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FULLY PRIMARY MODULES AND SOME VARIATIONS

A. NIKSERESHT* AND H. SHARIF

Abstract. Let R be a commutative ring and M be an R-module. We say that M is fully primary, if every proper submodule of M is primary. In this paper, we state some characterizations of fully primary modules. We also give some characterizations of rings over which every module is fully primary, and of those rings over which there exists a faithful fully primary module. Furthermore, we will introduce some variations of fully primary modules and consider similar questions about them.

1. Introduction

In this paper all rings are commutative with identity and all modules are unitary. Also R denotes a ring, M is an R-module, $\mathfrak{P}$ stands for a prime ideal of R and k is a positive integer. Moreover, by $N(R)$, J(R) and Ann(N) we mean the nilradical of R, the Jacobson radical of R and the annihilator of a submodule N of M, respectively.

In [10], rings in which every ideal is primary, called *generalized primary rings*, are studied and a characterization of Noetherian commutative generalized primary rings is given. In [5], this characterization is generalized to non-Noetherian commutative rings. There it is proved that R is generalized primary if and only if either R is a one dimensional local domain (in this note, local does not necessarily imply Noetherian) or a zero dimensional local ring (see [5, Theorem 2.4]).

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$*$Corresponding author .

The notion of k-primary ideals is investigated in [14]. A proper ideal I of R is called *k-primary* for a $k \in \mathbb{N}$, when for every two ideals A and B of R with $AB \subseteq I$ we have either $A \subseteq I$ or $B^k \subseteq I$ (note that in [14], it is assumed that this k is minimum, but here for the sake of simplicity of notations, we have dropped this assumption). According to [14, Theorem 4], every ideal of R is k-primary if and only if R is zero-dimensional local with maximal ideal $\mathfrak{M}$, where $\mathfrak{M}^k = 0$.

In this paper, we generalize this concept to modules. We say that a proper submodule N of M is *k-primary* [*weak k-primary*], if from $IK \subseteq N$ where I is an ideal [a principal ideal] of R and K is a submodule [a cyclic submodule] of M, we can deduce that either $K \subseteq N$ or $I^k \subseteq (N : M) = \{r \in R | rM \subseteq N\}$. It is easy to see that if N is a k-primary [weak k-primary] submodule, then $(N : M)$ is a k-primary [weak k-primary] ideal of R (that is, as an R-submodule of R). We say that N is *residually primary* [resp. *residually k-primary*, *residually weak k-primary*], if $(N : M)$ is a primary [resp. k-primary, weak k-primary] ideal in R.

We also say that M is *fully primary* [resp. *fully k-primary*, *fully weak k-primary*, *residually fully primary*, *residually fully k-primary*, *residually fully weak k-primary*], when every proper submodule of M is primary [resp. k-primary, weak k-primary, residually primary, residually k-primary, residually weak k-primary].

The main aim of this research is to study fully primary modules, fully k-primary modules, etc. and also rings over which there exist such modules or rings over which every module is so. For example, we will show that if there exists a faithful R-module M such that either (i) M is fully k-primary [fully weak k-primary], (ii) M is finitely generated and residually fully k-primary [residually fully weak k-primary] or (iii) $k \geq 2$ and every nonzero proper submodule of M is $(k-1)$-primary [weak $(k-1)$-primary], then every R-module is fully k-primary [fully weak k-primary]. Also we shall give a characterization of modules, in which every nonzero proper submodule is primary, k-primary or weak k-primary (see, for example (4.4)). It should be noted that some of the results of this paper was presented in the 43^{rd} Annual Iranian Mathematics Conference (see [8]).

2. Fully Primary Modules

At the first of this section, we consider fully primary modules, that is, modules in which every proper submodule is primary. For example, every vector space and in particular every finitely generated divisible module is fully primary. On the other hand, the $\mathbb{Z}$-module $\mathbb{Q}$ is not fully primary, despite being divisible. Also $\mathbb{Z}_n$ is a fully primary $\mathbb{Z}$-module, if and only if n is a prime power. Before giving a characterization of fully primary modules, we state an easy remark which will be used throughout this paper without any further mention.

Remark 2.1.
(i) It is well-known that if $\sqrt{I}$ is maximal for an ideal I of R, then I is primary. A quite similar argument shows that if $\sqrt{(N:M)}$ is a maximal ideal for a submodule N of M, then N is primary.
(ii) If N is a $\mathfrak{P}$-primary [resp. k-primary, weak k-primary] submodule of M and K a submodule of M not contained in N, then $N \cap K$ as a submodule of K is $\mathfrak{P}$-primary [resp. k-primary, weak k-primary]. In particular, if M is fully primary [fully k-primary, fully weak k-primary], then all of its submodules are so.
(iii) If $K \subseteq N$ are two submodules of M, then N is $\mathfrak{P}$-primary [resp. k-primary, weak k-primary] in M if and only if $\frac{N}{K}$ is $\mathfrak{P}$-primary [resp. k-primary, weak k-primary] in $\frac{M}{K}$. Therefore, every homomorphic image of a fully primary [fully k-primary, fully weak k-primary] module is fully primary [fully k-primary, fully weak k-primary].

Lemma 2.2. *Let N be a submodule of M with $(N:M) \subseteq \mathfrak{P}$ where $\mathfrak{P}$ is a prime ideal of R. Then rN is not a primary submodule of M, for each $r \in R \setminus \mathfrak{P}$ such that $N \neq rN$.*

Proof. Let $x \in N \setminus rN$. Since $rx \in rN$ and $r \notin \sqrt{(rN:M)} \subseteq \sqrt{(N:M)} \subseteq \mathfrak{P}$, if rN is primary, we must have $x \in rN$, a contradiction. □

Recall that a proper submodule N of M is said to be *prime*, if from $rm \in N$ where $r \in R$ and $m \in M$, we can deduce that $m \in N$ or $r \in (N:M)$. In this case, $\mathfrak{P} = (N:M)$ is a prime ideal and N is called a $\mathfrak{P}$-prime submodule. In [10], it is proved that if R is a generalized primary ring (that is, R is a fully primary R-module), then Spec(R) is a chain and in particular, R is a local ring. The following lemma, generalizes this result.

Lemma 2.3. *Suppose that M is a finitely generated faithful fully primary R-module. Then* Spec(R) *is a chain and R is a local ring.*

Proof. Assume that $\mathfrak{P}$ is a prime ideal of R. By [4, Lemma 4], there is a $\mathfrak{P}$-prime submodule P of M, because M is finitely generated. Thus $(\mathfrak{P}M:M) \subseteq (P:M) = \mathfrak{P}$ and hence $(\mathfrak{P}M:M) = \mathfrak{P}$. Let $b \in R \setminus \mathfrak{P}$. We will show $\mathfrak{P} \subseteq Rb$. Choose arbitrary elements $a \in \mathfrak{P}$ and $m \in M$.

If $x = am \notin Rbm$, then $bx \in Rabm$ and $x \notin Rabm$. Therefore, since $Rabm$ is primary, we get $b \in \sqrt{(Rabm : M)} \subseteq \sqrt{(Ram : M)} \subseteq \sqrt{(\mathfrak{P}M : M)} = \mathfrak{P}$, which is a contradiction. So $am \in Rbm$, say $am = bm'$ where $m' \in M$. Now since $b \notin \sqrt{(Ram : M)} \subseteq \mathfrak{P}$ and $bm' \in Ram$, we conclude that $m' \in Ram$. Hence $m' = ram$ for some $r \in R$ and $am = rbam$, that is, $am \in baM$. Since m was arbitrary we have $aM = baM$ and by Nakayama's lemma, there exists $r_0 \in R$ such that $r_0aM = 0$ and $r_0 \equiv 1 \pmod{Rb}$. Thus $a \equiv r_0a = 0 \pmod{Rb}$, that is, $a \in Rb$. Since $a \in \mathfrak{P}$ was arbitrary, we get $\mathfrak{P} \subseteq Rb$, as claimed.

Assume that $\mathfrak{P}'$ is a prime ideal of R not contained in $\mathfrak{P}$. Then by choosing $b \in \mathfrak{P}' \setminus \mathfrak{P}$, we see that $\mathfrak{P} \subseteq Rb \subseteq \mathfrak{P}'$. Consequently, $\mathfrak{P}$ is comparable to every other prime ideal of R and since $\mathfrak{P}$ was chosen arbitrarily, the set of prime ideals of R is a chain, in particular, R is local. □

Theorem 2.4. *An R-module M is fully primary if and only if either of the following holds.*

- **(i)** $\frac{R}{\mathrm{Ann(M)}}$ *is a 1-dimensional local domain, the zero submodule of M is prime and for each submodule $0 \neq N$ of M, we have $(N : M) \not\subseteq \mathrm{Ann(M)}$.*
- **(ii)** $\frac{R}{\mathrm{Ann(M)}}$ *is a 0-dimensional local ring.*

Proof. ($\Leftarrow$): Just note that for every submodule N of M in case (ii) and every nonzero submodule N of M in case (i), $\sqrt{N : M}$ is the unique maximal ideal of R containing $(N : M)$.

($\Rightarrow$): Clearly we can assume that M is faithful. First we show that for each nonzero proper submodule N of M, $\mathfrak{P} = \sqrt{(N : M)}$ is a maximal ideal. On the contrary, suppose that $\mathfrak{P}$ is not maximal and $\mathfrak{M}$ is a maximal ideal of R containing $\mathfrak{P}$. Let $r \in \mathfrak{M} \setminus \mathfrak{P}$ and $0 \neq x \in N$. Then $rRx = Rx$ by (2.2). Thus for some $r' \in R$, we have $(1 - rr')x = 0$. Since the zero submodule of M is primary, we get $1 - rr' \in \sqrt{(0 : M)} = \mathrm{N(R)} \subseteq \mathfrak{M}$, a contradiction. Consequently, $\mathfrak{P}$ is maximal.

Next we show that R is either 0-dimensional or a 1-dimensional domain. If M is finitely generated, then R is local by (2.3) and since $\mathfrak{P} = \sqrt{(\mathfrak{P}M : M)}$ is maximal for each prime ideal $0 \neq \mathfrak{P}$ of R, the claim follows.

Now consider the case that M is not necessarily finitely generated. Let $0 \neq M'$ be a finitely generated submodule of M. Then $\mathrm{Ann(M')(M' : M)} = 0$, because M is faithful. But since $0 = (0 : M)$ is a primary ideal, either $\mathrm{Ann(M')} = 0$ or $(M' : M) \subseteq \mathrm{N(R)}$. In the former case, M' is a finitely generated faithful fully primary R-module, whence according to the above argument, R is a 0-dimensional local ring or a 1-dimensional local domain. In the latter case, that is, if $(M' : M) \subseteq \mathrm{N(R)}$, since $\sqrt{(M' : M)}$ is maximal, we see that R is a 0-dimensional local ring.

Thus to complete the proof, we just need to show that if $(R, \mathfrak{M})$ is a 1-dimensional local domain, then the zero submodule is prime and $(N : M) \neq 0$ for each nonzero submodule of M. This last assertion follows from the fact that $\sqrt{(N : M)} = \mathfrak{M}$. Also if $rm = 0$ for $r \in R$ and $0 \neq m \in M$, then as the zero submodule is primary, we get that $r \in \mathrm{N(R)} = 0 = (0 : \mathrm{M})$, that is, the zero submodule of M is prime. □

Corollary 2.5. *There exists a faithful fully primary R-module, if and only if R is either a 0-dimensional local ring or a 1-dimensional local domain.*

Corollary 2.6. *Every R-module is fully primary, if and only if R is a 0-dimensional local ring.*

Proof. Let $(R, \mathfrak{M})$ be a 1-dimensional local domain, $M = R \oplus R$, $N = 0 \oplus \mathfrak{M}$ and $0 \neq x \in \mathfrak{M}$. Then $x(0,1) \in N$, although $(0,1) \notin N$ and $x \notin \sqrt{(N : M)} = 0$. Hence N is not a primary submodule of M. □

Corollary 2.7. *Suppose that $(R, \mathfrak{M})$ is a 1-dimensional local domain, $\mathfrak{M}$ is principal and M is a faithful finitely generated R-module. Then M is fully primary if and only if $M \cong R$.*

Proof. $(\Leftarrow)$: Obvious. $(\Rightarrow)$: By Cohen's theorem, R is Noetherian and since $\mathfrak{M}$ is principal, R is a discrete valuation domain. Therefore, every finitely generated torsion-free R-module is free. According to (2.4), M is torsion-free and hence free. But it is easy to see that if rank of $M \geq 2$, then M is not fully primary. Consequently, rank of $M = 1$ and $M \cong R$, as asserted. □

Suppose that $(R, \mathfrak{M})$ is a 1-dimensional local domain and M is a finitely generated faithful fully primary R-module. (2.7) shows that if $\mathfrak{M}$ can be generated by one element, M also can be generated by one element. One might guess that in general, M can be generated by at most n elements, where n is the number of generators of $\mathfrak{M}$. But the following example shows that this is not the case.

Example 2.8. Let n be an arbitrary positive integer and $k \in \mathbb{N}$ be such that $(k+1)(n-1) < kn$. Let $R_0 = \mathbb{Z}_2[X^k, X^{k+1}]$ and set R to be the localization of R_0 at $\langle X^k, X^{k+1} \rangle$. Then since $\mathbb{Z}_2[X]$ is integral over R_0, we see that R is a 1-dimensional local domain. If $\mathfrak{M} = \langle x^k, x^{k+1} \rangle$ where x denotes the image of X

in R, then $\mathfrak{M}^{n-1} = \langle x^{(n-1)k}, x^{(n-1)k+1}, \ldots, x^{(n-1)(k+1)}\rangle$ and $\mathfrak{M}^n = \langle x^{nk}, x^{nk+1}, \ldots, x^{n(k+1)}\rangle$. Thus, by the way we have chosen k, dim $\frac{\mathfrak{M}^{n-1}}{\mathfrak{M}^n} = n$ as a vector space over $\mathbb{Z}_2 \cong \frac{R}{\mathfrak{M}}$. Therefore, by Nakayama's lemma, we see that the minimum number of generators of $\mathfrak{M}^{n-1}$ is n. Since R is a fully primary R-module, its submodule $\mathfrak{M}^{n-1}$ is also fully primary and clearly faithful. Note that $\mathfrak{M}$ is generated by 2 elements.

The fully primary modules constructed above are finitely generated. The authors could not find any faithful fully primary module M over a 1-dimensional Noetherian local domain, which is not finitely generated, on the other hand, we could not prove that such a module does not exist. Therefore, we pose the following question. Note that if $(R, \mathfrak{M})$ is a non-Noetherian 1-dimensional local domain (for example, if R is 1-dimensional valuation domain which is not discrete), then $\mathfrak{M}$ is a faithful fully primary R-module which is not finitely generated.

Question 2.9. *Is there any faithful fully primary module M over a 1-dimensional Noetherian local domain, such that M is not finitely generated?*

3. Almost Fully Primary Modules

Next we study modules in which every proper submodule, except possibly the zero submodule, is primary. We shall call such modules, *almost fully primary* or *a.f. primary* for short. First we need a lemma.

Lemma 3.1. *Let $R = R_1 \times R_2$ for nontrivial rings R_1 and R_2 and M be a faithful R-module. Then*

(i) *$M = M_1 \oplus M_2$ where M_i is an R_i-module ($i = 1, 2$).*
(ii) *M is a.f. primary if and only if $M \cong R$ and R_i's are fields.*

Proof. (i): Take $M_1 = (1, 0)M$ and $M_2 = (0, 1)M$.

(ii) ($\Leftarrow$): Obvious. ($\Rightarrow$): Now assume that M is a.f. primary. If M_1 is not simple and N_1 is a nontrivial submodule of M_1 and $m \in M_1 \setminus N_1$, then $(0, 1)(m, 0) \in N = N_1 \oplus 0 \neq 0$ but neither $(m, 0) \in N$ nor $(0, 1) \in \sqrt{(N : M)}$, a contradiction. So M_1 and similarly M_2 are simple modules over R_1 and R_2, respectively. Because M is faithful, $\mathrm{Ann(M_1)} \cap \mathrm{Ann(M_2)} = 0$ and hence $M \cong R \cong \frac{R}{\mathrm{Ann(M_1)}} \times \frac{R}{\mathrm{Ann(M_2)}}$ and the result follows. □

Now we give a characterization of a.f. primary modules.

Theorem 3.2. *Let M be a faithful R-module. Then M is a.f. primary, if and only if one of the following holds:*

(i) *$M \cong R$ and R is a direct product of two fields.*

(ii) *R is a zero dimensional local ring.*
(iii) *R has exactly two different prime ideals $\mathfrak{P} \subseteq \mathfrak{M}$ with $\mathfrak{M}\mathfrak{P} = 0$. Also $\mathfrak{P}M$ is a $\mathfrak{P}$-prime submodule of M and is either zero or simple. Moreover, for each nonzero submodule $N \neq \mathfrak{P}M$ of M, $(N : M) \not\subseteq \mathfrak{P}$.*

Proof. It is easy to see that in each of these cases M is a.f. primary. For the converse, by (3.1), we can assume that R is indecomposable and show that (ii) or (iii) holds. We consider two cases:

Case 1: For every nonzero proper submodule N of M, $\mathfrak{M} = \sqrt{(N : M)}$ is a maximal ideal. First we show that in this case, $\mathrm{Ann(N)} \subseteq \mathrm{N(R)}$ for every submodule N of M. If $\mathfrak{P}$ is any prime ideal of R different from $\mathfrak{M}$ and $r \in \mathfrak{M} \setminus \mathfrak{P}$, then there is an $n \in \mathbb{N}$ such that $r^n M \subseteq N$. Thus $r^n \mathrm{Ann(N)} = 0 \subseteq \mathfrak{P}$, since M is faithful. Hence $\mathrm{Ann(N)} \subseteq \mathfrak{P}$. If $\mathrm{Ann(N)}$ is not contained in $\mathfrak{M}$ and $A = \cap\{\mathfrak{P} | \mathfrak{M} \neq \mathfrak{P}$ is a prime ideal of $R\}$, then $\mathrm{Ann(N)} \subseteq \mathrm{A}$ and hence $A \not\subseteq \mathfrak{M}$. Therefore, we have $A + \mathfrak{M} = R$, $A \cap \mathfrak{M} = \mathrm{N(R)}$ and neither $A = \mathrm{N(R)}$ nor $\mathfrak{M} = \mathrm{N(R)}$. Consequently, $\frac{R}{\mathrm{N(R)}}$ is decomposable and by [2, Proposition 27.1], R is decomposable, against our assumption. From this we conclude that $\mathrm{Ann(N)} \subseteq \mathfrak{M}$, whence $\mathrm{Ann(N)} \subseteq \mathrm{N(R)}$.

Now suppose that $\mathfrak{P}$ is a nonzero prime ideal of R. Since M is faithful, there exists an $m \in M$ such that $\mathfrak{P}m \neq 0$. Because $\mathrm{Ann(Rm)} \subseteq \mathrm{N(R)} \subseteq \mathfrak{P}$, according to [4, Lemma 4], there is a $\mathfrak{P}$-prime submodule P of Rm. If $P = 0$, then $\mathfrak{P}Rm \subseteq P = 0$, a contradiction. So P is nonzero and hence primary in M and Rm with $\sqrt{(P : M)} \subseteq \sqrt{(P : Rm)} = \mathfrak{P}$. So by the assumption of this case, we see that $\mathfrak{P}$ is maximal. Since $\mathfrak{P}$ was an arbitrary nonzero prime ideal of R, this means that R is either a 0-dimensional local ring (that is, (ii) has come true) or a 1-dimensional local domain. If the latter case holds and $rx = 0$ for some $r \in R$ and $0 \neq x \in M$, then $r(Rx : M) = 0$ and hence $r = 0$, because $(Rx : M) \neq 0$ by the assumption of this case. Thus the zero submodule of M is 0-prime and (iii) holds with $\mathfrak{P} = 0$.

Case 2: There is a nonzero submodule N of M, such that $\mathfrak{P} = \sqrt{(N : M)}$ is not maximal. First we show that N is simple. Else, if N' is a nontrivial submodule of N, then $\overline{M} = \frac{M}{N'}$ is a fully primary module and hence it follows from (2.4) that for each nonzero proper submodule K of $\overline{M}$, $\sqrt{(K : \overline{M})}$ is a maximal ideal. In particular, this is true for $K = \frac{N}{N'}$ and hence $\mathfrak{P}$ is maximal, a contradiction. Therefore, N is simple.

Also $\frac{M}{N}$ is a faithful fully primary $\frac{R}{\mathfrak{P}}$-module, hence by (2.4), we get that $\frac{R}{\mathfrak{P}}$ is a 1-dimensional local domain and $\frac{M}{N}$ is torsion-free, that is, N

is a $\mathfrak{P}$-prime submodule of M. Set $\mathfrak{M} = \mathrm{Ann(N)}$, which is a maximal ideal of R. Since $\mathfrak{M}\mathfrak{P}M \subseteq \mathfrak{M}N = 0$, we have $\mathfrak{M}\mathfrak{P} = 0$, therefore, $\mathfrak{P}$ is contained in every prime ideal of R, except possibly $\mathfrak{M}$. If $\mathfrak{P} \nsubseteq \mathfrak{M}$, then by an argument similar to the first paragraph of case 1, we see that R is decomposable, which is against our assumption. So $\mathfrak{P}$ is contained in all prime ideals of R and because $\frac{R}{\mathfrak{P}}$ is a 1-dimensional local domain, we conclude that the only prime ideals of R are $\mathfrak{P}$ and $\mathfrak{M}$. Moreover, $0 \neq \mathfrak{P}M \subseteq N$ and since N is simple, it follows that $N = \mathfrak{P}M$. Consequently, (iii) holds. □

Remark 3.3. Case 1 of the proof of (3.2) shows that if R is indecomposable and $\sqrt{(N:M)}$ is a maximal ideal for each nonzero proper submodule N of M, then the zero submodule of M is also primary and M is a fully primary module.

Corollary 3.4. *Every nonzero proper ideal of R is primary, if and only if either (i) R is a 0-dimensional local ring or (ii) R is a 1-dimensional local domain or (iii) R is a direct product of two fields or (iv) R has exactly two prime ideals $\mathfrak{P} \subseteq \mathfrak{M}$ such that $\mathfrak{P}$ is a minimal ideal and $\mathfrak{M}\mathfrak{P} = 0$.*

Example 3.5. Let R be the localization of $\frac{K[X,Y]}{\langle X^2, XY\rangle}$ at $\langle X, Y\rangle$, where K is a filed and suppose that x, y denote the images of X, Y in R, respectively. Then clearly the only prime ideals of R are Rx and $\mathfrak{M} = Rx + Ry$, and also $\mathfrak{M}Rx = 0$, whence Rx is simple. Therefore, every nonzero ideal of R is primary. Thus the R-module R serves as an example of case (iii) of (3.2), with $\mathfrak{P} \neq 0$. Note that zero is not primary in R. Also $Ry \supsetneqq Ry^2 \supsetneqq Ry^3 \cdots$ is a chain of ideals of R not containing Rx. Hence, although Rx is the unique minimal ideal of R, it is not the smallest ideal.

Let N be a submodule of M, then the set $\mathrm{E}(N) = \{rm | \exists n \in \mathbb{N} \quad r^n m \in N\}$ is called the envelope of N in M. Let $R\mathrm{E}(N)$ denote the submodule of M generated by $\mathrm{E}(N)$ and $\mathrm{rad}(N)$ denote the intersection of all prime submodules of M containing N (if there is no such submodule, we set $\mathrm{rad}(N) = M$). It is easy to see that $N \subseteq R\mathrm{E}(N) \subseteq \mathrm{rad}(N)$. An R-module M is said to satisfy the radical formula (s.t.r.f.), when for all submodules N of M, $R\mathrm{E}(N) = \mathrm{rad}(N)$ (see, for example, [6, 11, 12]).

Lemma 3.6 ([12, Lemma 1.3])**.** *If N is a primary submodule of M, then $RE(N) = N + \sqrt{(N:M)}M$.*

Corollary 3.7. *If M is a.f. primary, then M s.t.r.f.*

Proof. By passing to $\frac{R}{\mathrm{Ann(M)}}$ we suppose that M is faithful and hence satisfies the conditions of (3.2). If case (i) or (ii) of (3.2) holds, then

R is 0-dimensional and by [11, Theorem 2.8], every R-module s.t.r.f.. Thus assume that case (iii) of (3.2) is true. For each nonzero proper submodule $N \neq \mathfrak{P}M$ of M, $(N : M) \not\subseteq \mathfrak{P}$ (in the notations of (3.2)) and hence $\sqrt{(N : M)} = \mathfrak{M}$. Therefore, $R\mathrm{E}(N) = N + \mathfrak{M}M$, whence either $(R\mathrm{E}(N) : M) = \mathfrak{M}$ and $R\mathrm{E}(N)$ is a prime submodule of M or $R\mathrm{E}(N) = M$ (see [12, Lemma 1.4]). So $\mathrm{rad}(N) \subseteq R\mathrm{E}(N)$, as required. Also since $\mathfrak{P}M$ is a prime submodule, $R\mathrm{E}(\mathfrak{P}M) = \mathrm{rad}(\mathfrak{P}M) = \mathfrak{P}M$. Moreover, $\mathfrak{P}^2 = 0$. Hence $\mathfrak{P}M \subseteq R\mathrm{E}(0)$, but $\mathrm{rad}(0) \subseteq \mathfrak{P}M$ and consequently $\mathrm{rad}(0) = R\mathrm{E}(0)$. □

4. Fully k-Primary and Weak k-Primary Modules

Now we turn our attention to fully k-primary and fully weak k-primary modules.

Lemma 4.1. *Assume that M is a faithful fully weak k-primary R-module. If $a \in \mathrm{J(R)}$, then $a^k = 0$.*

Proof. Let $a \in \mathrm{J(R)}$ with $a^k \neq 0$ and $0 \neq m \in M$. If $m = ram$ for some $r \in R$, then $(1 - ra)m = 0$ and by the choice of a, we have $m = 0$, a contradiction. So $m \notin Ram$. But $am \in Ram$ and Ram is weak k-primary, whence $a^k M \subseteq Ram$. Let $m' \in M$ be arbitrary, then $a^k m' = ram$ for some $r \in R$, that is, $a(a^{k-1}m' - rm) = 0$. By zero being primary, we get either $a^k M = 0$, against our assumption, or $a^{k-1}m' \in Rm$. Since m' was arbitrary we conclude that $N = a^{k-1}M \subseteq Rm$. But $0 \neq m$ was also arbitrary, so N is contained in every nonzero submodule of M and hence it is simple. If $\mathfrak{M} = \mathrm{Ann(N)}$, then $\mathfrak{M}a^{k-1}M = 0$. Thus $\mathfrak{M}a^{k-1} = 0$. But $\mathfrak{M}$ is a maximal ideal and $a \in \mathfrak{M}$, therefore, $a^k \in \mathfrak{M}a^{k-1} = 0$. □

Theorem 4.2. *For a ring R, the following are equivalent.*

- **(i)** *Every R-module is fully weak k-primary [fully k-primary].*
- **(ii)** *R is a fully weak k-primary [fully k-primary] R-module.*
- **(iii)** *There exists a faithful fully weak k-primary [faithful fully k-primary] R-module.*
- **(iv)** *$(R, \mathfrak{M})$ is a 0-dimensional local ring and $r^k = 0$ for all $r \in \mathfrak{M}$ [$\mathfrak{M}^k = 0$].*

Proof. (i) ⇒ (ii) ⇒ (iii): Trivial. Also if (iv) holds, then for each proper principal ideal [proper ideal] I of R and every submodule N of M, we have $I^k \subseteq (N : M)$. So (iv) ⇒ (i).

(iii) ⇒ (iv): First we consider the case that a faithful fully weak k-primary R-module exists. By (2.4), $(R, \mathfrak{M})$ is either a 1-dimensional local domain, or a 0-dimensional local ring. Also according to (4.1), we have $r^k = 0$ for all $r \in \mathfrak{M}$. Note that this can not happen in a 1-dimensional domain. Thus the result on fully weak k-primary modules follows.

Now to prove the fully k-primary case, assume that M is a faithful fully k-primary module. Note that M is also fully weak k-primary, so R is a 0-dimensional local ring and $r^k = 0$ for all $r \in \mathfrak{M}$. Suppose that $\mathfrak{M}^k \neq 0$, say $r_1 r_2 \cdots r_k \neq 0$ for some r_i's in $\mathfrak{M}$. Set $I = \langle r_1, r_2, \ldots, r_k \rangle$. Clearly $I^{k^2} = 0$ and hence $I(I^{k^2-1}M) = 0$. Since $0 \neq r_1 r_2 \cdots r_k \in I^k$, we deduce that $I^{k^2-1} = 0$. By a similar argument I^{k^2-2}=0, I^{k^2-3}=0, ... and finally $I^k = 0$, a contradiction. Consequently, $\mathfrak{M}^k = 0$. $\square$

Note that in the case $k = 1$, both of the notions of k-primary and weak k-primary submodules, coincide with the notion of prime submodules. Thus (4.2), generalizes the well-known fact that if every proper submodule of M is prime, then M is a vector space over $\frac{R}{\mathrm{Ann(M)}}$. But when $k > 1$, even if $k = 2$, the two concepts of k-primary submodules and weak k-primary submodules do not coincide, as the following example shows.

Example 4.3. Let $R = \frac{\mathbb{Z}_2[X,Y]}{\langle X^2, Y^2 \rangle}$. Then $(R, Rx + Ry)$ is a 0-dimensional local ring, where x, y denote the images of X, Y in R, respectively. For each nonunit element of R such as $f = ax + by + cxy$, we have $f^2 = 2abxy = 0$, but $0 \neq xy \in (Rx + Ry)^2$. Therefore, R is a fully weak 2-primary R-module, which is not fully 2-primary. Note that every nonzero proper ideal of R is 2-primary and Rxy is the unique smallest ideal of R.

We say that M is almost fully k-primary (a.f. k-primary, for short) or almost fully weak k-primary (a.f. weak k-primary, for short), if every nonzero proper submodule of M is k-primary or weak k-primary, respectively.

Theorem 4.4. *M is a.f. k-primary [a.f. weak k-primary], if and only if either of the following holds.*

- **(i)** *M is cyclic and $\frac{R}{\mathrm{Ann(M)}}$ is a direct product of two fields.*
- **(ii)** *M is fully k-primary [fully weak k-primary].*
- **(iii)** *M is fully $(k+1)$-primary [fully weak $(k+1)$-primary] and has a unique smallest submodule N such that $\frac{M}{N}$ is fully k-primary [fully weak k-primary].*

Proof. ($\Leftarrow$): Immediate from (2.1) and (3.2).

($\Rightarrow$): We consider the k-primary submodules, the assertion on weak k-primary submodules is proved similarly. By passing to $\frac{R}{\mathrm{Ann(M)}}$ we can assume that M is faithful. Also according to (3.1), we can assume that M is indecomposable. Because M is a.f. primary, case (ii) or case (iii) of (3.2), must hold. If (iii) of that theorem holds, then $\frac{M}{\mathfrak{P}M}$ is a faithful fully k-primary module over the 1-dimensional domain $\frac{R}{\mathfrak{P}}$, which contradicts (4.2). Therefore, $(R,\mathfrak{M})$ is a 0-dimensional local ring.

Suppose that M is not fully k-primary. It is easy to see that the intersection of every chain of k-primary submodules is again k-primary. Thus by Zorn's lemma, every submodule of M contains a minimal k-primary submodule. But since every nonzero proper submodule of M is k-primary and the zero submodule is not k-primary, we conclude that minimal k-primary submodules are minimal submodules.

Let N be a minimal submodule of M. Then clearly $\frac{M}{N}$ is a fully k-primary $\frac{R}{(N:M)}$-module. Hence it follows from (4.2), that $\mathfrak{M}^k M \subseteq N$. If $\mathfrak{M}^k = 0$, then M is fully k-primary, against our assumption. Whence $0 \neq \mathfrak{M}^k M \subseteq N$, and because of minimality of N, we get $N = \mathfrak{M}^k M$. Consequently, $\mathfrak{M}^k M$ is the only minimal submodule of M and since every submodule of M contains a minimal submodule, N is the unique smallest submodule of M. Also note that $\mathfrak{M}^{k+1}M = \mathrm{Ann(N)}\mathfrak{M}^{\mathrm{k}}\mathrm{M} \subseteq \mathrm{Ann(N)N} = 0$. So $\mathfrak{M}^{k+1} = 0$ and by (4.2), M is fully $(k+1)$-primary. □

Now we can deduce Theorem 10 of [14] as a corollary to (4.4) above. Also note that in the case $k = 1$, the next corollary coincides with [13, Theorem 2.3].

Corollary 4.5. *Every nonzero proper ideal of R is k-primary [weak k-primary], if and only if R is either a direct product of two fields or a 0-dimensional local ring with maximal ideal $\mathfrak{M}$ such that $\mathfrak{M}^k$ [for any $r \in \mathfrak{M}$, Rr^k] is zero or the unique smallest ideal of R.*

Indeed, this corollary is stronger than [14, Theorem 10], since that theorem, in addition to the three cases of (4.5), has the following extra case: "$(R,\mathfrak{M})$ is a local ring and $\mathfrak{M}^k = \mathfrak{M}^{k+1}$ is a minimal nonzero ideal of R, and there exists exactly one nonzero ideal of R that does not contain $\mathfrak{M}^k$." Therefore, by comparing (4.5) and [14, Theorem 10], we conclude the following result (in fact, it is not much hard to prove this result directly and without any use of (4.4)).

Corollary 4.6. *There does not exist any commutative local ring $(R,\mathfrak{M})$ such that $\mathfrak{M}^k = \mathfrak{M}^{k+1}$ is a minimal nonzero ideal of R and R has exactly one nonzero ideal that does not contain $\mathfrak{M}^k$.*

5. Residually Fully Primary Modules

Next, we consider residually fully primary modules. As we will see in (5.2), the existence of a faithful residually fully primary, (even k-primary) R-module is not a much restrictive condition on a ring R (compare with (3.2)). Before that we need the following lemma.

Lemma 5.1. *If I is a k-primary [weak k-primary] ideal of R, then $\left(\sqrt{I}\right)^k \subseteq I$ [$r^k \in I$, for each $r \in \sqrt{I}$].*

Proof. Suppose that I is a k-primary ideal of R and $r_1, r_2, \ldots, r_k \in \sqrt{I}$ with $r_1 r_2 \cdots r_k \notin I$. Set $J = \langle r_1, r_2, \ldots, r_k \rangle$. Then $J^n \subseteq I$ for some $n \in \mathbb{N}$ and hence, as I is k-primary, we conclude that $J^k \subseteq I$. But this is in contradiction with the choice of r_i's, and the result follows. The proof of the statement on weak k-primary ideals is similar. $\square$

Recall that M is called $\mathfrak{P}$-*secondary*, if $\mathfrak{P} = \sqrt{\mathrm{Ann(M)}}$ and for each $r \in R \setminus \mathfrak{P}$, $rM = M$. It is easy to see that in this case Ann(M) is $\mathfrak{P}$-primary. Note that every divisible module over a domain is 0-secondary (see, for example, [7, Appendix to §6]).

Example 5.2. Suppose that M is a $\mathfrak{P}$-secondary module and N is a proper submodule of M. Then it is easy to see that $(N : M)$ is a $\mathfrak{P}$-primary ideal of R. Therefore, M is residually fully primary. Moreover, if Ann(M) is k-primary [weak k-primary], then using (5.1), it can be readily checked that M is residually fully k-primary [residually fully weak k-primary].

Now let R be a ring, with primary [resp. k-primary, weak k-primary] zero ideal and $M = T(R)$ be the total quotient ring of R. Then M is a faithful 0-secondary R-module, and hence is residually fully primary [resp. k-primary, weak k-primary]. In particular, if R is any integral domain and Q is its quotient field, then Q is a faithful residually fully k-primary R-module, for every $k \in \mathbb{N}$. Note that Q is a fully primary R-module, if and only if , $R = Q$.

In the following theorem, we give a characterization of residually fully primary modules.

Theorem 5.3. *An R-module M is residually fully primary, if and only if for every pair of finitely generated (or equivalently, principal) ideals I and J of R, one of the following occurs:*

(i) $IM = IJM$ (ii) $JM = IJM$ (iii) $\sqrt{\mathrm{(IM : M)}} = \sqrt{\mathrm{(JM : M)}}$.

Proof. ($\Rightarrow$): Since $IJ \subseteq (IJM : M)$, so $IM \subseteq IJM$ and case (i) holds or for all $j \in J$, there is an $s \in \mathbb{N}$, such that $j^s M \subseteq IJM \subseteq IM$. Note that in the latter case, there exists an $n \in \mathbb{N}$, such that $J^n M \subseteq IM$, for J is finitely generated. Similarly we have $JM = IJM$ or $I^{n'}M \subseteq IM$ for some $n' \in \mathbb{N}$. Therefore, if neither of cases (i) and (ii) is true, then $I^{n'}M \subseteq JM$ and $J^n M \subseteq IM$, for some $n, n' \in \mathbb{N}$. Now if $r \in \sqrt{(IM : M)}$, then for some $t \in \mathbb{N}$, we have $r^t M \subseteq IM$. Consequently, $r^{tn'}M \subseteq r^{t(n'-1)}IM \subseteq r^{t(n'-2)}I^2 M \subseteq \cdots \subseteq I^{n'}M \subseteq JM$, that is, $r \in \sqrt{(JM : M)}$. Thus $\sqrt{(IM : M)} \subseteq \sqrt{(JM : M)}$ and by a similar argument the inverse inclusion also holds.

($\Leftarrow$): Let N be a proper submodule of M and $r_1 r_2 M \subseteq N$ and assume that $r_2 M \nsubseteq N$. We must show that $r_1^n M \subseteq N$, for some $n \in \mathbb{N}$. Apply the assumption with $I = Rr_1$ and $J = Rr_2$. Clearly $r_2 M \neq r_1 r_2 M$ and if $r_1 M = r_1 r_2 M$, then $r_1 M \subseteq N$. Therefore, we can assume case (iii) is true. Hence $Rr_1 \subseteq \sqrt{(r_1 M : M)} = \sqrt{(r_2 M : M)}$ and for some $n \in \mathbb{N}$, we have $r_1^n M \subseteq r_2 M$. So $r_1^{n+1} M \subseteq r_1 r_2 M \subseteq N$, as required. Note that we have used the assumption of the statement just in the case that I and J are principal. $\square$

Lemma 5.4. *If M is residually fully k-primary [residually fully weak k-primary], then for every ideal [principal ideal] I of R, we have $I^k M = I^{k+1} M$.*

Proof. Obviously $I^{k+1}M \subseteq I^k M$. Since $I \subseteq \sqrt{(I^{k+1}M : M)}$, by (5.1), $I^k M \subseteq I^{k+1} M$, as asserted. $\square$

Corollary 5.5. *An R-module M is residually fully k-primary [residually fully weak k-primary], if and only if for every pair of ideals [principal ideals] I and J of R, one of the following occurs:*

$$\text{(i) } IM = IJM \quad \text{(ii) } JM = IJM \quad \text{(iii) } \mathrm{I^k M = J^k M}.$$

Proof. Suppose that M is residually fully k-primary and cases (i) and (ii) do not hold for ideals I and J of R. Then according to (5.3), $I \subseteq \sqrt{(IM : M)} = \sqrt{(JM : M)}$ and by (5.1), $I^k M \subseteq JM$. Thus by (5.4), $I^k M = I^{k^2} M \subseteq J^k M$. Similarly $J^k M \subseteq I^k M$.

Conversely, if case (iii) holds for a pair of ideals I and J of R, then by an argument similar to the first paragraph of (5.3), one can see that $\sqrt{(IM : M)} = \sqrt{(JM : M)}$. Hence the result follows from (5.3). A similar argument deals with the weak k-primary case. $\square$

Despite (5.2), we will show that the existence of a finitely generated faithful fully primary R-module, is an strong condition on R.

Theorem 5.6. *The following are equivalent for a ring R.*

(i) *Every R-module is residually fully primary.*

(ii) *There exists a finitely generated faithful residually fully primary R-module.*

(iii) *R is either a 0-dimensional local ring or a 1-dimensional local domain.*

Proof. (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii): Clear. (ii) $\Rightarrow$ (iii): Let M be a nonzero finitely generated faithful residually fully primary R-module. By [4, Lemma 4], $(\mathfrak{P}M : M) = \mathfrak{P}$ for all prime ideals $\mathfrak{P}$ of R. Therefore, if $\mathfrak{P}$ and $\mathfrak{Q}$ are prime ideals of R, then $\mathfrak{P} \cap \mathfrak{Q} = (\mathfrak{P}M : M) \cap (\mathfrak{Q}M : M) = (\mathfrak{P}M \cap \mathfrak{Q}M : M)$ is primary and whence prime. Thus either $\mathfrak{P} \subseteq \mathfrak{Q}$ or $\mathfrak{Q} \subseteq \mathfrak{P}$, that is, the set of prime ideals of R is a chain, in particular, $(R, \mathfrak{M})$ is a local ring.

Now assume that I and J are two nonzero proper ideals of R. We will show that $\sqrt{(IM : M)} = \sqrt{(JM : M)}$. First assume that I and J are finitely generated. If $IM = IJM$, then it follows from the Nakayama's lemma, that $IM = 0$, which means $I = 0$, a contradiction. So $IM \neq IJM$ and on the same lines $JM \neq IJM$. Therefore, according to (5.3), $\sqrt{(IM : M)} = \sqrt{(JM : M)}$. Now consider the general case that I and J are not necessarily finitely generated. Let $\mathcal{A} = \{I_f \subseteq I | 0 \neq I_f \text{ is a finitely generated ideal of } R\}$. Then we have $IM = \bigcup_{I_f \in \mathcal{A}} I_f M$, therefore, $\sqrt{(IM : M)} = \bigcup_{I_f \in \mathcal{A}} \sqrt{(I_f M : M)}$, for M is finitely generated. A similar formula holds for $\sqrt{(JM : M)}$. But $\sqrt{(I_f M : M)} = \sqrt{(J_f M : M)}$ for every pair of nonzero finitely generated ideals $I_f \subseteq I$ and $J_f \subseteq J$. Consequently, $\sqrt{(IM : M)} = \sqrt{(JM : M)}$.

By Nakayama's lemma, $\mathfrak{M}M \neq M$ and hence $\sqrt{(IM : M)} = \sqrt{(\mathfrak{M}M : M)} = \mathfrak{M}$ for each nonzero proper ideal I of R. Particularly, $\mathfrak{P} = \sqrt{(\mathfrak{P}M : M)} = \mathfrak{M}$ for each nonzero prime ideal $\mathfrak{P}$ of R, and the result follows. $\square$

Note that if $(R, \mathfrak{M})$ is a 1-dimensional local domain, then there exist finitely generated faithful R-modules which are residually fully primary but not fully primary, for example $R \oplus R$. Moreover, if R is Noetherian and $\mathfrak{M}$ is not principal, then $R \oplus R$ is a finitely generated residually fully primary R-module which does not s.t.r.f., by [6, Corollary 2.7]. But if we replace primary with k-primary, then such a module does not exist. To show this we need the following lemma, the proof of which is obvious.

Lemma 5.7. *Assume that N is a submodule of M. Then N is k-primary [weak k-primary] if and only if it is primary and residually k-primary [residually weak k-primary].*

Corollary 5.8. *For a finitely generated module, being residually fully k-primary [residually fully weak k-primary] is equivalent to being fully k-primary [fully weak k-primary].*

Proof. Suppose that M is a finitely generated faithful residually fully weak k-primary R-module. By (5.4), for each nonunit $r \in R$, we have $r^k M = r^{k+1} M$ and by Nakayama's lemma $r^k M = 0$, whence $r^k = 0$. Therefore, R is a 0-dimensional local ring. So M is fully primary and the result follows from (5.7). □

Corollary 5.9. *If there exists a finitely generated faithful R-module which is residually fully k-primary [residually fully weak k-primary], then every R-module is fully k-primary [fully weak k-primary].*

6. Integral Extensions and Fully Primary Modules

Corollary 6.1. *Let $R \subseteq T$ be an integral ring extension. Suppose that $\mathcal{P}$ is one of the properties of being: fully primary, fully k-primary, fully weak k-primary, residually fully primary, residually fully k-primary, or residually fully weak k-primary. Then*

- **(i)** *If every T-module has property $\mathcal{P}$, then every R-module has property $\mathcal{P}$.*
- **(ii)** *If T has property $\mathcal{P}$ as a T-module, then R has this property as an R-module.*
- **(iii)** *If there exist a faithful T-module with property $\mathcal{P}$, then there exists a faithful R-module with property $\mathcal{P}$.*

Proof. (i): Assume that every T-module is fully primary. Then by (2.6), $(T, \mathfrak{M})$ is a 0-dimensional local ring. It is well-known that R and T have the same Krull dimensions. Also according to Theorem 5.10 and Corollary 5.8 of [3], $(R, \mathfrak{M}')$ is a local ring, where $\mathfrak{M}' = \mathfrak{M} \cap R$. Therefore, R is 0-dimensional local and hence by (2.6), every R-module is fully primary. For the case of k-primary, just note that if $\mathfrak{M}^k = 0$, then $\mathfrak{M}'^k = 0$.

The proofs of (i) on other properties, and also the proof of (ii) are similar.

(iii): If $\mathcal{P}$ is being fully primary, fully k-primary or fully weak k-primary and there exists a faithful T-module with property $\mathcal{P}$, then T itself has this property as a T-module, by (4.2) and (2.5). Hence the result follows from (ii). For the cases that $\mathcal{P}$ stands for a "residual" property, note that for every ideal I of R and every T-module M, we have $IM = (IT)M$. Consequently, it follows from (5.3) and (5.5) that if M has property $\mathcal{P}$ as a T-module, then it has property $\mathcal{P}$ as an R-module. Note that the proof of (iii) for the "residual" properties does not use the integrality of the extension $R \subseteq T$, and hence the "residually" form of (iii) holds for every ring extension. $\square$

We end this paper by an example which shows that the properties of (6.1) do not necessarily ascend from R to T.

Example 6.2. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers and $n \in \mathbb{N}$. Then $\frac{\mathbb{Z}[i]}{\langle n \rangle} \cong \mathbb{Z}_n[i]$, which is called the ring of Gaussian integers modulo n. It is proved in [1] that for a prime integer p and a positive integer t, $S = \mathbb{Z}_{p^t}[i]$ is a local ring if and only if $p = 2$ or $p \equiv 3 \pmod 4$, since $p = 2$ ramifies in $\mathbb{Z}[i]$ and $p \equiv 3 \pmod 4$ remains prime. Note that in each of these cases, S has a unique t-nilpotent maximal ideal and hence every S-module is fully t-primary.

Now let $p \equiv 1 \pmod 4$. Then p is not a Gaussian prime and in fact splits (see, for example [9]). So there exist primes $a + ib$ and $a - ib$ such that $p = (a + ib)(a - ib)$. Thus the ideals $\langle a + bi \rangle$ and $\langle a - bi \rangle$ are the only maximal ideals in $\mathbb{Z}[i]$ containing p, for $\mathbb{Z}[i]$ is a PID. Therefore, $\mathbb{Z}_p[i] \cong \frac{\mathbb{Z}[i]}{\langle p \rangle} \cong \left(\frac{\mathbb{Z}[i]}{\langle a+bi \rangle}\right) \times \left(\frac{\mathbb{Z}[i]}{\langle a-bi \rangle}\right)$. Hence $T = \mathbb{Z}_p[i]$ is not even a residually fully primary T-module. But clearly every $\mathbb{Z}_p$-module is fully k-primary, for every $k \in \mathbb{N}$. Note that $R = \mathbb{Z}_p \subseteq T$ is an integral extension.

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References

1. E. A. Abu Osba, S. Al-Addasi, N. Abu Jaradeh, *Zero divisor graph for the ring of Gaussian integers modulo n*, Comm. Algebra, **36** (2008), 3865–3877.
2. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
3. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, London, 1969.
4. A. Azizi and H. Sharif, *On prime submodules*, Honam Math. J., **21**(1) (1999), 1–12.
5. R. Chaudhuri, *A note on generalized primary rings*, Mat. Vesnik., **13**(28) (1976), 375–377.
6. K. H. Leung and S. H. Man, *On commutative Noetherian rings which satisfy the radical formula*, Glasgow Math. J., **39** (1997), 285–293.
7. H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1992.

8. A. Nikseresht and H. Sharif, *Fully primary modules*, Proceedings of the 43^{rd} Annual Iraninan Mathematics Conference, 27–30 August 2012, University of Tabriz, Tabriz, Iran, 107–110.
9. P. Samuel, *Algebraic Theorey of Numbers*, Hermann, London, 1972.
10. M. Satyanarayana, *Generalized primary rings*, Math. Ann., **179** (1969), 109–114.
11. H. Sharif, Y. Sharifi and S. Namazi, *Rings satisfying the radical formula*, Acta Math. Hungar., **71** (1996), 103–108.
12. P. F. Smith, *Primary modules over commutative rings*, Glasgow Math. J., **43** (2001), 103–111.
13. H. Tsutsui, *Fully prime rings II*, Comm. Algebra, **24**(9) (1996), 2981–2989.
14. H. Yasuyuki and H. Tsutsui, *Fully k-priamry rings*, Comm. Algebra, **37**(7) (2009), 2225–2235.

Ashkan Niksersht
Department of Mathematics, Shiraz University , P.O.Box 71454, Shiraz, Iran.
Email: ashkan_nikseresht@yahoo.com

Habib Sharif
Department of Mathematics, Shiraz University , P.O.Box 71454, Shiraz, Iran.
Email: sharif@susc.ac.ir

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مدول های تماماً اولیه و مشابه با آن ها

اشکان نیک سرشت[*] و حبیب شریف
دانشکده ریاضی، دانشگاه شیراز، شیراز، ایران

چکیده
فرض کنید R یک حلقه جا بجایی و M یک R- مدول باشد. می گوییم M تماماً اولیه است، اگر هر زیر مدول محض M اولیه باشد. در این مقاله توصیف هایی از مدول های تماماً اولیه بیان می کنیم. همچنین توصیف هایی از حلقه هایی که بر روی آن ها هر مدول تماماً اولیه است و آنهایی که مدول وفادار تماماً اولیه ای بر رویشان وجود دارد ارائه می دهیم. اضافه بر آن، چند نوع مدول مشابه با مدول های تماماً اولیه را معرفی کرده و مسائل مشابهی را در مورد آن ها بررسی می کنیم.

[*] مؤلف مسئول
Email: ashcan_nikseresht@yahoo.com