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ARENS REGULARITY AND DERIVATIONS OF HILBERT MODULES WITH THE CERTAIN PRODUCT

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ABSTRACT. Let A be a C^*-algebra and E be a left Hilbert A-module. In this paper we define a product on E that making it into a Banach algebra and show that under the certain conditions E is Arens regular. We also study the relationship between derivations of A and E.

1. Introduction and preliminaries

The notion of Hilbert C^*-module is a natural generalization that of Hilbert space arising by replacing of the field of scalars $\mathbb{C}$ by a C^*-algebra. For commutative C^*-algebras, such generalization was described for the first time in the work of I. Kaplansky [6] and the general theory of Hilbert C^*-modules appeared in the basic papers of W. L. Paschke [10] and M. A. Rieffel [11]. Let us recall these notions with more details.

Let A be a C^*-algebra and E be a linear space which is a left A-module with a compatible scalar multiplication. The space E is called a left pre-Hilbert A-module if there exists an A-valued inner product ${}_E\langle .,. \rangle : E \times E \longrightarrow A$ with the following properties:

(i) ${}_E\langle x, x\rangle \geq 0$ and ${}_E\langle x, x\rangle = 0$ if and only if $x = 0$;
(ii) ${}_E\langle \lambda x + y, z\rangle = \lambda\, {}_E\langle x, z\rangle + {}_E\langle y, z\rangle$;
(iii) ${}_E\langle a.x, y\rangle = a\, {}_E\langle x, y\rangle$;
(iv) ${}_E\langle x, y\rangle^* = {}_E\langle y, x\rangle$ for all $x, y, z \in E, a \in A, \lambda \in \mathbb{C}$.

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From the validity of a useful version of the classical Cauchy-Schwartz inequality it follows that $||x|| = ||_E\langle x, x\rangle||^{\frac{1}{2}}$ is a norm on E making it into a normed left A-module [7]. The left pre-Hilbert module E is called left Hilbert A-module if it is complete with respect to the above norm. One interesting example of left Hilbert C^*-modules is any C^*-algebra A as a left Hilbert A-module via $_A\langle a, b\rangle = ab^*(a, b \in A)$.

The left Hilbert A-module E is called full if the closed linear span $_E\langle E, E\rangle$ of all elements of the form $_E\langle x, y\rangle$ $(x, y \in E)$ is equal to A. Likewise, a right Hilbert A-module with an A-valued inner product $\langle ., .\rangle_E$ can be defined. The reader is referred to [7] for more details on Hilbert C^*-modules .

For a normed space X, we denote by X' the topological dual of X. Now, let X, Y and Z be normed spaces and let $f : X \times Y \longrightarrow Z$ be a bounded bilinear map. In [2], R. Arens showed that f has two natural but different extensions f''' and $f^{r'''r}$ from $X'' \times Y''$ to Z''. The adjoint $f' : Z' \times X \longrightarrow Y'$ of f is defined by $< f'(z', x), y >=< z', f(x, y) >$ for all $x \in X, y \in Y, z' \in Z'$, which is also a bounded bilinear map. By setting $f'' = (f')'$ and continuing in this way, the mapping $f'' : Y'' \times Z' \longrightarrow X'$, $f''' : Y'' \times Z' \longrightarrow X'$ may be defined similarly.
We also denote by f^r the reverse map of f, that is, the bounded bilinear map $f^r : Y \times X \longrightarrow Z$ defined by $f^r(y, x) = f(x, y)$ for all $x \in X, y \in Y$, and it may be extended as above to $f^{r'''r} : X'' \times Y'' \longrightarrow Z''$.

The map f is called Arens regular when the equality $f''' = f^{r'''r}$ holds. Two natural extensions of the multiplication map $\pi : X \times X \longrightarrow X$ of a Banach algebra X, π''' and $\pi^{r'''r}$, are actually the so-called first and second Arens products, which will be denoted by $\Box$ and $\Diamond$, respectively. The Banach algebra X is said to be Arens regular if the multiplication map π is Arens regular. For example $L^1(G)$ is Arens regular if and only if G is finite [13].

A derivation of an algebra A is a linear mapping D from A into itself such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. For a fixed $b \in A$, the mapping $a \longmapsto ba - ab$ is clearly a derivation, which is called an inner derivation implemented by b.

Throughout this paper A denotes a C^*-algebra. We recall that every Hilbert module is a Banach space but the algebraic properties of Hilbert modules are our interesting subject. So in this note we utilize the A-valued inner product of Hilbert module E and define a product on E that making it into a Banach algebra. Our goal is finding the conditions under which E is Arens regular. We also study derivations of E and give some conditions under which innerness of derivations on A implies the innerness of derivations on E and vice-versa. Finally we give a necessary and sufficient condition under which every derivation of $C(X,H)$ is zero.

2. Arens regularity of Hilbert modules

In this section we introduce a product on a left Hilbert A-module that making it into a Banach algebra and study Arens regularity of this Banach algebra.

Let E be a left Hilbert A-module, and let e be an arbitrary element in E with $||e||=1$. Then by a direct calculation the map $\pi_e : E\times E \longrightarrow E$ defined by $\pi_e(x,y) = {}_E\langle x,e\rangle.y$ is a product on E that making it into a Banach algebra. We denote this Banach algebra by (E,π_e).

Example 2.1. Let X be a compact Hausdorff space and H be a Hilbert space. Then $E = C(X,H)$, the space of all continuous H-valued functions on X, is a Banach space and it is a left Banach $C(X)$-module with the module action defined by $\pi_l(f,\Lambda)(x) = f(x)\Lambda(x)$ for all $f\in C(X), \Lambda\in E, x\in X$. Also we define a $C(X)$-valued inner product ${}_E\langle .,.\rangle$ on E by ${}_E\langle \Lambda,\Gamma\rangle(x) = {}_H\langle \Lambda(x),\Gamma(x)\rangle$for all $\Lambda,\Gamma\in E, x\in X$. It is easy to verify that E is a left $C(X)$-Hilbert module.

Now let h be an arbitrary element of Hilbert space H with $||h||=1$. The map $\Lambda_0 : X \longrightarrow H$ defined by $\Lambda_0(x) = h$ for all $x\in X$ is a continuous H-valued function on X, therefore we have $\Lambda_0\in E$ and it is easy to see that ${}_E\langle \Lambda_0,\Lambda_0\rangle = 1_{C(X)}$. So π_{Λ_0} is a product on E that making it into a Banach algebra denoted by (E,π_{Λ_0}).

Theorem 2.2. [8] *For a bounded bilinear map $f : X\times Y \longrightarrow Z$ the following statements are equivalent:*

(i) *f is regular;*
(ii) *$f'''' = f^{r''''r}$;*
(iii) *$f''''(Z',X'') \subseteq Y'$;*
(iv) *the linear map $x \longmapsto f'(z',x) : X \longrightarrow Y'$ is weakly compact for every $z'\in Z'$.*

Theorem 2.3. *Let E be a left Hilbert A-module and let for all $x'\in E'$ the bounded linear map $T_{x'} : A \longrightarrow E'$ defined by $T_{x'}(a) = \pi_l'(x',a)$ be weakly compact . Then the Banach algebra (E,π_e) is Arens regular.*

Proof. Let $\varphi : E \longrightarrow A$ be defined by $\varphi(x) = {}_E\langle x,e\rangle$, then φ is a bounded linear map and let $\pi_l : A\times E \longrightarrow E$ be the left module action of A on E, thus $\pi_e(x,y) = \pi_l(\varphi(x),y)$. Now suppose that $x,y\in E, x'\in$

E', x'' and $y'' \in E''$. So we have:

$$\begin{aligned} < \pi_e'(x',x),y> = < x',\pi_e(x,y)> &= < x',\pi_l(\varphi(x),y)> \\ &= < \pi_l'(x',\varphi(x)),y> . \end{aligned}$$

$$\begin{aligned} < \pi_e''(x'',x'),x> = < x'',\pi_e'(x',x)> &= < x'',\pi_l'(x',\varphi(x))> \\ &= < \pi_l''(x'',x'),\varphi(x)> \\ &= < \varphi^*(\pi_l''(x'',x')),x> . \end{aligned}$$

$$\begin{aligned} < \pi_e'''(x'',y''),x'> &= < x'',\pi_e''(y'',x')> \\ &= < x'',\varphi^*(\pi_l''(y'',x'))> \\ &= < \varphi^{**}(x''),\pi_l''(y'',x')> \\ &= < \pi_l'''(\varphi^{**}(x''),y''),x'> . \end{aligned}$$

Therefore $\pi_e'''(x'',y'') = \pi_l'''(\varphi^{**}(x''),y'')$ (1). Now

$$\begin{aligned} < \pi_e^{r\prime}(x',x),y> = < x',\pi_e(y,x)> &= < x',\pi_l(\varphi(y),x)> \\ &= < x',\pi_l^r(x,\varphi(y))> \\ &= < \pi_l^{r\prime}(x',x),\varphi(y)> \\ &= < \varphi^*(\pi_l^{r\prime}(x',x)),y> . \end{aligned}$$

$$\begin{aligned} < \pi_e^{r\prime\prime}(x'',x'),x> = < x'',\pi_e^{r\prime}(x',x)> &= < x'',\varphi^*(\pi_l^{r\prime}(x',x))> \\ &= < \varphi^{**}(x''),\pi_l^{r\prime}(x',x)> \\ &= < \pi_l^{r\prime\prime}(\varphi^{**}(x''),x'),x> . \end{aligned}$$

$$\begin{aligned} < \pi_e^{r\prime\prime\prime r}(x'',y''),x'> &= < \pi_e^{r\prime\prime\prime}(y'',x''),x'> \\ &= < y'',\pi_e^{r\prime\prime}(x'',x')> \\ &= < y'',\pi_l^{r\prime\prime}(\varphi^{**}(x''),x')> \\ &= < \pi_l^{r\prime\prime\prime}(y'',\varphi^{**}(x'')),x'> \\ &= < \pi_l^{r\prime\prime\prime r}(\varphi^{**}(x''),y''),x'> . \end{aligned}$$

So we have $\pi_e^{r\prime\prime\prime r}(x'',y'') = \pi_l^{r\prime\prime\prime r}(\varphi^{**}(x''),y'')$ (2).
Now, since for all $x' \in E'$ the bounded linear mapping $a \longmapsto \pi_l'(x',a)$ from A to E' is weakly compact, so applying Theorem (2.2) for π_l shows that π_l is regular, and finally by (1), (2) we have $\pi_e'''(x'',y'') = \pi_e^{r\prime\prime\prime r}(x'',y'')$ for all $x'',y'' \in E''$, thus (E,π_e) is Arens regular. $\square$

Example 2.4. Let Y be a Banach space and X be a compact Hausdorff space. Then $C(X,Y)$, the space of all continuous Y-valued functions on X, is a Banach space and $\mathcal{M}(X,Y)$, the Banach space of all countably additive Y-valued measures with regular finite variation defined on the Borel σ-algebra $\mathcal{B}_X$ of X, is the topological dual of $C(X,Y)$ [3].

In particular when H is a Hilbert space $\mathcal{M}(X,H)$ is the topological dual of $C(X,H)$. It is proved that if Y^* is weakly sequentially complete then $\mathcal{M}(X,Y^*)$ is weakly sequentially complete [12]. Now since the Hilbert spaces are reflexive, so the topological dual of $C(X,H)$ is weakly sequentially complete, therefore by [1,Theorem 4.2] we have for all $x' \in E'$ the bounded linear mapping $a \longmapsto \pi_l'(x',a)$ from A to E' is weakly compact. Thus applying the above Theorem shows that $(C(X,H),\pi_{\Lambda_0})$ is an Arens regular Banach algebra.

Definition 2.5. Let E be a left Hilbert A-module and e be an arbitrary element in E with $||e||=1$. We define the set $A_e := \{\ {}_E\langle x,e\rangle : x \in E\}$.

It is easy to verify that A_e is a left ideal in A, but it is not closed in general. Indeed, let $A=\{f:[0,1]\longrightarrow \mathbb{C} : f \text{ is continuous }, f(1)=0\}$. Then, $f:[0,1]\longrightarrow\mathbb{C}$ defined by $f(x)=x-1$ is an element of A and $A_f=\{\ {}_A\langle g,f\rangle : g\in A\}=\{gf^*: g\in A\}$ is not closed, because $f\in\overline{A_f}$ and $f\notin A_f$.

Now we give some conditions under which A_e is a closed ideal in A. For instance if e be a element of E such that ${}_E\langle e,e\rangle = 1_A$ then $A_e=A$, because for all $a\in A$ we have $a=a1_A=a\ {}_E\langle e,e\rangle = {}_E\langle a.e,e\rangle$.

The following definition of a Hilbert bimodule is orginally due to Brown, Mingo and Shen [4].

Definition 2.6. Let E be an A-bimodule. E is said to be a Hilbert A-bimodule, when E is a left and right Hilbert A-module and satisfies the relation ${}_E\langle x,y\rangle.z = x.\langle y,z\rangle_E$.

Proposition 2.7. *Let A be unital and E be a Hilbert A-bimodule. If e be an element of E such that $\langle e,e\rangle_E \in Inv(A)$ then A_e is closed.*

Proof. Let $b\in\overline{A_e}$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}\subseteq E$ such that ${}_E\langle x_n,e\rangle$ convergence to b. Thus the sequence $({}_E\langle x_n,e\rangle)_{n\in\mathbb{N}}\subseteq A$ is Cauchy. Now we have:

$$\begin{aligned}
||x_n-x_m|| &= ||(x_n-x_m)\langle e,e\rangle_E\langle e,e\rangle_E^{-1}||\\
&\le ||x_n.\langle e,e\rangle_E - x_m.\langle e,e\rangle_E||\,||\langle e,e\rangle_E^{-1}||\\
&= ||{}_E\langle x_n,e\rangle.e - {}_E\langle x_m,e\rangle.e||\,||\langle e,e\rangle_E^{-1}||\\
&\le ||{}_E\langle x_n,e\rangle - {}_E\langle x_m,e\rangle||\,||e||\,||\langle e,e\rangle_E^{-1}||.
\end{aligned}$$

So the sequence $(x_n)_{n\in\mathbb{N}}\subseteq E$ is Cauchy and by the completeness of E there exists an element $x\in E$ such that x_n convergence to x. Now by continuity of A-valued inner product we conclude that ${}_E\langle x_n,e\rangle$ convergence to ${}_E\langle x,e\rangle$. Thus $b={}_E\langle x,e\rangle$ and A_e is closed. □

The following useful Proposition is well-known and its proof is straightforward.

Proposition 2.8. *Let X and Y be Banach algebras and T be a continuous homomorphism from X onto Y. If X is Arens regular then Y is.*

Theorem 2.9. *Let A be unital and E be a Hilbert A-bimodule, $||e|| = 1$ and $\langle e, e\rangle_E \in Inv(A)$. Then the Banach algebra (E, π_e) is Arens regular.*

Proof. In Proposition (2.7) we saw that under the above conditions A_e is a closed ideal in A. Now since A is Arens regular so A_e is. We define the map $T : A_e \longrightarrow (E, \pi_e)$ by $T({}_E\langle x, e\rangle) = x$ for all $x \in E$. T is well-defined because if ${}_E\langle x, e\rangle = {}_E\langle y, e\rangle$ we have:

$$\begin{aligned} x - y &= (x-y).(\langle e, e\rangle_E \langle e, e\rangle_E^{-1}) \\ &= ((x-y).\langle e, e\rangle_E).\langle e, e\rangle_E^{-1} \\ &= ({}_E\langle x, e\rangle.e - {}_E\langle y, e\rangle.e).\langle e, e\rangle_E^{-1}. \end{aligned}$$

And T is continuous because

$$\begin{aligned} ||x_n - x|| &= ||(x_n - x).\langle e, e\rangle_E \langle e, e\rangle_E^{-1}|| \\ &\leq ||(x_n - x).\langle e, e\rangle_E|| ||\langle e, e\rangle_X^{-1}|| \\ &= ||{}_E\langle x_n - x, e\rangle.e|| ||\langle e, e\rangle_E^{-1}|| \\ &\leq ||{}_E\langle x_n - x, e\rangle|| ||e|| ||\langle e, e\rangle_E^{-1}||. \end{aligned}$$

It is easy to see that T is linear. So it is enough that we show that T is multiplicative

$$\begin{aligned} T({}_E\langle x, e\rangle {}_E\langle y, e\rangle) = T({}_E\langle {}_E\langle x, e\rangle.y, e\rangle) &= {}_E\langle x, e\rangle.y \\ &= \pi_e(x, y) \\ &= \pi_e(T({}_E\langle x, e\rangle), T({}_E\langle y, e\rangle)). \end{aligned}$$

By Proposition (2.8) since T is onto, the Banach algebra (E, π_e) is Arens regular. □

3. Derivations of (E, π_e)

Let E be a left Hilbert A-module, and let e be an element in E with $||e|| = 1$ and (E, π_e) be the Banach algebra introduced in previous section.

Lemma 3.1. *Let E be a full Hilbert A-module and let $a \in A$. Then $a = 0$ if and only if $x.a = 0$ for all $x \in E$ [9].*

Theorem 3.2. *Let A be unital and E be a left Hilbert A-module and let $D : A \longrightarrow A$ and $\delta : (E, \pi_e) \longrightarrow (E, \pi_e)$ be derivations of Banach algebras such that $\delta(a.x) = D(a).x + a.\delta(x)$. Suppose that δ is inner implemented by y, then*

(i) *if E is full then D is inner.*
(ii) *if A is unital and there exists $z \in E$ such that ${}_E\langle z, y\rangle \in Inv(A)$, then D is inner.*

Proof. Let a be an arbitrary element of A. Then for all $x \in E$, $\delta(a.x) = D(a).x + a.\delta(x)$. So for all $x \in E$

$$\begin{aligned} D(a).x &= \delta(a.x) - a.\delta(x) \\ &= \pi_e(y, a.x) - \pi_e(a.x, y) - a.(\pi_e(y, x) - \pi_e(x, y)) \\ &= {}_E\langle y, e\rangle.(a.x) - {}_E\langle a.x, e\rangle.y - a.({}_E\langle y, e\rangle.x - {}_E\langle x, e\rangle.y) \\ &= {}_E\langle y, e\rangle a.x - a\ {}_E\langle x, e\rangle.y - a\ {}_E\langle y, e\rangle.x + a\ {}_E\langle x, e\rangle.y \\ &= {}_E\langle y, e\rangle a.x - a\ {}_E\langle y, e\rangle.x. \end{aligned}$$

Hence $D(a).x = ({}_E\langle y, e\rangle a - a\ {}_E\langle y, e\rangle).x$ for all $x \in E$.
(i) Since for all $x \in E$ we have $(D(a) - ({}_E\langle y, e\rangle a - a\ {}_E\langle y, e\rangle)).x = 0$ and E is full, applying Lemma (3.1) for left Hilbert modules shows that $D(a) = {}_E\langle y, e\rangle a - a\ {}_E\langle y, e\rangle$ and D is a inner derivation implemented by ${}_E\langle y, e\rangle$.
(ii)Since for all $x \in E$ in particular for z, $D(a).x = {}_E\langle y, e\rangle a.x - a\ {}_E\langle y, e\rangle.x$, we conclude that ${}_E\langle D(a).z, y\rangle = {}_E\langle({}_E\langle y, e\rangle a - a\ {}_E\langle y, e\rangle).z, y\rangle$ and so
$D(a)\ {}_E\langle z, y\rangle = ({}_E\langle y, e\rangle a - a\ {}_E\langle y, e\rangle){}_E\langle z, y\rangle$. Now since ${}_E\langle z, y\rangle \in Inv(A)$ we obtain that $D(a) = {}_E\langle y, e\rangle a - a\ {}_E\langle y, e\rangle$. Thus D is a inner derivation implemented by ${}_E\langle y, e\rangle$. □

Theorem 3.3. *Let E be a Hilbert A-bimodule, $\langle e, e\rangle_E \in Inv(A)$ and all derivations of A_e be inner, then every derivation of (E, π_e) is inner.*

Proof. Let δ be an arbitrary derivation of (E, π_e). We define the mapping D on A_e by $D({}_E\langle x, e\rangle) = {}_E\langle \delta(x), e\rangle$ for all $x \in E$. It is easy to verify that D is linear, also for all $x, y \in E$ we have:

$$\begin{aligned} D({}_E\langle x, e\rangle{}_E\langle y, e\rangle) &= D({}_E\langle {}_E\langle x, e\rangle.y, e\rangle) \\ &= {}_E\langle \delta({}_E\langle x, e\rangle.y), e\rangle \\ &= {}_E\langle \delta(\pi_e(x, y)), e\rangle \\ &= {}_E\langle \pi_e(\delta(x), y) + \pi_e(x, \delta(y)), e\rangle \\ &= {}_E\langle {}_E\langle \delta(x), e\rangle.y, e\rangle + {}_E\langle {}_E\langle x, e\rangle.\delta(y), e\rangle \\ &= D({}_E\langle x, e\rangle){}_E\langle y, e\rangle + {}_E\langle x, e\rangle D({}_E\langle y, e\rangle). \end{aligned}$$

So D is a derivation of A_e and since every derivation $D : A_e \longrightarrow A_e$ is inner, there exists $t \in E$ such that $D({}_E\langle x, e\rangle) = {}_E\langle t, e\rangle{}_E\langle x, e\rangle - {}_E\langle x, e\rangle{}_E\langle t, e\rangle = {}_E\langle \pi_e(t, x) - \pi_e(x, t), e\rangle$. Thus ${}_E\langle \delta(x), e\rangle = {}_E\langle \pi_e(t, x) - \pi_e(x, t), e\rangle$ and so
${}_E\langle \delta(x) - (\pi_e(t, x) - \pi_e(x, t)), e\rangle.e = 0$. Now since E is a Hilbert bimodule

we have$(\delta(x)-(\pi_e(t,x)-\pi_e(x,t))).\langle e,e\rangle_E=0$ and by invertiblity of $\langle e,e\rangle_E$ we conclude that $\delta(x)=\pi_e(t,x)-\pi_e(x,t)$ and δ is inner. □

If in the above theorem we add the conditions under which $A=A_e$, for example ${}_E\langle e,e\rangle=1_A$, then we obtain relationship between A and E.

Now suppose that X is a compact Hausdorff space and H is a Hilbert space. For $E=C(X,H)$ and Λ_0 in Example (2.1) we have ${}_E\langle\Lambda_0,\Lambda_0\rangle=1_{C(X)}$, so for every $f\in C(X)$ we have $f=f{}_E\langle\Lambda_0,\Lambda_0\rangle={}_E\langle f.\Lambda_0,\Lambda_0\rangle$. Thus $C(X)=\{{}_E\langle\Lambda,\Lambda_0\rangle:\Lambda\in E\}$. Also we notice that Λ_0 is a left unit for Banach algebra (E,π_{Λ_0}). So we have:

Theorem 3.4. *Every derivation of $(C(X,H),\pi_{\Lambda_0})$ is zero if and only if Λ_0 is unit element of $(C(X,H),\pi_{\Lambda_0})$.*

Proof. Let d be an arbitrary derivation of Banach algebra $(E,\pi_{\Lambda_0})=(C(X,H),\pi_{\Lambda_0})$. We define the mapping D on $C(X)$ by $D({}_E\langle\Lambda,\Lambda_0\rangle)={}_E\langle d(\Lambda),\Lambda_0\rangle$ for all $\Lambda\in E$. With the same proof of the above Theorem we have D is a derivation of $C(X)$. Now since $C(X)$ is a commutative C^*-algebra, D is zero [5] and so $D({}_E\langle\Lambda,\Lambda_0\rangle)=0$ for all $\Lambda\in E$. Now since Λ_0 is unit element of E for all $\Lambda\in E$ we have $d(\Lambda)=\pi_{\Lambda_0}(d(\Lambda),\Lambda_0)={}_E\langle d(\Lambda),\Lambda_0\rangle.\Lambda_0=D({}_E\langle\Lambda,\Lambda_0\rangle).\Lambda_0=0$ and so $d\equiv 0$.
For the converse, consider the inner derivation d_{Λ_0} on E defined by $d_{\Lambda_0}(\Lambda)=\pi_{\Lambda_0}(\Lambda_0,\Lambda)-\pi_{\Lambda_0}(\Lambda,\Lambda_0)$ for all $\Lambda\in E$. Since every derivation of (E,π_{Λ_0}) is zero thus $d_{\Lambda_0}=0$. So for all $\Lambda\in E$ we have $\pi_{\Lambda_0}(\Lambda_0,\Lambda)=\pi_{\Lambda_0}(\Lambda,\Lambda_0)$ and it shows that $\pi_{\Lambda_0}(\Lambda,\Lambda_0)=\Lambda$ and so Λ_0 is unit element of (E,π_{Λ_0}). □

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منظم آرنز بودن و مشتق روی مدول های هیلبرت با یک ضرب مشخص

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چکیده
فرض کنید $\boldsymbol{A}$ یک $\boldsymbol{C}^{G}$- جبر و $\boldsymbol{E}$ یک $\boldsymbol{A}$- مدول هیلبرت چپ باشد. در این مقاله ضربی را روی $\boldsymbol{E}$ تعریف می کنیم که آن را به یک جبر باناخ تبدیل می کند و نشان خواهیم داد که تحت شرایط مشخص، $\boldsymbol{E}$ منظم آرِنز است. همچنین رابطه بین مشتق ها روی $\boldsymbol{A}$ و $\boldsymbol{E}$ را بررسی می کنیم.

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