf. [4], Proposition 3.3), hence $\sigma(I_\sigma) \subseteq I_\sigma$. Since σ is a semi-prime operation on $\mathcal{I}(A)$, $I_\sigma \subseteq \sigma(I_\sigma)$, thus $I_\sigma = \sigma(I_\sigma)$.
(ii) Let n and q be large enough integers such that for an ideal I of A, $I \neq \{0\}$, we have $(I^k)_\sigma = \sigma(I^{n+k}) : \sigma(I^n)$ and $\sigma(I^{q+1}) : I = \sigma(I^q)$, for all $k \geq 1$. It is obvious that $(I^{nq})_\sigma = \sigma(I^{n+nq}) : \sigma(I^n) = \sigma(I^{n(1+q)}) : \sigma(I^n)$. We put $J = I^n$, then $(J^q)_\sigma = \sigma(J^{q+1}) : \sigma(J)$. It follows that $\sigma(J)(J^q)_\sigma \subseteq \sigma(J^{q+1})$. Since $J \subseteq \sigma(J)$, we have $J(J^q)_\sigma \subseteq \sigma(J^{q+1})$ and $(J^q)_\sigma \subseteq \sigma(J^{q+1}) : J = \sigma(J^q)$, as q is large enough, thus $(I^{nq})_\sigma \subseteq \sigma(I^{nq})$. By [4], Proposition 3.2, $I^m \subseteq (I^m)_\sigma$ for all $m \geq 1$, hence $\sigma(I^m) \subseteq \sigma((I^m)_\sigma) = (I^m)_\sigma$ (we refer to (i)). It follows that $\sigma(I^m) \subseteq (I^m)_\sigma$, for all $m \geq 1$, in particular, $\sigma(I^{nq}) \subseteq (I^{nq})_\sigma$. Therefore $(I^{nq})_\sigma = \sigma(I^{nq})$ and $Ass\big(A/(I^{nq})_\sigma\big) = Ass\big(A/\sigma(I^{nq})\big) = A_\sigma(I)$.
(iii) Let $P \in \mathcal{S}_n^\sigma(I) = Ass\big(A/\sigma(I^n)\big) \setminus A_\sigma(I)$.
(a) Suppose that A is a local ring with maximal ideal P. There is $x \notin \sigma(I^n)$ such that $P = \sigma(I^n) : x$. Let us assume that $(I^n)_\sigma : x$ is a proper ideal of A. We have

$$P = \sigma(I^n) : x \subseteq (I^n)_\sigma : x \subseteq P$$

hence $(I^n)_\sigma : x = P$ and $P \in Ass\big(A/(I^n)_\sigma\big)$. Since $(Ass\big(A/(I^n)_\sigma\big))_{n\in\mathbb{N}^*}$ is an increasing sequence and stabilizes to $A_\sigma(I)$ (cf. [4]), $P \in A_\sigma(I)$. This contradicts the fact that $P \in \mathcal{S}_n^\sigma(I)$, thus $(I^n)_\sigma : x = A$ and $x \in (I^n)_\sigma$. It follows that $P \in Ass\big((I^n)_\sigma/\sigma(I^n)\big)$.
(b) Suppose that A is not a local ring with maximal ideal P. It is well-known that A_P is a local ring with maximal ideal PA_P. We have $PA_P \in Ass[A_P/\sigma(I^n)A_P]$ and $PA_P \notin Ass[A_P/\sigma(I^k)A_P]$, $k \gg 0$. That is, $PA_P \in Ass[A_P/\hat{\sigma}_P(I^nA_P)]$ and $PA_P \notin Ass[A_P/\hat{\sigma}_P(I^kA_P)]$, $k \gg 0$. Hence, $PA_P \in Ass[A_P/\hat{\sigma}_P(I^nA_P)] \setminus Ass[A_P/\hat{\sigma}_P(I^kA_P)]$, $k \gg 0$. By (a), we obtain $PA_P \in Ass[(I^nA_P)_{\hat{\sigma}_P}/\hat{\sigma}_P(I^nA_P)]$. We have

$$\begin{aligned}(I^nA_P)_{\hat{\sigma}_P} &= \hat{\sigma}_P(I^{n+k}A_P) :_{A_P} \hat{\sigma}_P(I^kA_P) = \sigma(I^{n+k})A_P :_{A_P} \sigma(I^k)A_P \\ &= [\sigma(I^{n+k}) :_A \sigma(I^k)]A_P, \quad k \gg 0\end{aligned}$$

The first equality follows immediately from the definition. Let us prove the second equality. Indeed, let $w \in [\sigma(I^{n+k}) :_A \sigma(I^k)]A_P$. There exist $\alpha \in \sigma(I^{n+k}) :_A \sigma(I^k)$ and $s \in S = A\backslash P$ such that $w = \frac{\alpha}{s}$. For every $v \in \sigma(I^k)A_P$ there is $y \in \sigma(I^k)$ and $t \in S$ such that $v = \frac{y}{t}$. We have $wv = \frac{\alpha}{s}\frac{y}{t} = \frac{\alpha y}{st}$ with $\alpha y \in \sigma(I^{n+k})$ and $st \in S$, hence $wv \in \sigma(I^{n+k})A_P$, therefore $w \in \sigma(I^{n+k})A_P :_{A_P} \sigma(I^k)A_P$ and

$$[\sigma(I^{n+k}) :_A \sigma(I^k)]A_P \subseteq \sigma(I^{n+k})A_P :_{A_P} \sigma(I^k)A_P.$$

Conversely, let $\frac{\alpha}{s} \in \sigma(I^{n+k})A_P :_{A_P} \sigma(I^k)A_P$ and $(\frac{y_1}{1}, ..., \frac{y_r}{1})$ be a finite system of generators of $\sigma(I^k)A_P$. For all $i = 1, \dots, r$ we have $\frac{\alpha}{s}\frac{y_i}{1} = \frac{\alpha y_i}{s} \in \sigma(I^{n+k})A_P$. Hence there exists $u_i \in S$ such that $u_i\,\alpha\, y_i \in \sigma(I^{n+k})$. We put $u = u_1u_2...u_r$. For all $i = 1, \dots, r$ we have $u\,\alpha\, y_i \in \sigma(I^{n+k})$, thus $\alpha u \in \sigma(I^{n+k}) :_A \sigma(I^k)$, it follows that $\frac{\alpha}{s} = \frac{\alpha u}{su} \in [\sigma(I^{n+k}) :_A \sigma(I^k)]A_P$ and $\sigma(I^{n+k})A_P :_{A_P} \sigma(I^k)A_P \subseteq [\sigma(I^{n+k}) :_A \sigma(I^k)]A_P$ so that we get

$$\sigma(I^{n+k})A_P :_{A_P} \sigma(I^k)A_P = [\sigma(I^{n+k}) :_A \sigma(I^k)]A_P.$$

Consequently,

$$Ass[(I^nA_P)_{\hat{\sigma}_P}/\hat{\sigma}_P(I^nA_P)] = Ass[[\sigma(I^{n+k}) :_A \sigma(I^k)]A_P/[\sigma(I^n)]A_P].$$

Since $PA_P \in Ass[(I^nA_P)_{\hat{\sigma}_P}/\hat{\sigma}_P(I^nA_P)]$, it follows that

$$PA_P \in Ass[[\sigma(I^{n+k}) :_A \sigma(I^k)/\sigma(I^n)]A_P] = Ass[[(I^n)_\sigma/\sigma(I^n)]A_P],$$

hence $P \in Ass[(I^n)_\sigma/\sigma(I^n)]$ and $\mathcal{S}_n^\sigma(I) \subseteq Ass[(I^n)_\sigma/\sigma(I^n)]$. □

Remark 3.5. By Lemma 3.3, if $\sigma = sat$ then $\hat{\sigma}_P$ exists for every $P \in Spec(A)$.

The following proposition is a generalization of [9], Lemma 2.5.

Proposition 3.6. *Let H be an ideal containing I, $V = \{P_1, P_2, ..., P_n\}$ be a finite set of associated prime ideals of I such that every P_i is isoled in V. Suppose that $\sigma(I)A_Q \subsetneq \sigma(H)A_Q$ for every $Q \in V$. Let $P \in V$ and σ_P be a semi-prime operation on $\mathcal{I}(A_P)$ such that $\sigma_P(KA_P) = \sigma(K)A_P$ for all $K \in \mathcal{I}(A)$. We put $J = \sigma(I) + P_1 \dots P_n\,\sigma(H)$. Then*

(i) *$V \subseteq Ass\big(A/\sigma(J)\big)$,*
(ii) *If $Q \in \mathcal{S}_1^\sigma(J)$ and Q contains no $P \in V$ then $Q \in \mathcal{S}_1^\sigma(H)$.*

Proof. Let $P \in V$, P is a minimal and maximal element in V. We have $JA_P = \sigma(I)A_P + PA_P\sigma(H)A_P$. Since $\sigma(I)A_P \subsetneq \sigma(H)A_P$, we have $JA_P \subsetneq \sigma(H)A_P = \sigma_P(HA_P)$. We also have $\sigma_P(JA_P) \subsetneq \sigma_P(HA_P)$ and

$\sigma(J)A_P \subsetneq \sigma(H)A_P$, since σ_P is a semi-prime operation on $\mathcal{I}(A_P)$. It follows that $\sigma(J)A_P : \sigma(H)A_P$ is a proper ideal of the local ring A_P. We have $PA_P\sigma(H)A_P \subseteq \sigma(I)A_P + PA_P\sigma(H)A_P = JA_P$, therefore $PA_P \subseteq \sigma(J)A_P : \sigma(H)A_P$ and $PA_P = \sigma(J)A_P : \sigma(H)A_P$, since PA_P is the maximal ideal of A_P. Hence $PA_P \in Ass\big(A_P/\sigma(J)A_P\big)$ and $P \in Ass\big(A/\sigma(J)\big)$. This proves that $V \subseteq Ass\big(A/\sigma(J)\big)$.
ii) Suppose $Q \in \mathcal{S}_1^\sigma(J)$ and Q contains no $P \in V$. Since $PA_Q = A_Q$, we have $JA_Q = \sigma(I)A_Q + \sigma(H)A_Q$. It follows that $\sigma(H)A_Q \subseteq JA_Q$ and $\sigma(H)A_Q \subseteq \sigma(J)A_Q$. We have $JA_Q = \sigma(I)A_Q + \sigma(H)A_Q$ and $\sigma(I)A_Q \subseteq \sigma(H)A_Q$, therefore $JA_Q \subseteq \sigma(H)A_Q = \sigma_Q(HA_Q)$. It follows that $\sigma(J)A_Q \subseteq \sigma(H)A_Q$ and $\sigma(J)A_Q = \sigma(H)A_Q$. Since $Q \in \mathcal{S}_1^\sigma(J) = Ass\big(A/\sigma(J)\big)\setminus A_\sigma(J)$, we have $QA_Q \in Ass\big(A_Q/\sigma(J)A_Q\big)\setminus A_{\sigma_Q}(JA_Q) = Ass\big(A_Q/\sigma(H)A_Q\big) \setminus A_{\sigma_Q}(HA_Q)$, hence $Q \in Ass\big(A/\sigma(H)\big) \setminus A_\sigma(H) = \mathcal{S}_1^\sigma(H)$. □

Theorem 3.7. *Let I be a nonzero ideal of the ring A.*

(i) *For all $k \geq 1$, $(I^k)_\sigma \subseteq (I^{k-1})_\sigma$.*
(ii) *$\big((I^n)_\sigma\big)_{n\in\mathbb{N}}$ is a filtration on the ring A.*
(iii) *Let $n \geq 1$ be an integer, J be an ideal of A such that $J \subseteq (I^n)_\sigma$. If $P \in Ass\big(A/\sigma(J)\big)$ then $P \in Ass\big((I^{n-1})_\sigma/\sigma(J)\big)$. In particular, $Ass\big((I^n)_\sigma/\sigma(J)\big) \subseteq Ass\big((I^{n-1})_\sigma/\sigma(J)\big)$.*
(iv) *Let $n \geq 1$ be an integer. If $J \subseteq (I^n)_\sigma$ then for every integer $0 \leq k < n$, we have $Ass\big(A/\sigma(J)\big) = Ass\big((I^k)_\sigma/\sigma(J)\big)$.*

Proof. (i) Let $k \in \mathbb{N}^*$ and $x \in (I^k)_\sigma$, we have $x\sigma(I^n) \subseteq \sigma(I^{n+k}) \subseteq \sigma(I^{n+k-1})$ for n large enough, hence $x \in (I^{k-1})_\sigma$.
(ii) It is obvious that $(I^0)_\sigma = A_\sigma = A$. We also have $(I^n)_\sigma \subseteq (I^{n-1})_\sigma$ (we refer to (i)) and $(I^p)_\sigma(I^q)_\sigma \subseteq (I^{p+q})_\sigma$ (cf. [4], Proposition 3.2).
(iii) It is clear for $n = 1$. Assume that $n > 1$. If $P \in Ass\big(A/\sigma(J)\big)$ then there exists $x \in A \setminus \sigma(J)$ such that $P = \sigma(J) : x$. It follows that $xP \subseteq \sigma(J)$ and $x \in \sigma(J) : P$. Since $I \subseteq P$, we have $x\sigma(I) \subseteq \sigma(J)$ and $x \in \sigma(J) : \sigma(I)$. We also have $J \subseteq (I^n)_\sigma$, so that $\sigma(J) \subseteq (I^n)_\sigma$, since $(I^n)_\sigma$ is σ-closed. Therefore $x \in (I^n)_\sigma : \sigma(I) = (I^{n-1})_\sigma$ ([4], Proposition 3.4), hence $P = \sigma(J) : x$ with $x \in (I^{n-1})_\sigma$ and $x \notin \sigma(J)$. It follows that $P \in Ass\big((I^{n-1})_\sigma/\sigma(J)\big)$, in particular $Ass\big((I^n)_\sigma/\sigma(J)\big) \subseteq Ass\big((I^{n-1})_\sigma/\sigma(J)\big)$ (we refer to (i)).
(iv) By (i) and (iii), we have $Ass\big(A/\sigma(J)\big) \subseteq Ass\big((I^{n-1})_\sigma/\sigma(J)\big) \subseteq ... \subseteq Ass\big(I_\sigma/\sigma(J)\big) \subseteq Ass\big(A/\sigma(J)\big)$. □

Theorem 3.7, *(iii)* is a generalization of [9], 4.15.2

Proposition 3.8. *Let A be a commutative ring with identity and σ be a semi-prime operation on $\mathcal{I}(A)$. Let I and J be ideals of A. Assume* *that there exists a regular element u of A such that $uI = J$. For all $n \in \mathbb{N}^*$ we have $Ass\big(A/\sigma(I^n)\big) \subseteq Ass\big(A/\sigma(J^n)\big)$.*

Proof. Let $P \in Ass\big(A/\sigma(I^n)\big)$. There exists $x \in A \setminus \sigma(I^n)$ such that $P = \sigma(I^n) : x$. For every $a \in \sigma(I^n) : x$, we have $ax \in \sigma(I^n)$. Therefore $axu^n \in u^n\sigma(I^n) = \sigma((uI)^n) = \sigma(J^n)$ and $a \in \sigma(J^n) : xu^n$. Conversely, if $b \in \sigma(J^n) : xu^n$ then $bxu^n \in \sigma(J^n) = u^n\sigma(I^n)$. Since u is a regular element, u^n is a regular element. It follows that $bx \in \sigma(I^n)$, hence $b \in \sigma(I^n) : x$. We have $\sigma(I^n) : x = \sigma(J^n) : xu^n$, therefore $P = \sigma(I^n) : x = \sigma(J^n) : xu^n$ and $P \in Ass\big(A/\sigma(J^n)\big)$. □

Corollary 3.9. *Let A be an Artinian ring and σ be a prime operation on $\mathcal{I}(A)$. Let x be a regular element of a (regular) ideal I such that the principal ideal (x) is a reduction of I. Then there exists an integer $r > 0$ such that*

(i) *for all $n \in \mathbb{N}^*$, $Ass(A/\sigma(I^{rn})) \subseteq Ass(A/\sigma(I^{(r+1)n}))$,*
(ii) $\mathcal{S}_r^\sigma(I) \subseteq Ass\big((I^{r-1})_\sigma/\sigma(I^{r+1})\big)$.

Proof. (i) Follows from Proposition 3.8.
(ii) By Theorem 3.4, *(iii)* we have $\mathcal{S}_r^\sigma(I) \subseteq Ass\big((I^r)_\sigma/\sigma(I^r)\big)$ for all $r \in \mathbb{N}^*$. Since $I^r \subseteq (I^r)_\sigma$, it follows from Theorem 3.7, *(iii)* that $Ass\big((I^r)_\sigma/\sigma(I^r)\big) \subseteq Ass\big((I^{r-1})_\sigma/\sigma(I^r)\big)$. Since $xI^r = I^{r+1}$, it follows from Proposition 3.8 that $Ass(A/\sigma(I^r)) \subseteq Ass(A/\sigma(I^{r+1}))$. Now we show that $Ass((I^{r-1})_\sigma/\sigma(I^r)) \subseteq Ass((I^{r-1})_\sigma/\sigma(I^{r+1}))$. Let $P \in Ass((I^{r-1})_\sigma/\sigma(I^r))$. There exists $y \in (I^{r-1})_\sigma \setminus \sigma(I^r)$ such that $P = \sigma(I^r) : y$. Since $(\sigma(I^n) : y)_{n\in\mathbb{N}}$ is a decreasing sequence of ideals of the Artinian ring A, $\sigma(I^r) : y = \sigma(I^{r+1}) : y$ for r large enough. It follows that $P \in Ass((I^{r-1})_\sigma/\sigma(I^{r+1}))$, hence $Ass((I^{r-1})_\sigma/\sigma(I^r)) \subseteq Ass((I^{r-1})_\sigma/\sigma(I^{r+1}))$. □

Proposition 3.10. *([1], Prop. 4) For all $n \in \mathbb{N}^*$, there is an ideal $J_{(n)}$ of the ring A such that $\mathcal{S}_n^\sigma(I) = Ass\big(J_{(n)}/\sigma(I^n)\big)$.*

Proof. We refer to [1], Chap.4, Proposition 4. □

Proposition 3.11. *Let $k, m \in \mathbb{N}$ such that $k < m$. There exist $J_{(k)}$, $J_{(m)} \in \mathcal{I}(A)$ such that $Ass\big(\frac{J_{(k)}\cap J_{(m)}}{\sigma(I^m)}\big) \subseteq \mathcal{S}_m^\sigma(I)$.*

Proof. We use the fact that $\frac{J_{(k)}\cap J_{(m)}}{\sigma(I^m)} \subseteq J_{(m)}/\sigma(I^m)$. □

4. $\sigma(f_I)$-superficial elements of an ideal

Throughout this section $(A, \mathcal{M})$ is a Noetherian local ring with infinite residue field $K = \frac{A}{\mathcal{M}}$ and I is an $\mathcal{M}$-primary ideal of the ring A. Let σ be a semi-prime operation on $\mathcal{I}(A)$. We put $\sigma(f_I) = (\sigma(I^n))_{n\in\mathbb{N}}$, which is the σ-closure of the I-adic filtration $f_I = (I^n)_{n\in\mathbb{N}}$.

Definition 4.1. An element $x \in I$ is said to be $\sigma(f_I)$-superficial if there exists an integer n_0 such that $\big(\sigma(I^{n+1}) :_A x\big) \cap \sigma(I^{n_0}) = \sigma(I^n)$, for all $n \geq n_0$.

Proposition 4.2. *Let $x \in I$ be a $\sigma(f_I)$-superficial. For all $n \geq 1$ we have*

(i) $\big((I^{n+1})_\sigma : x\big) = (I^n)_\sigma$,
(ii) $(x) \cap (I^{n+1})_\sigma = x(I^n)_\sigma$.

Proof. Suppose that $x \in I$ is a $\sigma(f_I)$-superficial element.
(i) By [4], Proposition 3.2, we have $x(I^n)_\sigma \subseteq I(I^n)_\sigma \subseteq (I^{n+1})_\sigma$ for all $n \geq 1$, hence $x(I^n)_\sigma \subseteq (I^{n+1})_\sigma$ and $(I^n)_\sigma \subseteq \big((I^{n+1})_\sigma : x\big)$, for all $n \geq 1$. Conversely, let $a \in \big((I^{n+1})_\sigma : x\big)$, then $ax \in (I^{n+1})_\sigma = \sigma(I^{n+1+k}) : \sigma(I^k)$, $\forall\, k \gg 0$. It follows that $a\sigma(I^k) \subseteq \big(\sigma(I^{n+1+k}) : x\big)$, $\forall\, k \gg 0$. Since $x \in I$ is a $\sigma(f_I)$-superficial element, there exists an integer k_0 such that $\big(\sigma(I^{m+1}) :_A x\big) \cap \sigma(I^{k_0}) = \sigma(I^m)$, for all $m \geq k_0$. For k large enough, we obtain $a\sigma(I^k) \subseteq \big(\sigma(I^{n+1+k}) : x\big)$ and $a\sigma(I^k) \subseteq \sigma(I^{k_0})$. Therefore $a\sigma(I^k) \subseteq \big(\sigma(I^{n+k+1}) : x\big) \cap \sigma(I^{k_0}) = \sigma(I^{n+k})$ with $n + k > k_0$, thus $a \in \sigma(I^{n+k}) : \sigma(I^k) = (I^n)_\sigma$, $\forall\, k \gg 0$. This proves that $\big((I^{n+1})_\sigma : x\big) = (I^n)_\sigma$, for all $n \geq 1$.
(ii) Let $n \in \mathbb{N}^*$ and $y \in (x) \cap (I^{n+1})_\sigma$. There exists $a \in A$ such $y = ax$. Since $y = ax \in (I^{n+1})_\sigma$, $a \in (I^{n+1})_\sigma : x$. By (i), we have $a \in (I^n)_\sigma$ and $ax \in x(I^n)_\sigma$, hence $(x) \cap (I^{n+1})_\sigma \subseteq x(I^n)_\sigma$. Conversely, we have $x(I^n)_\sigma \subseteq I(I^n)_\sigma \subseteq I_\sigma(I^n)_\sigma \subseteq (I^{n+1})_\sigma$, it follows that $x(I^n)_\sigma \subseteq (x) \cap (I^{n+1})_\sigma$. Hence $(x) \cap (I^{n+1})_\sigma = x(I^n)_\sigma$. □

By Theorem 3.4, (ii), there exist large enough integers n such that $(I^n)_\sigma = \sigma(I^n)$. Set $\rho^I_\sigma(A) = \min\{\, n \,|\, (I^i)_\sigma = \sigma(I^i) \text{ for all } i \geq n\}$. The fact that such an integer $\rho^I_\sigma(A)$ may exist follows from [8], 2.6.

Corollary 4.3. *If $x \in I$ is a $\sigma(f_I)$-superficial, then $\sigma(I^{i+1}) : x = \sigma(I^i)$ for all $i \geq \rho^I_\sigma(A)$.*

Proof. Let $x \in I$ be a $\sigma(f)$-superficial element. By Proposition 4.2, $(I^{i+1})_\sigma : x = (I^i)_\sigma$, $\forall\, i \geq 1$. For all $i \geq \rho^I_\sigma(A)$, $(I^i)_\sigma = \sigma(I^i)$. It follows that $\sigma(I^{i+1}) : x = \sigma(I^i)$ for all $i \geq \rho^I_\sigma(A)$. □

Lemma 4.4. *Let $n \in \mathbb{N}^*$. If $x \in I$ is a $\frac{A}{\sigma(I^{n+1})}$-regular element then $\sigma(I^{n+k}) : x^k = \sigma(I^{n+1}) : x$ for all $k \geq 1$.*

Proof. Let $n, k \in \mathbb{N}^*$. If $a \in \sigma(I^{n+k}) : x^k$ then $ax^k \in \sigma(I^{n+k}) \subseteq \sigma(I^{n+1})$. It follows that $x\big(ax^{k-1} + \sigma(I^{n+1})\big) = \overline{0}$ and since $x \in I$ is a $\frac{A}{\sigma(I^{n+1})}$-regular element, $ax^{k-1} \in \sigma(I^{n+1})$. By iterating we get $ax \in \sigma(I^{n+1})$ et $a \in \sigma(I^{n+1}) : x$. Conversely, if $a \in \sigma(I^{n+1}) : x$ then $ax^k \in I^{k-1}\sigma(I^{n+1}) \subseteq \sigma(I^{n+k})$ and $a \in \sigma(I^{n+k}) : x^k$. □

Lemma 4.5. *Let $n \geq \rho^I_\sigma(A)$. If $x \in I$ is both a $\sigma(f)$-superficial and a $\frac{A}{\sigma(I^{n+1})}$-regular element then $\sigma(I^{n+k}) : x^k = \sigma(I^n)$ for all $k \geq 1$.*

Proof. Follows from Corollary 4.3 and Lemma 4.4. □

In [3], Lemma 1, the author proved that if A is a Noetherian ring and $k \geq 1$ such that I is an ideal of A containing a $\frac{A}{\sigma(I^{k+1})}$-regular element then there exists an integer $m_0 > k$ such that $\sigma(I^{m_0+1}) : I = \sigma(I^{m_0})$. He also proves Theorem 5 [3], assuming that condition $(E_\sigma) \quad \sigma(I^{n+1}) : I = \sigma(I^n) \ \ \forall n \gg 0$ (these are the Ratliff-Rush ideals if $\sigma = Id$).

Theorem 4.6. *If $x \in I$ is a $\sigma(f_I)$-superficial element, then $\sigma(I^{n+1}) : I = \sigma(I^n)$, for all $n \geq \rho^I_\sigma(A)$.*

Proof. If $I = xA$ and x is $\sigma(f)$-superficial element, then $\sigma(I^{n+1}) : I = \sigma(I^{n+1}) : x = \sigma(I^n)$ for all $n \geq \rho^I_\sigma(A)$. Suppose that $I \neq xA$ and $x \in I$ is a $\sigma(f)$-superficial element. Let $n \geq \rho^I_\sigma(A)$ be an integer and $a \in \sigma(I^{n+1}) : I$, then $aI \subseteq \sigma(I^{n+1})$ and $ax \in \sigma(I^{n+1})$, hence $a \in \sigma(I^{n+1}) : x = \sigma(I^n)$ by Corollary 4.3. It follows that $\sigma(I^{n+1}) : I \subseteq \sigma(I^n)$, pour tout $n \geq \rho^I_\sigma(A)$. Conversely, let $n \geq 1$ be an integer. If $a \in \sigma(I^n)$, then $aI \subseteq I\sigma(I^n) \subseteq \sigma(I)\sigma(I^n) \subseteq \sigma(I^{n+1})$, thus $a \in \sigma(I^{n+1}) : I$ and $\sigma(I^n) \subseteq \sigma(I^{n+1}) : I$. It follows that $\sigma(I^{n+1}) : I = \sigma(I^n)$ for all $n \geq \rho^I_\sigma(A)$. □

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References

1. N. Bourbaki, *Algèbre commutative*, Chap. 4, Masson, Paris 1985.
2. M. Brodmann, *Asymptotic stability of* $Ass(M/I^nM)$, Proc. Amer. Math. Soc., **74** (1979), 16-18.
3. E.K. Ambroise, *Opérations de clôture sur les sous-modules d'un module*, Annales Mathématiques Africaines, **3** (2012), 89-100.
4. E.K. Ambroise, *Filtrations, opérations de clôture et la suite* $\{Ass(A/(f^{(k)})_\sigma)\}_{k\in\mathbb{N}}$, Annales Mathématiques Africaines, **4** (2013), 117-124.
5. D. Kirby, *Closure operations on ideals and submodules*, J. London. Maths. Soc., **44** (1969), 283-291.
6. H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
7. S. McAdam, *Primes Associated to an Ideal*, Contemporary Mathematics, Amer. Math. Soc., **102**, 1989.
8. S. McAdam, *Asymptotic prime divisors*, no. **1023**, Berlin HeidelbergNewYork: Springer-Verlag, 1983.

9. S. McAdam and L.J. Ratliff, *Sporadic and irrelevant prime divisors*, Trans. Amer. Math. Soc., (1) **303** (1987), 311–324.
10. L.J. Ratliff and D.E. Rush, *Two notes on reductions of ideals*, Indiana Univ. Math. J., (6) **27** (1978), 929-934.
11. T.J. Puthenpurakal and F. Zulfeqarr, *Ratliff-Rush filtrations associated with ideals and modules in a noetherian ring*, J. Algebra, (2) **311** (2007), 551-583.

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