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SELF-COGENERATOR MODULES AND THEIR APPLICATIONS

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ABSTRACT. Let R be a ring and M be a right R-module. In this paper, we give some properties of self-cogenerator modules. If M is self-cogenerator and $S = End_R(M)$ is a cononsingular ring, then M is a $\mathcal{K}$-module. It is shown that every self-cogenerator Baer is dual Baer.

1. Introduction

Throughout this paper, R is an associative ring with identity, modules are right and unitary and $S = End_R(M)$ is the ring of all endomorphisms of M. $Rad(M)$, $Soc(M)$ will indicate Jacobson radical of M, Socle of M. A submodule K of M is denoted by $K \leq M$. A submodule K of M is called *essential* in M (denoted by $K \subseteq^e M$), if $K \cap L \neq 0$ for every nonzero submodule L of M, and a submodule K of M is called *small* in M (denoted by $K \ll M$), if $N + K \neq M$ for any proper submodule N of M. A nonzero module is said to be *uniform* if each nonzero submodule is essential. It is said to be *hollow* if each proper submodule is small.

Recall that the singular submodule $Z(M)$ of a module M is the set of $m \in M$ such that $mI = 0$ for some essential right ideal I of R. If $Z(M) = 0$ ($Z(M) = M$) then M is *nonsingular* (*singular*) module (see [4]).

A ring R (module M) is called a right *dual* ring (right dual module) if every right ideal I of R is an annihilator, that is, $r_R l_R(I) = I$

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($r_M l_S(N) = N$ for each submodule N). By [2, Lemma 24.4], for each submodule N of M we have $r_M l_S(N)/N = Rej_{M/N}(M) = \bigcap\{kerh \mid h \in Hom(M/N, M)\}$. Thus M is a dual module if and only if for each submodule N of M, we have $Rej_{M/N}(M) = 0$ if and only if M/N is cogenerated by M for each submodule N of M. Recall that a module M is said to be a *self-cogenerator* if it cogenerates each of its factors, that is, $N = r_M l_S(N)$ for all submodules N of M (see [12]). So self-cogenerator and dual modules are same. Clearly semisimple and cogenerator modules are self-cogenerator.

Let $S = End_R(M)$ be a ring and let ${}_SM$ be a left S-module. Then for any $X \subseteq M$ and $Y \subseteq S$, the left annihilator of X in S and the right annihilator of Y in M are

$l_S(X) = \{s \in S \mid sx = 0 \text{ for all } x \in X\}$ and $r_M(Y) = \{m \in M \mid ym = 0 \text{ for all } y \in Y\}$, respectively.

For all undefined notions, we refer the reader to [2], [3] and [9].

2. Properties of self-cogenerator modules

In this section, we give some basic properties of self-cogenerator modules that will use in section 3. It is shown that every self-cogenerator module whose endomorphism rings is PP, is continuous (Theorem 2.8). Suppose that R is a commutative ring and M an R-module. We show that, if for any submodule N of M, there exists a two sided ideal I of R such that $N = r_M(I)$, then M is a self-cogenerator R-module (Theorem 2.10).

Recall that an R-module M is called *coretractable* if, for any proper submodule K of M, there exists a nonzero homomorphism $f : M \longrightarrow M$ with $f(K) = 0$, that is, $Hom_R(M/K, M) \neq 0$ (see [1]). Recall that an R-module M is cosemisimple if for any $K \leq M$, $Rad(M/K) = 0$ (see [12]).

Proposition 2.1. *Any self-cogenerator module is coretractable.*

Proof. By [1, Lemma 4.1]. □

A ring R is said to be right *Kasch* if every simple right R-module can be embedded in R_R. By [1, Theorem 2.14], if R is a right Kasch ring, then R_R is a coretractable module. But it is clear that R_R is not self-cogenerator.

Proposition 2.2. *Let M be coretractable. If M is cosemisimple, then M is a self-cogenerator module.*

Proof. By [1, Corollary 4.2]. □

We have the following implications:
cogenerator module$\Longrightarrow$ self-cogenerator module$\Longrightarrow$ coretractable module

Proposition 2.3. *Let M be a self-cogenerator module. Then:*
(1) *If $l_S(N) \subseteq^e {}_SS$, then $N \ll M$.*
(2) *If $I \subseteq^e {}_SS$, then $r_M(I) \ll M$.*
(3) *If $N \ll M$, then $l_S(N) \subseteq^e {}_SS$.*
(4) *$Soc({}_SS) \subseteq l_S(Rad(M))$.*
(5) *$Z({}_SM) = Rad(M)$.*

Proof. (1),(2) follow from [1, Proposition 4.5].
(3) For $b \in S$, let $l_S(N) \cap Sb = 0$. By [15, Proposition 2.5], we have $r_M(Sb \cap l_S(N)) = r_M(b) + N$. So $M = r_M(Sb \cap l_S(N)) = r_M(b) + N$, then $r_M(b) + N = M$ and since N is a small submodule, $r_M(b) = M$. Thus $b = 0$, that is, $l_S(N) \subseteq^e {}_SS$.
(4) We have $l_S(Rad(M)) = l_S(\sum_{N \ll M} N) = \bigcap_{N \ll M} l_S(N)$. By (3), $l_S(N) \subseteq^e {}_SS$ for any $N \ll M$. Thus $Soc({}_SS) \subseteq l_S(Rad(M))$.
(5) Let $x \in Z({}_SM) = \{x \in M \mid l_S(x) = l_S(xR) \subseteq^e {}_SS\}$. Suppose $xR + L = M$, then $l_S(xR + L) = l_S(M) = 0$, so $l_S(xR) \cap l_S(L) = 0$. By assumption, $l_S(L) = 0$, then $L = M$. Therefore $xR \ll M$ and thus $x \in Rad(M)$. Conversely, let $x \in Rad(M)$. By (3), $l_S(xR) \subseteq^e {}_SS$, so $x \in Z({}_SM)$. □

Corollary 2.4. *Let M be a self-cogenerator R-module and $\nabla(M) = \{f \in S \mid f(M) \ll M\}$. Then $Z({}_SS) = \nabla(M)$.*

Proof. By [1, Corollary 4.8], we have $Z({}_SS) \subseteq \nabla(M)$. Now suppose that $f \in \nabla(M)$. Then by Proposition 2.3, $l_S(f) = l_S(f(M)) \subseteq^e {}_SS$. Therefore, $f \in Z({}_SS)$ and $\nabla(M) \subseteq Z({}_SS)$. □

Following [12, p. 261], an R-module M is called *semi-injective* if for any $f \in S$, $Sf = l_S(ker(f)) = l_S(r_M(f))$ (equivalently, for any monomorphism $f : N \longrightarrow M$, where N is a factor module of M, and for any homomorphism $g : N \longrightarrow M$, there exists $h : M \longrightarrow M$ such that $hf = g$).

Proposition 2.5. *Let M be a semi-injective self-cogenerator module and N a proper submodule of M. If M/N is hollow, then $l_S(N)$ is a uniform ideal of S.*

Proof. Assume M/N is a non-zero hollow module. Then $l_S(N) \neq 0$. Suppose $f, g \in l_S(N)$ and $fS \cap gS = 0$. Then $l_S(ker(f) + ker(g)) = l_S(ker(f) \cap l_S(ker(g)) = fS \cap gS = 0$. Since M is self-cogenerator, $ker(f) + ker(g) = M$, but since $N \subseteq ker(f) \cap ker(g)$, we conclude that

$M/N = ker(f)/N + ker(g)/N$. As M/N is hollow, $ker(f) = M$ and $f = 0$ or $ker(g) = M$ and $g = 0$. Thus $l_S(N)$ is a uniform ideal of S. □

Consider the following properties for an R-module M:

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is also a direct summand.

(C_3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

An R-module M is called *continuous* if it has (C_1) and (C_2), M is called *quasi-continuous* if it has (C_1) and (C_3), and M is called an *extending* if it has property (C_1).

Lemma 2.6. *Let M be a self-cogenerator module. Then M is quasi-continuous.*

Proof. Let N_1 and N_2 be submodules of M such that $N_1 \cap N_2 = 0$. Since M is self-cogenerator, $0 = N_1 \cap N_2 = r_M l_S(N_1) \cap r_M l_S(N_2) = r_M[l_S(N_1 + N_2)]$. Then $0 = r_M[l_S(N_1 + N_2)]$ and $l_S(0) = l_S r_M[l_S(N_1 + N_2)]$. So $S = l_S(N_1 + N_2)$, and $S = l_S(N_1) + l_S(N_2)$. Hence M is quasi-continuous by [13, Theorem 8]. □

The converse of Lemma 2.6 does not hold in general, because the $\mathbb{Z}$-module $\mathbb{Z}_{\mathbb{Z}}$ is quasi-continuous but is not self-cogenerator.

Example 2.7. ([9, Example 2.9]) Let $S = R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is field and $M = {}_R R_R$. Then M_R is not quasi-continuous. It is clear that M is not self-cogenerator. But $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is the jacobson radical of R. Since J and R/J are semisimple, so they are self-cogenerator.

A ring R is called left PP-ring, if every cyclic left ideal is projective, equivalently, the left annihilator of each element of R is a direct summand of ${}_R R$.

Theorem 2.8. *Let M be self-cogenerator and S a PP-ring. Then M is continuous.*

Proof. By Lemma 2.6, M is quasi-continuous. To prove that M is continuous, by [9, Lemma 3.14], it is enough to show that every essential monomorphism $f \in S$ is an isomorphism. Let $f \in S$ a monomorphism, with $f(M) \subseteq^e M$. By [12, 39.11], $f(M)$ is a direct summand. Therefore $f(M) = M$. □

Corollary 2.9. *Let M be a self-cogenerator module and S a PP-ring, $\Delta(M) = \{f \in S \mid Kerf \subseteq^e M\}$. Then $J(S) = \Delta(M)$.*

Proof. It follows from Theorem 2.8 and [9, Proposition 3.15]. □

Theorem 2.10. *Let R be a commutative ring and M an R-module. If for any submodule N of M, there exists a two sided ideal I of R such that $N = r_M(I)$, then M is self-cogenerator R-module.*

Proof. Let M be an R-module with stated property and $N \leq M$. Then there exists an ideal I of R such that $N = r_M(I)$. For each $a \in I$, define the map $f_a : M \longrightarrow M$ by $m \mapsto am$. Since R is a commutative ring, f_a is an R-endomorphism. It is clear that for each $a \in I$, $N \subseteq ker(f_a)$ and we have $\bigcap\{kerf \mid f \in S, N \subseteq kerf\} \subseteq \bigcap_{a\in I} kerf_a = N$. So $\bigcap\{kerf \mid f \in S, N \subseteq kerf\} = N$ if and only if M/N cogenerated by M. □

Example 2.11. We know that the $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module is self-cogenerator module (p is a prime number). Choose $N = \mathbb{Z}(1/p + \mathbb{Z})$ and set $I = \mathbb{Z}p^i$, $i \geq 0$. It is clear that $N = r_M(I)$. Therefore, by Theorem 2.10, M is self-cogenerator.

Corollary 2.12. *Let R be a commutative ring and M an R-module. If for any submodule N of M, there exists a two sided ideal I of R such that $N = r_M(I)$ and S is a domain, then M is hollow.*

Proof. Let L and K be proper submodules of M, such that $L+K = M$. By Theorem 2.10, M is self-cogenerator. So there exist $0 \neq f, g \in S$ such that $L \subseteq ker(f)$ and $K \subseteq ker(g)$. Now since S is a domain, we have $fg \neq 0$ and $f, g \neq 0$. Then $(fg)(L + K) = (fg)(L) + (fg)(K) = (fg)(M)$. It follows that $(fg)(K) = (fg)(M)$ so that $(fg)(M) = (fg)(K) \subseteq g(K) = 0$. Hence $fg = 0$. But this is a contradiction. Therefore $L = M$ or $K = M$, as required. □

3. Applications

In this section, we consider applications of self-cogenerator and self-generator modules in other modules, in particular in *Baer*, *dual Baer* and extending modules. This is the focus of our investigations in this paper. We provide some additional motivation as follows. In [5], Kaplansky introduced the concept of a Baer ring. A ring R is called Baer if the right annihilator of any nonempty subset of R is generated by an idempotent. According to [10], M is called a Baer module if the right annihilator in M of any left ideal of S is a direct summand of M. Following [7], a module M is called a dual Baer module if for every right ideal I of S, $\sum_{\phi\in I} Im\phi$ is a direct summand of M.

Recall from [11], a module M is called *$\mathcal{K}$-nonsingular* if, for every $0 \neq \phi \in End_R(M)$, $Ker\phi$ is not essential in M. In [6], a module M is called *$\mathcal{T}$-noncosingular* if, $\forall \phi \in End_R(M)$, $Im\phi \ll M$ implies $\phi = 0$. A module M is said to be *$\mathcal{K}$-module* if, $D_S(N) = \{\varphi \in S \mid Im\varphi \subseteq N\} = 0$ implies $N \ll M$. A ring R is said to be *cononsingular* if $\forall I \leq R$, $rI \neq 0$ $\forall 0 \neq r \in R$ implies that $I \subseteq^e R$ and a module M is *$\mathcal{K}$-cononsingular* if, for all $N \leq M$, $l_S(N) = 0$ implies $N \subseteq^e M$.

Lemma 3.1. *([14, Lemma 1]) If $I \leq S$ and $N \leq M$, then:*
(1) *$D_S(N)M \subseteq N$.*
(2) *$N \subseteq r_M(l_S(N))$.*
(3) *$l_S(N)D_S(N) = 0$.*
(4) *$D_S(r_M(I)) = r_S(I)$.*
(5) *$l_S(IM) = l_S(I)$.*

Lemma 3.2. *([14, Lemma 2]) Let M be an R-module. Then:*
(1) *If $D_S(N)M = N$, then $l_S(N) = l_S(D_S(N))$.*
(2) *If $r_M(l_S(N)) = N$, then $D_S(N) = r_S(l_S(N))$.*

Recall from [12], a module M is said to be *self-generator* if it generates each of its submodules, i.e. $N = Hom(M, N)M$ for all $N \leq M$. We call M *weakly self-generator*, if M generates $r_M(I)$ for every $I \leq {}_SS$. It is clear that every self-generator module is weakly self-generator. It is easy to check that a module M is weakly self-generator if and only if M generates any submodule N with M/N being cogenerated by M if and only if $r_M(I) = r_S(I)M$ for every left ideal I of S (Lemma 3.1).

Proposition 3.3. *The following are equivalent for a module M:*
(1) *M is a Baer module;*
(2) *M is weakly self-generator and S is a Baer ring.*

Proof. (1) $\Longrightarrow$ (2) Let I be a left ideal of S. Since M is a Baer module, $r_M(I) = eM$ for some $e^2 = e \in S$. Then $IeM = 0$, so $Ie = 0$. Thus $e \in r_S(I)$ and $eM \subseteq \sum_{\phi \in r_S(I)} Im\phi = r_S IM$. Therefore $r_M(I) = r_S(I)M$ and M is weakly self-generator. Now from [10, Theorem 4.1], S is a Baer ring.

(2) $\Longrightarrow$ (1) Let I be a left ideal of S. We have $r_S(I) = eS$ for some $e^2 = e \in S$ because S is a Baer ring. Since M is weakly self-generator, $r_M(I) = r_S(I)M = eSM = eM$. □

Corollary 3.4. *Let M be weakly self-generator. If M is a dual Baer module, then M is Baer.*

Proof. Let I be a left ideal of S. Since M is dual Baer, $r_S(I)M = eM$ for some $e^2 = e \in S$. But $r_M(I) = r_S(I)M$, so M is Baer. □

Let M be a module. We call M *weakly self-cogenerator*, if $r_M l_S(I) = IM$ for each $I \leq {}_S S$ and $IM \lneq M$.

Every self-cogenerator module is weakly self-cogenerator

Proposition 3.5. *The following are equivalent for a module M:*

(1) *M is dual Baer module;*

(2) *M is weakly self-cogenerator and S is a Baer ring.*

Proof. (1) $\Longrightarrow$ (2) Let I be an ideal of S. Since M is dual Baer, there exists $e^2 = e \in S$ such that $IM = eM$. Thus $(1-e) \in l_S(I)$. Let $m \in r_M(l_S(I))$. Then $(1-e)m = 0$, so $m \in eM = IM$. Therefore M is weakly self-cogenerator. Second part, follows from [8, Theorem 3.6].

(2) $\Longrightarrow$ (1) Let I be an ideal of S and $l_S(I) = Se$, for some $e^2 = e \in S$. Hence $\forall \phi \in I$, $e\phi = 0$, so $\phi = (1-e)\phi$ and $\phi M \subseteq (1-e)M$. Thus $IM \subseteq (1-e)M$. But $(1-e)M \subseteq r_M(Se) = r_M(l_S(I)) = IM$ because M is a weakly self-cogenerator module. Therefore M is a dual Baer module. □

Example 3.6. (1) If G is a divisible abelian group, then G is dual Baer and so $End(G)$ is a Baer ring.

(2) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{\mathbb{Z}}$. Then $End(M) \cong \mathbb{Z}$ is a Baer ring but M is not a dual Baer module.

Proposition 3.7. *Let M be a weakly self-cogenerator module. If M is a Baer module, then M is dual Baer.*

Proof. Let I be an ideal of S. Since M is Baer, $r_M(l_S(I)) = eM$ for some $e^2 = e \in S$. But $IM = r_M(l_S(I))$, so M is dual Baer. □

Corollary 3.8. *Let M be a self-cogenerator module. If M is a Baer module, then M is dual Baer.*

Proof. It follows by Proposition 3.7. □

Lemma 3.9. *Let M be an R-module and S a cononsingular ring. Then:*

(1) *If M is a self-generator, then M is $\mathcal{K}$-cononsingular.*

(2) *If M be a self-cogenerator, then M is $\mathcal{K}$-module.*

Proof. (1) Let $N \leq M$ such that $l_S(N) = 0$. First, we show that $D_S(N) \subseteq^e S$. Let $0 \neq \phi \in S$ and $\phi l_S(N) = 0$. Then $\phi \in l_S(D_S(N))$. Since M is a self-generator, $l_S(D_S(N)) = l_S(N)$ (Lemma 3.2). Thus $\phi \in l_S(N) = 0$. It follows that $\phi D_S(N) \neq 0$, $\forall 0 \neq \phi \in S$. Hence, by cononsingularity of S, $D_S(N) \subseteq^e S$. Next, let $K \leq M$ such that $K \cap N = 0$. Then $D_S(N) \cap D_S(K) = 0$. Hence $D_S(K) = 0$. As M is a self-generator, $K = 0$ and so $N \subseteq^e M$. This shows that M is $\mathcal{K}$-cononsingular.

(2) Let $N \leq M$ such that $D_S(N) = 0$. First, we show that $l_S(N) \subseteq^e S$. Let $0 \neq \phi \in S$ and $l_S(N)\phi = 0$. Then $\phi \in r_S(l_S(N))$. Since M is a self-cogenerator, $r_S(l_S(N)) = D_S(N)$ (Lemma 3.2). Thus $\phi \in D_S(N) = 0$. It follows that $l_S(N)\phi \neq 0$, $\forall 0 \neq \phi \in S$. Hence, by cononsingularity of S, $l_S(N) \subseteq^e S$. Next, let $K \leq M$ such that $K + N = M$. Then $l_S(N) \cap l_S(K) = 0$. Hence $l_S(K) = 0$. As M is a self-cogenerator, $K = M$ and so $N \ll M$. This proves that M is $\mathcal{K}$-module. □

Theorem 3.10. *([10, Theorem 2.12]) A module M is extending and $\mathcal{K}$-nonsingular if and only if M is Baer and $\mathcal{K}$-cononsingular.*

Proposition 3.11. *Let M be a self-generator module and S is extending and nonsingular. Then M is extending.*

Proof. Since S is a nonsingular extending ring, by Theorem 3.10, it is Baer. Then by Proposition 3.3, M is Baer module and by Lemma 3.9(1), M is $\mathcal{K}$-cononsingular. From Theorem 3.10, M is extending. □

Remark 3.12. Let M be a self-cogenerator module and S a Baer ring. Then M is $\mathcal{T}$-noncosingular.

Proof. It follows by Proposition 3.5, Since every dual Baer module is $\mathcal{T}$-noncosingular. □

Example 3.13. The $\mathbb{Z}$-module $\mathbb{Q}_{\mathbb{Z}}$ is $\mathcal{T}$-noncosingular. Although it is not self-cogenerator, since $Hom_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$.

We conclude this paper by investigating the connection of the $\mathcal{T}$-noncosingularity and $\mathcal{K}$-nonsingularity of a module to its endomorphism ring.

Proposition 3.14. *Let M be a self-cogenerator module. The following hold:*

(1) M is $\mathcal{T}$-noncosingular if and only if ${}_SS$ is $\mathcal{K}$-nonsingular.

(2) M is $\mathcal{K}$-nonsingular and S a PP-ring if and only if ${}_SS$ is $\mathcal{T}$-nonconsingular.

Proof. (1) It follows from Corollary 2.4 and [11, Proposition 2.7].

(2) It follows from Corollary 2.9 and [6, Corollary 2.7]. □

Following example presents an application of Proposition 3.14.

Example 3.15. The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is self-cogenerator and $\mathcal{T}$-noconsingular. By [10, Example 4.3], S is Baer. So by [10, Lemma 2.15], S is $\mathcal{K}$-nonsingular.

Theorem 3.16. *Let M be a self-cogenerator module and S a Baer ring. If M is discrete module, then M is $\mathcal{K}$-nonsingular if and only if M is $\mathcal{T}$-nonconsingular.*

Proof. By [9, Theorem 5.4] and Corollaries 2.4, 2.9, we have $Z({}_SS) = J({}_SS)$. This completes the proof. □

Let M be $\mathbb{Z}$-module $\mathbb{Z}_p$ (p is a prime number). Then M is $\mathcal{K}$-nonsingular and $\mathcal{T}$-noncosingular.

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References

1. B. Amini, M. Ershad, and H. Sharif, *Coretractable modules*, J. Aust. Math. Soc., **86** (2009), 289-304.
2. F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
3. J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules. Supplements and Projectivity in Module Theory*, Frontiers in Math. Boston: Birkhäuser, 2006.
4. R.E. Johnson, *Structure theory of faithful rings. I. Closure operations on lattices. II. Restricted rings*, Trans. Amer. Math. Soc., **84** (1957), 508-522.
5. I. Kaplansky, *Rings of Operators*, Math. Lecture Note Ser., W. A. Benjamin, New York, 1968.
6. D. Keskin Tütüncü and R. Tribak, *On $\mathcal{T}$-noconsingular module*s, Bull. Aust. Math. Soc., **80** (2009), 462-471.
7. D. Keskin Tütüncü and R. Tribak, *On dual Baer modules*, Glasg. Math. J., **52** (2010), 261–269.
8. D. Keskin Tütüncü, P. F. Smith, and S. E. Toksoy, *On dual Baer modules*, Contemporary Mathematics, **609** (2014), 173-184.
9. S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Notes Series 147, Cambridge, University Press, 1990.
10. S.T. Rizvi and C.S. Roman, *Baer and Quasi-Baer Modules*, Comm. Algebra, **32** (2004), 103-123.
11. S.T. Rizvi and C.S. Roman, *On $\mathcal{K}$-nonsingular modules and applications*, Comm. Algebra, **35** (2007), 1–22.
12. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.
13. R. Wisbauer, M.F. Yousif, and Y. Zhou, *Ikeda-Nakayama modules*, Beitr. Algebra Geom. (1) **43** (2002), 111-119.
14. K.G. Wolfson, *Baer rings of endomorphisms*, Math. Ann. **143** (1961), 19-28.
15. D. Zhou, D. Li, and L. Guo, *Annihilator conditions relative to a class of modules*, Thai J. Math., (3) **8** (2010), 419-428.

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