Journal of Algebra and related topics
Vol. 1, No 1, (2013), pp 41-55

ON GRADED ALMOST SEMIPRIME SUBMODULES

F. FARZALIPOUR

ABSTRACT. Let G be a group with identity e. Let R be a G-graded commutative ring with a non-zero identity and M be a graded R-module. In this article, we introduce the concept of graded almost semiprime submodules. Also, we investigate some basic properties of graded almost semiprime and graded weakly semiprime submodules and give some characterizations of them.

1. Introduction

In the resent years a good deal of researches have done concerning graded ring and graded modules. Particularly, there is a wide variety of applications of graded algebras in geometry and physics (see [15]). Furthermore, in physical sense and in studying supermanifold, supersymmetries and quantizations of systems with symmetry, graded ring and modules play a key role (see [2]). Having the vast heritage of ring theory available, a number of authors have tried to extend and generalize many classical notions and definitions, see for example, [3]-[14]. Graded prime and graded primary ideals of a commutative graded ring R with a non-zero identity have been introduced and studied by M. Refai and K. Al-Zoubi in [17]. Graded prime and graded weakly prime submodules of a graded R-module have been studied by S. Ebrahimi Atani in [3] and [4]. Also, graded semiprime and graded weakly semiprime submodules of graded R-modules have been studied in [9] and [11]. Here we study a number of results of graded weakly

MSC(2010): Primary: 13A02; Secondary: 16W50
Keywords: graded almost semiprime, graded multiplication module, graded weakly semiprime.
Received: 13 August 2013, Accepted: 7 October 2013.

semiprime and graded almost semiprime submodules. First, we define the graded almost semiprime submodules of a graded R-module. We give some results concerning this class of graded submodules and some characterizations of them (see sec. 2). Also, we study and characterize other properties of graded weakly semiprime submodules (see sec. 3). Before we state some results let us introduce some notation and terminology. Let G be a group with identity e. A ring (R, G) is called a G-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g \in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for each g and h in G. In this case, R_e is a subring of R and $1_R \in R_e$. For simplify, we will denote the graded ring (R, G) by R. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g \in G} M_g$ such that for all $g, h \in G$; $R_g M_h \subseteq M_{gh}$. Any element of R_g or M_g for any $g \in G$, is said to be a homogeneous element of degree g. A submodule $N \subseteq M$, where M is G-graded, is called G-graded if $N = \bigoplus_{g \in G}(N \cap M_g) = \bigoplus_{g \in G} N_g$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. We write $h(R) = \cup_{g \in G} R_g$ and $h(M) = \cup_{g \in G} M_g$. A graded ring R is called graded integral domain, if whenever $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded ideal I of R is said to be graded maximal if $I \neq R$ and there is no graded ideal J of R such that $I \subsetneq J \subsetneq R$. A graded module M over a G-graded ring R is called graded finitely generated if $M = \sum_{i=1}^{n} R x_{g_i}$ where $x_{g_i} \in h(M)$. A graded R-module M is called graded cyclic if $M = R x_g$ where $x_g \in h(M)$. A graded R-module M is called a graded second module provided that for every element $r \in h(R)$, either $rM = M$ or $rM = 0$. A graded R-module M is called a graded multiplication module provided that, for every graded submodule N of M, there exists a graded ideal I of R so that $N = IM$ (or equivalently, $N = (N : M)M$). A graded submodule N of a graded R-module M is called a graded pure (graded RD-) submodule if $IN = N \cap IM$ ($rN = N \cap rM$) for any graded ideal I of R (for any $r \in h(R)$). A graded ideal I of a graded ring R is called graded multiplication, if it is multiplication as graded R-modules. Graded multiplication modules and ideals have been studied extensively in [7] ,[8], [13] and [14]. A graded R-module M is called a graded cancellation module if for all graded ideals I and J of R, $IM = JM$ implies that $I = J$. Let N be a graded R-submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R (see [3]). A graded R-module M is called faithful, if $Ann(M) = (0 : M) = 0$. A proper graded submodule N of M is called graded prime, if whenever

$rm \in N$ where $r \in h(R)$ and $m \in h(M)$, then $m \in N$ or $r \in (N : M)$. A graded R-module M is said to be graded prime, if the zero graded submodule of M is a graded prime submodule. A proper graded submodule N of a graded R-module M is called graded semiprime if whenever $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^k m \in N$, then $rm \in N$. If R is a graded ring and M a graded R-module, the subset $T^g(M)$ of M is defined by $T^g(M) = \{m \in M : rm = 0$ for some $0 \neq r \in h(R)\}$. If R is a graded integral domain, then $T^g(M)$ is a graded submodule of M [3]. We say that M is graded torsion, if $T^g(M) = M$ and we say that M is graded torsion free, if $T^g(M) = 0$.

2. Graded Almost Semiprime Submodules

Definition 2.1. (i) Let R be a commutative G-graded ring. A proper graded ideal I of R is called graded almost semiprime if whenever $a_g^k b_h \in I_{g^k h} - I^2 \bigcap R_{g^k h}$ for $a_g, b_h \in h(R)$ and $k \in \mathbb{Z}^+$, then $a_g b_h \in I_{gh}$.
(ii) Let R be a commutative G-graded ring and M be a graded R-module. A proper graded submodule N of M is called graded almost semiprime if whenever $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r_g^k m_h \in N_{g^k h} - (N : M)N \cap M_{g^k h}$, then $r_g m_h \in N_{gh}$.
(iii) Let N be a graded submodule of a graded R-module M. We say that N_g is a g-almost semiprime submodule of R_e-module M_g, if $N_g \neq M_g$ and whenever $r_e^k m_g \in N_g - (N_g :_{R_e} M_g)N_g$ for some $r_e \in R_e$, $m_g \in M_g$ and $k \in \mathbb{Z}^+$, then $r_e m_g \in N_g$.

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.2. *Let M be a graded module over a graded ring R. Then the following hold:*

(i) *If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals.*
(ii) *If N is a graded submodule, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M.*
(iii) *If N and K are graded submodules of M, then $N + K$ and $N \bigcap K$ are also graded submodules of M and $(N : M)$ is a graded ideal of R.*
(iv) *Let $\{N_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Lemma 2.3. *Let N be a graded R-submodule of M and I a graded ideal of R. Then $(N :_M I) = \{m \in M | mI \subseteq N\}$ is a graded submodule of M.*

Proof. We have $(N :_M I)_g = (N :_M I) \cap M_g \subseteq (N :_M I)$ for all $g \in G$. Then $\bigoplus_{g\in G}(N :_M I) \subseteq (N :_M I)$. Let $m = \sum_{g\in G} m_g \in (N :_M I)$. It is enough to show that $m_g I \subseteq N$ for all $g \in G$. So without loss of generality we may assume that $m = \sum_{i=1}^{n} m_{g_i}$ where $m_{g_i} \neq 0$ for all $i = 1, ..., n$ and $m_{g_i} = 0$ for all $g_i \notin \{g_1, ..., g_n\}$. Let $a \in I$. As I is a graded ideal, so $a = \sum_{i=1}^{k} a_{g_i}$ where $0 \neq a_{g_i} \in I \cap R_{g_i}$. Therefore, $(\sum_{i=1}^{n} m_{g_i})a_{g_i} \in N$ $(1 \leq i \leq m)$. Since N is a graded submodule we conclude that $m_{g_i}a_{g_i} \in N$, so $m_{g_i}I \subseteq N$. Hence $m_g I \subseteq N$ for all $g \in G$. So $\bigoplus_{g\in G}(N :_M I) = (N :_M I)$. □

Let M be a graded R-module and N a graded submodule of M. N is called idempotent in M if $N = (N : M)N$. Thus any proper idempotent graded submodule of M is graded almost semiprime. If M is a graded multiplication R-module and $N = IM$ and $K = JM$ are two graded submodules of M, then the product NK of N and K is defined as $NK = (IM)(JM) = (IJ)M$, see [1]. In particular, we have $N^2 = NN = [(N : M)M][(N : M)M] = (N : M)^2M$. If further, M is a graded cancellation R-module, then by using Lemma 2.11, $(N : M)N = ((N : M)N : M)M = (N : M)^2M = N^2$. So in this case, a graded submodule N is idempotent in M if and only if $N = N^2$.

Example 2.4. It is clear that every graded semiprime submodule is graded almost semiprime. But the converse is not true in general. For example, let $R = \mathbb{Z} = R_0$ be as $\mathbb{Z}$-graded ring and $M = \mathbb{Z}_{24}\times\mathbb{Z}_{24}$ be the $\mathbb{Z}$-graded R-module with $M_0 = \mathbb{Z}_{24}\times\{0\}$ and $M_1 = \{0\}\times\mathbb{Z}_{24}$. Consider the graded submodule $N =< 8 > \times < 8 >$ with $N_0 =< 8 > \times\{0\}$ and $N_1 = \{0\}\times < 8 >$. Then $(N : M)N = N$, and so N is a graded almost semiprime submodule of M. But N is not graded semiprime in M, because $2^2(2, 0) \in N_0$, but $2(2, 0) \notin N_0$.

In the graded semiprime submodules case, N is a graded semiprime submodule of M, if and only if N/K is so in M/K for any graded submodule $K \subseteq N$ [9]. But the covers part may not be true in the case of graded almost semiprime submodules. For example, for any graded non almost semiprime submodule N of M, we have $N/N = 0$ is a graded almost semiprime submodule of M/N. But we have the following Theorem:

Theorem 2.5. *Let N and K be graded submodules of a graded R-module M with $K \subseteq (N : M)N$. Then N is a graded almost semiprime submodule of M if and only if N/K is a graded almost semiprime submodule of the graded R-module M/K.*

Proof. Let N be a graded almost semiprime submodule of M and assume that $r_g \in h(R)$, $m_h + K \in h(M/K)$ and $k \in \mathbb{Z}^+$ such that

$r_g^k(m_h+K) \in (N/K)_{g^kh} - (N/K : M/K)N/K \cap (M/K)_{g^kh}$. It is clear that $(N/K :_R M/K) = (N :_R M)$, and so $r_g^k m_h \in N_{g^kh} - (N : M)N \cap M_{g^kh}$. Therefore $r_g m_h \in N_{gh}$ since N is graded almost semiprime. Therefore, $r_g(m_h+K) \in (N_{gh}+K)/K = (N/K)_{gh}$, hence N/K is a graded almost semiprime submodule. Conversely, let N/K be a graded almost semiprime submodule of M/K and assume that $r_g^k m_h \in N_{g^kh} - (N : M)N \cap M_{g^kh}$ for some $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Hence $r_g^k(m_h+K) \in (N/K)_{g^kh} - (N/K : M/K)N/K \cap (M/K)_{g^kh}$, because if, $r_g^k(m_h+K) \in (N/K : M/K)N/K \cap (M/K)_{g^kh} = (N : M)(N/K) \cap (M/K)_{g^kh} = ((N : M)N+K)/K \cap (M/K)_{g^kh} = (N : M)N/K \cap (M/K)_{g^kh}$ since $K \subseteq (N : M)N$, so $r_g^k m_h \in (N : M)N$, a contradiction. Therefore $r_g(m_h+K) \in (N/K)_{gh}$, so $r_g m_h \in N_{gh}$, as required. $\square$

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fractions $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in h(R), s \in S \text{ and } g = (degs)^{-1}(degr)\}$.
Let M be a graded module over a graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in h(M), s \in S \text{ and } g = (degs)^{-1}(degm)\}$. We write $h(S^{-1}R) = \bigcup_{g\in G}(S^{-1}R)_g$ and $h(S^{-1}M) = \bigcup_{g\in G}(S^{-1}M)_g$. One can prove that the graded submodules of $S^{-1}M$ are of the form $S^{-1}N = \{\beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S\}$ and that $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N : M) = \emptyset$ ([13]).

Theorem 2.6. *Let $S \subseteq h(R)$ be a multiplicative closed subset of a graded ring R with $S \cap (N : M) = \emptyset$. Then $S^{-1}N$ is a graded almost semiprime submodule of the graded $S^{-1}R$-module $S^{-1}M$.*

Proof. Let N be a graded almost semiprime submodule of M. Since $(N : M) \cap S = \emptyset$, then $S^{-1}N \neq S^{-1}M$.
Assume that $(r_{g_1}/s_{h_1})^k(m_{g_2}/t_{h_2}) \in (S^{-1}N)_{(g_1^kg_2)(h_1^kh_2)^{-1}} - (S^{-1} :_{S^{-1}R} S^{-1}M)S^{-1}N \bigcap (S^{-1}M)_{(g_1^kg_2)(h_1^kh_2)^{-1}}$ where $r_{g_1}/s_{h_1} \in h(S^{-1}R)$, $m_{g_2}/t_{h_2} \in h(S^{-1}M)$ and $k \in \mathbb{Z}^+$. Hence $r_{g_1}^k m_{g_2}/s_{h_1}^k t_{h_2} = n_{g_1^kg_2}/s'_{h_1^kh_2}$ for some $n_{g_1^kg_2} \in N_{g_1^kg_2}$ and $s'_{h_1^kh_2} \in S$, and so there exists $t_{h_3} \in S$ such that $r_{g_1}^k s'_{h_1^kh_2} t_{h_3} m_{g_2} = s_{h_1}^k t_{h_2} t_{h_3} n_{g_1^kg_2} \in N$. If $r_{g_1}^k s'_{h_1^kh_2} t_{h_3} m_{g_2} \in (N : M)N$, then $r_{g_1}^k m_{g_2}/s_{h_1}^k t_{h_2} = r_{g_1}^k s'_{h_1^kh_2} t_{h_3} m_{g_2}/s_{h_1}^k t_{h_2} t_{h_3} s'_{h_1^kh_2} \in S^{-1}((N :_R M)N) = S^{-1}(N :_R M)S^{-1}N \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)S^{-1}N$, a contradiction. So

$r_{g_1}^k(s'_{h_1^k h_2} t_{h_3} m_{g_2}) \in N_{g_1^k h_1^k h_2 h_3 g_2} - (N : M)N \bigcap M_{g_1^k h_1^k h_2 h_3 g_2}$, and hence $r_{g_1} s'_{h_1^k h_2} t_{h_3} m_{g_2} \in N_{g_1 h_1^k h_2 h_3 g_2}$ since N is graded almost semiprime. Therefore

$r_{g_1} m_{g_2}/s_{h_1} t_{h_2} = r_{g_1} s'_{h_1^k h_2} t_3 m_{g_2}/s_{h_1} t_{h_2} s'_{h_1^k h_2} t_3 \in (S^{-1}N)_{(g_1 g_2)(h_1 h_2)^{-1}}$, hence $S^{-1}N$ is a graded almost semiprime submodule of $S^{-1}M$. □

Proposition 2.7. *Let M be a graded R-module and N be a graded almost semiprime submodule of M. Then*

(i) *If M is a graded second R-module, then N is a graded second module.*
(ii) *If M is a graded second R-module, then N is a graded RD-submodule of M.*

Proof. (i) Let N be a graded almost semiprime submodule of M. Let $r_g \in h(R)$. If $r_g M = 0$, then $r_g N \subseteq r_g M = 0$. Let $r_g M = M$. Now It is enough to show that $N \subseteq r_g N$. First, we show that $(N : M)N = 0$. Since N is a proper graded submodule of M, so for any $r \in (N : M)$, we have $rM = 0$, because, we can write $r = \sum_{h \in G} r_h$. Since $(N : M)$ is a graded ideal of R, so $r_h \in (N : M)$ for any $h \in G$. Hence $r_h M \subseteq N$ and since M is graded second, we have $r_h M = 0$, so $r_h N = 0$ for any $h \in G$, and $(N : M)N = 0$. Let $n = \sum_{i=1}^n n_{g_i} \in N$ where $n_{g_i} \neq 0$. Since $r_g M = M$, so for any $1 \leq i \leq n$, $n_{g_i} = r_g m_{g_i g^{-1}}$ for some $m_{g_i g^{-1}} \in h(M)$, and $m_{g_i g^{-1}} = r_g m'_{g_i (g^{-1})^2}$ for some $m'_{g_i (g^{-1})^2} \in h(M)$. Hence $0 \neq n_{g_i} = (r_g)^2 m'_{g_i (g^{-1})^2} \in N_{g_i} - (N : M)N \bigcap M_{g_i}$, as N is graded almost semiprime so $m_{g_i g^{-1}} = r_g m'_{g_i (g^{-1})^2} \in N$. Hence $n_{g_i} = r_g m_{g_i g^{-1}} \in r_g N$ for any $1 \leq i \leq n$, so $n \in r_g N$ and $N \subseteq r_g N$. Therefore $r_g N = N$, hence N is graded second.

(ii) Let $r_g \in h(R)$. If $r_g M = 0$, then $r_g N = 0$, so $r_g N = 0 = N \cap r_g M$. Suppose that $r_g M = M$, so by (i), $r_g N = N$, therefore $r_g N = N \cap r_g M$. □

In the following Theorems, we give other characterizations of graded almost semiprime submodules.

Theorem 2.8. *Let M be a graded R-module and N a proper graded submodule of M. Then the following are equivalent:*

(i) *N is a graded almost semiprime submodule of M.*
(ii) *For $r_g \in h(R)$ and $k \in \mathbb{Z}^+$; $(N_h :_M r_g^k) = (N_{hg^{1-k}} :_M r_g) \cup ((N : M)N \cap M_h :_M r_g^k)$.*
(iii) *For $r_g \in h(R)$ and $k \in \mathbb{Z}^+$; $(N_h :_M r_g^k) = (N_{hg^{1-k}} :_M r_g)$ or $(N_h :_M r_g^k) = ((N : M)N \cap M_h :_M r_g^k)$.*

Proof. $(i) \Rightarrow (ii)$ Let $m_{hg^{-k}} \in (N_h :_M r_g^k)$, then $r_g^k m_{hg^{-k}} \in N_h$. If $r_g^k m_{hg^{-k}} \notin (N :_R M)N \cap M_h$, as N is graded almost semiprime, $r_g m_{hg^{-k}} \in N_{hg^{1-k}}$, so $m_{hg^{-k}} \in (N_{hg^{1-k}} :_M r_g)$. Let $r_g^k m_{hg^{-k}} \in (N :_R M)N \cap M_h$, then $m_{hg^{-k}} \in ((N :_R M)N \cap M_h :_M r_g^k)$, hence $(N_h :_M r_g^k) \subseteq (N_{hg^{-k}} :_M r_g) \cup ((N : M)N \cap M_h :_M r_g^k)$. The other containment holds for any graded submodule N.

$(ii) \Rightarrow (iii)$ It is well known that if a graded submodule is the union of two graded submodules, then it is equal to one of them.

$(iii) \Rightarrow (i)$ Let $r_g^k m_h \in N_{g^k h} - (N :_R M)N \cap M_{g^k h}$ for some $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Hence $m_h \in (N_{g^k h} :_M r_g^k)$ and $m_h \notin ((N :_R M)N \cap M_{g^k h} :_M r_g^k)$, so by assumption, $m_h \in (N_{gh} :_M r_g)$ and $r_g m_h \in N_{gh}$. Therefore N is graded almost semiprime. □

Theorem 2.9. *Let M be a graded R-module and N be a proper graded submodule of M. Then N is graded almost semiprime in M if and only if for any graded submodule $K = \bigoplus_{h \in G} K_h$ of M, $a_g \in h(R)$ and $k \in \mathbb{Z}^+$ with $a_g^k K_h \subseteq N_{g^k h}$ and $a_g^k K_h \nsubseteq (N :_R M)N \cap M_{g^{kh}}$, we have $a_g K_h \subseteq N_{gh}$.*

Proof. Assume that N is graded almost semiprime. Let $a_g^k K_h \subseteq N_{g^k h}$ and $a_g^k K_h \nsubseteq (N : M)N \cap M_{g^k h}$. Then $K_h \subseteq (N_{g^k h} : a_g^k)$. Since $K_h \nsubseteq ((N : M)N \cap M_{g^k h} : a_g^k)$, so by Theorem 2.8, $K_h \subseteq (N_{gh} : a_g)$, and hence $a_g K_h \subseteq N_{gh}$, as needed. Conversely, let $r_g^k m_h \in N_{g^k h} - (N : M)M \cap M_{g^k h}$ where $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Hence $r_g^k (R_e m_h) \subseteq N_{g^k h}$ and $r_g^k (R_e m_h) \nsubseteq (N : M)N \cap M_{g^k h}$, so by hypothesis, $r_g (R_e m_h) \subseteq N_{gh}$, hence $r_g m_h \in N_{gh}$ $(1 \in R_e)$, as needed. □

Theorem 2.10. *Let M be a graded finitely generated faithful graded multiplication R-module. Then M is a graded cancellation module.*

Proof. Let I, J be graded ideals of R such that $IM \subseteq JM$. Let $a = a_{g_1} + a_{g_2} + ... + a_{g_n} \in I$ where $0 \neq a_{g_i} \in I \cap R_{g_i}$. It suffices to show that for each i, $a_{g_i} \in J$. Let $a \in \{a_{g_1}, a_{g_2}, ..., a_{g_n}\}$, it is clear that $K = \{r \in R \mid ra \in J\}$ is a graded ideal of R. Suppose $K \neq R$. Then there exists a graded maximal ideal P of R such that $K \subseteq P$ by [5, Lemma 2.3]. If $PM = M$, then as M is graded finitely generated, we conclude $(1 - p)M = 0$ for some $p \in P$, which is a contradiction since M is faithful. Thus $M \neq PM$, so by [16, Theorem 7], there exist $m \in h(M)$ and $q \in h(R) - P$ such that $qM \subseteq Rm$. In particular, $qam \in qJM \subseteq Jm$, so that there exists $c \in J$ such that $(qa - c)m = 0$.

But $qAnn(m) \subseteq AnnM = 0$. Thus $q(qa - c) = 0$ and this implies that $q^2 a \in J$ so that $q^2 \in K \subseteq P$. Hence $q \in P$ since every graded maximal is graded prime, a contradiction. So $K = R$ and hence $a \in J$. Therefore $I \subseteq J$. □

Lemma 2.11. *Let N be a graded submodule of a graded finitely generated faithful graded multiplication (so graded cancellation) R-module. Then $(IN : M) = I(N : M)$ for every graded ideal I of R.*

Proof. As M is graded multiplication R-module, then $I(N : M)M = IN = (IN : M)M$. The result follows because M is a graded cancellation module. □

Theorem 2.12. *Let $M = \bigoplus_{g \in G} M_g$ be a graded R-module and N a graded submodule of M. Let M_g be a finitely generated faithful multiplication R_e-module and N_g a proper submodule of M_g. Then the following are equivalent:*

(i) *N_g is g-almost semiprime in M_g.*
(ii) *$(N_g :_{R_e} M_g)$ is almost semiprime in R_e.*
(iii) *$N_g = P_e M_g$ for some almost semiprime ideal P_e of R_e.*

Proof. $(i) \Rightarrow (ii)$ Suppose that N_g is a g-almost semiprime submodule of M_g. Let $a_e, b_e \in R_e$ and $k \in \mathbb{Z}^+$ such that $a_e^k b_e \in (N_g :_{R_e} M_g) - (N_g :_{R_e} M_g)^2$. Then $a_e^k(b_e M_g) \subseteq N_g$ and $a_e^k(b_e M_g) \nsubseteq (N_g : M_g)N_g$. Indeed, if $a_e^k(b_e M_g) \subseteq (N_g : M_g)N_g$, then by Lemma 2.11, $a_e^k b_e \in ((N_g : M_g)N_g : M_g) = (N_g : M_g)^2$, a contradiction. Now, N_g g-almost semiprime implies that $a_e(b_e M_g) \subseteq N_g$ by Theorem 2.9, so $a_e b_e \in (N_g : M_g)$, hence $(N_g : M_g)$ is almost semiprime in R_e.

$(ii) \Rightarrow (i)$ Let $r_e^k m_g \in N_g - (N_g : M_g)N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in \mathbb{Z}^+$. Then $r_e^k(R_e m_g : M_g) \subseteq (R_e(r_e^k m_g) : M_g) \subseteq (N_g : M_g)$. Moreover, $r_e^k(R_e m_g : M_g) \nsubseteq (N_g : M_g)^2$ because otherwise, if $r_e^k(R_e m_g : M_g) \subseteq (N_g : M_g)^2 = ((N_g : M_g)N_g : M_g)$, then $r_e^k(R_e m_g) = r_e^k(R_e m_g : M_g)M_g \subseteq (N_g : M_g)N_g$, a contradiction. As $(N_g : M_g)$ is an almost semiprime ideal of R_e, then $r_e(R_e m_g : M_g) \subseteq (N_g : M_g)$ by Theorem 2.9. Therefore $r_e(R_e m_g) = r_e(R_e m_g : M_g)M_g \subseteq (N_g : M_g)M_g = N_g$, and so $r_e m_g \in N_g$, as required.

$(ii) \Rightarrow (iii)$ We choose $P_e = (N_g :_{R_e} M_g)$.

$(iii) \Rightarrow (ii)$ Let $N_g = P_e M_g$ for some almost semiprime ideal P_e of R_e. Then $N_g = P_e M_g = (N_g : M_g)M_g$, since M_g is cancellation, we conclude $P_e = (N_g : M_g)$. □

Lemma 2.13. *Every graded cyclic R-module is a graded multiplication module.*

Proof. Let $M = Rx_g$ for some $x_g \in h(M)$. Let N be a graded submodule of M and let $n \in N$. Then $n = rx_g \in N$ for some $r \in R$, and so $r \in (N : x_g) = (N : M)$. Hence $n \in (N : M)M$, so $N = (N : M)M$. Therefore M is a graded multiplication module. $\square$

We know that [9], if N is a graded semiprime submodule of M, then $(N :_R M)$ is a graded semiprime ideal of R. But it may not be true in the case of graded almost semiprime submodules.

Example 2.14. Consider the $\mathbb{Z}$-graded ring $R = R_0 = \mathbb{Z}$ and the graded $\mathbb{Z}$-module $\mathbb{Z}_4 \times \mathbb{Z}_4$ with $M_0 = \mathbb{Z}_4 \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}_4$. Take $N = \{0\} \times \{0\}$. Certainly, N is graded almost semiprime, but $(N :_R M) = 4\mathbb{Z}$ is not a graded almost semiprime ideal of $\mathbb{Z}$. Because $2^2 \in (N : M)_0 - (N : M)^2 \cap R_0$, but $2 \notin (N : M)_0$.

Theorem 2.15. *Let M be a faithful graded cyclic (graded multiplication) R-module and N a proper graded submodule of M. Then the following are equivalent:*

(i) *N is graded almost semiprime in M.*
(ii) *$(N :_R M)$ is graded almost semiprime in R.*
(iii) *$N = PM$ for some graded almost semiprime ideal P of R.*

Proof. Let $M = Rm_g$ for some homogeneous element $m_g \in h(M)$.

$(i) \Rightarrow (ii)$ Suppose that N is a graded almost semiprime submodule of M. Let $a_{g'}, b_h \in h(R)$ and $k \in \mathbb{Z}^+$ such that $a_{g'}^k b_h \in (N : M)_{g'^k h} - (N : M)^2 \cap R_{g'^k h}$. Then $a_{g'}^k (b_h M) \subseteq N$ and $a_{g'}^k (b_h M) \nsubseteq (N : M)N$. Indeed, if $a_g^k b_h M \subseteq (N : M)N$, then by Lemma 2.11, $a_{g'}^k b_h \in ((N : M)N : M) = (N : M)^2$, a contradiction. So $a_{g'}^k b_h m_g \in N_{g'^k hg}$ and $a_{g'}^k b_h m_g \notin (N : M)N \cap M_{g'^k hg}$, because if $a_{g'}^k b_h m_g \in (N : M)N$, then $a_{g'}^k b_h \in ((N : M)N : m_g) = ((N : M)N : M)$, a contradiction. Thus $a_{g'} b_h m_g \in N_{g'hg} \subseteq N$ and so $a_{g'} b_h M \subseteq N$. Hence $a_g b_h \in (N : M) \cap R_{gh} = (N : M)_{gh}$, so $(N : M)$ is graded almost semiprime in R.

$(ii) \Rightarrow (i)$ Assume that $(N : M)$ is a graded almost semiprime ideal of R. Let $r_{g'}^k m_h \in N_{g'^k h} - (N : M)M \cap M_{g'^k h}$ where $r_{g'} \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Then $m_h = r_{hg^{-1}} m_g$ for some $r_{g^{-1}h} \in h(R)$. So $r_{g'}^k r_{hg^{-1}} m_g \in N$, hence $r_{g'}^k r_{hg^{-1}} \in (N : m_g) = (N : M) \cap R_{g'^k hg^{-1}} = (N : M)_{g'^k hg^{-1}}$. Thus $r_{g'} r_{hg^{-1}} \in (N : M)_{g'hg^{-1}}$ since $(N : M)$ is graded almost semiprime. Hence $r_{g'} r_{hg^{-1}} M \subseteq N$, and so $r_{g'} m_h = r_{g'} r_{hg^{-1}} m_g \in N_{g'h}$, as required.

$(ii) \Leftrightarrow (iii)$ We choose $P = (N : M)$ and the fact M is graded cancellation module.

□

Let I be a proper graded ideal of R. Then the G-radical of I, denoted by $Gr(I)$, is defined to be the intersection of all graded prime ideals of R containing I.

Let N be a proper graded submodule of M. Then the G-radical of N, denoted by $Gr(N)$, is defined to be the intersection of all graded prime submodules of M containing N. It is shown in [16], that if N is a proper graded submodule of a graded multiplication R-module M, then $Gr(N) = (Gr(N :_R M))M$.

Lemma 2.16. *For every proper graded ideal I of R, $Gr(I)$ is a graded almost semiprime ideal of R.*

Proof. Since $(Gr(I))^2 = Gr(I)$, so the proof is hold. □

Theorem 2.17. *Let M be a faithful graded cyclic R-module. Then for every proper graded submodule N of M, $Gr(N)$ is a graded almost semiprime submodule of M.*

Proof. Let N be a proper graded submodule of M. Hence by Lemma 2.16, $Gr(N :_R M)$ is a graded almost semiprime ideal of R. Therefore by Theorem 2.15, $Gr(N) = Gr((N : M)M) = (Gr(N :_R M))M$ is a graded almost semiprime submodule of M. □

3. Graded Weakly Semiprime Submodules

Definition 3.1. (i) Let R be a commutative G-graded ring. A proper graded ideal I of R is called graded weakly semiprime if whenever $0 \neq a_g^k b_h \in I_{g^k h}$ for some $a_g, b_h \in h(R)$ and $k \in \mathbb{Z}^+$, then $a_g b_h \in I_{gh}$.

(ii) Let M be a graded R-module. A proper graded submodule N of M is called graded weakly semiprime if whenever $0 \neq r_g^k m_h \in N_{g^k h}$ for some $r_g \in h(R)$, $m_h \in h(M)$ and $k \in Z^+$; then $r_g m_h \in N_{gh}$.

(iii) Let N be a graded submodule of a graded R-module M. We say that N_g is a g-weakly semiprime submodule of R_e-module M_g, if $N_g \neq M_g$ and whenever $0 \neq r_e^k m_g \in N_g$ for some $r_e \in R_e$, $m_g \in M_g$ and $k \in \mathbb{Z}^+$, then $r_e m_g \in N_g$.

Remark 3.2. Let M be a graded module over a graded ring R. Then graded semiprime submodules $\Rightarrow$ graded weakly semiprime submodules $\Rightarrow$ graded almost semiprime submodules.

Theorem 3.3. *Let M be a graded R-module and N a proper graded submodule of M. Then N is a graded almost semiprime submodule of M if and only if $N/(N:M)N$ is a graded weakly semiprime submodule of the graded R-module $M/(N:M)N$.*

Proof. Assume that N is a graded almost semiprime submodule of M. Let $r_g \in h(R)$, $m_h + (N:M)N \in h(M/(N:M)N)$ and $k \in \mathbb{Z}^+$ such that $0 \neq r_g^k(m_h + (N:M)N) \in (N/(N:M)N)_{g^kh}$. Hence $r_g^k m_h \in N_{g^kh} - (N:M)N \cap M_{g^kh}$, and so $r_g m_h \in N_{gh}$. Therefore $r_g(m_h + (N:M)N) \in (N/(N:M)N)_{gh}$, as needed.
Conversely, assume that $N/(N:M)N$ is graded weakly semiprime in $M/(N:M)N$. Let $r_g^k m_h \in N_{g^kh} - (N:M)N \cap M_{g^kh}$ where $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Then $0 \neq r_g^k(m_h + (N:M)N) \in (N/(N:M)N)_{g^kh}$, and hence $r_g(m_h + (N:M)N) \in (N/(N:M)N)_{gh}$. Therefore $r_g m_h \in N_{gh}$, as required. □

Proposition 3.4. *Let R be a graded integral domain and M be a graded torsion free R-module. Then every graded weakly semiprime submodule of M is graded semiprime.*

Proof. Let N be a graded weakly semiprime submodule of M. Let $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r_g^k m_h \in N_{g^kh}$. If $0 \neq r_g^k m_h$, then N graded weakly semiprime gives that $r_g m_h \in N_{gh}$. Suppose that $r_g^k m_h = 0$. If $r_g^k \neq 0$, then $m_h \in T^g(M) = 0$, so $r_g m_h \in N_{gh}$. If $r_g^k = 0$, then $r_g = 0$, and hence $r_g m_h \in N_{gh}$. Therefore N is graded semiprime. □

Proposition 3.5. *Let M be a graded prime R-module. Then every graded weakly semiprime submodule of M is graded semiprime.*

Proof. Let N be a graded weakly semiprime submodule of M. Let $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r_g^k m_h \in N_{g^kh}$. If $0 \neq r_g^k m_h$, then N graded weakly semiprime gives that $r_g m_h \in N_{gh}$. Suppose that $r_g^k m_h = 0$, then $r_g m_h = 0$ or $r_g^{k-1} M = 0$ since M is a graded prime module. By following this method, we get $r_g m_h = 0 \in N_{gh}$, hence N is a graded semiprime submodule of M. □

Proposition 3.6. *Let M be a graded second R-module and N a proper graded submodule of M. Then N is graded almost semiprime if and only if N is graded weakly semiprime.*

Proof. We know that every graded weakly semiprime is graded almost semiprime. Let N be a graded almost semiprime submodule of M and $0 \neq r_g^k m_h \in N_{g^kh}$ for some $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. By Proposition 2.7, we have $(N:M)N = 0$, hence $r_g^k m_h \in N_{g^kh} - (N:$

$M)N \cap M_{g^k h}$, and so $r_g m_h \in N_{gh}$. Therefore N is a graded weakly semiprime submodule of M. $\square$

Now, we get other characterizations of graded weakly semiprime submodules.

Theorem 3.7. *Let M be a graded R-module and N a proper graded submodule of M. Then the following are equivalent:*

(i) *N is a graded weakly semiprime submodule of M.*
(ii) *For $r_g \in h(R)$ and $k \in \mathbb{Z}^+$; $(N_h :_M r_g^k) = (0 :_M r_g^k) \cup (N_{hg^{1-k}} :_M r_g)$.*
(iii) *For $r_g \in h(R)$ and $k \in \mathbb{Z}^+$; $(N_h :_M r_g^k) = (0 :_M r_g^k)$ or $(N_h :_M r_g^k) = (N_{hg^{1-k}} :_M r_g)$.*

Proof. $(i) \Rightarrow (ii)$ Let $m_{hg^{-k}} \in (N_h :_M r_g^k)$, then $r_g^k m_{hg^{-k}} \in N_h$. If $r_g^k m_{hg^{-k}} \neq 0$, then since N is graded weakly semiprime implies $r_g m_{hg^{-k}} \in N_{hg^{1-k}}$, so $m_{hg^{-k}} \in (N_{hg^{1-k}} :_M r_g)$. Let $r_g^k m_{hg^{-k}} = 0$, then $m_{hg^{-k}} \in (0 :_M r_g^k)$, hence $(N_h :_M r_g^k) \subseteq (N_{hg^{1-k}} :_M r_g) \cup (0 :_M r_g^k)$. The other containment holds for any graded submodule N.

$(ii) \Rightarrow (iii)$ It is straightforward .

$(iii) \Rightarrow (i)$ Let $0 \neq r_g^k m_h \in N_{g^k h}$ where $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Hence $m_h \in (N_{g^k h} :_M r_g^k)$ and $m_h \notin (0 :_M r_g^k)$, so by assumption, $m_h \in (N_{gh} :_M r_g)$ and $r_g m_h \in N_{gh}$. Therefore N is graded weakly semiprime.

$\square$

Theorem 3.8. *Let M be a graded R-module and N be a proper graded submodule of M. Then N is graded weakly semiprime in M if and only if for any graded submodule $K = \bigoplus_{h \in G} K_h$ of M, $a_g \in h(R)$ and $k \in \mathbb{Z}^+$ with $0 \neq a_g^k K_h \subseteq N_{g^k h}$, we have $a_g K_h \subseteq N_{gh}$.*

Proof. Assume that N is graded weakly semiprime. Let $0 \neq a_g^k K_h \subseteq N_{g^k h}$, for some graded submodule K of M, $a_g \in h(R)$ and $k \in \mathbb{Z}^+$. Then $K_h \subseteq (N_{g^k h} : a_g^k)$. Since $K_h \nsubseteq (0 : a_g^k)$, so by Theorem 3.7, $K_h \subseteq (N_{gh} : g_g)$, and hence $a_g K_h \subseteq N_{gh}$. Conversely, let $0 \neq r_g^k m_h \in N_{g^k h}$ where $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Hence $0 \neq r_g^k (R_e m_h) \subseteq N_{g^k h}$, so by hypothesis, $r_g (R_e m_h) \subseteq N_{gh}$, hence $r_g m_h \in N_{gh}$, as needed. $\square$

Theorem 3.9. *Let $M = \bigoplus_{g \in G} M_g$ be a graded R-module and N a graded submodule of M. Let M_g be a finitely generated faithful multiplication R_e-module and N_g a proper submodule of M_g. Then the following are equivalent:*

(i) N_g *is* g*-weakly semiprime in* M_g.
(ii) $(N_g :_{R_e} M_g)$ *is weakly semiprime in* R_e.
(iii) $N_g = P_e M_g$ *for some weakly semiprime ideal* P_e *of* R_e.

Proof. $(i) \Rightarrow (ii)$ In this direction, we need M_g to be just a faithful module. Suppose that N_g is a g-weakly semiprime submodule of M_g. Let $a_e, b_e \in R_e$ and $k \in \mathbb{Z}^+$ such that $0 \neq a_e^k b_e \in (N_g :_{R_e} M_g)$. Then $0 \neq a_e^k(b_e M_g) \subseteq N_g$ since M_g is faithful. Now, N_g g-weakly semiprime implies that $a_e(b_e M_g) \subseteq N_g$ by Theorem 3.8, so $a_e b_e \in (N_g : M_g)$, hence $(N_g : M_g)$ is weakly semiprime in R_e.

$(ii) \Rightarrow (i)$ In this direction, we need M_g to be just a multiplication module. Let $0 \neq r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in \mathbb{Z}^+$. Then $r_e^k(R_e m_g : M_g) \subseteq (R_e(r_e^k m_g) : M_g) \subseteq (N_g : M_g)$. Moreover, $r_e^k(R_e m_g : M_g) \neq 0$ because otherwise, if $r_e^k(R_e m_g : M_g) = 0$, then $r_e^k(R_e m_g) = r_e^k(R_e m_g : M_g)M_g = 0$, a contradiction. As $(N_g : M_g)$ is a weakly semiprime ideal of R_e, then $r_e(R_e m_g : M_g) \subseteq (N_g : M_g)$. Therefore $r_e(R_e m_g) = r_e(R_e m_g : M_g)M_g \subseteq (N_g : M_g)M_g = N_g$, and so $r_e m_g \in N_g$, as required.

$(ii) \Leftrightarrow (iii)$ We choose $P_e = (N_g :_{R_e} M_g)$.

□

Theorem 3.10. *Let* M *be a faithful graded cyclic* R*-module and* N *be a proper graded submodule of* M*. Then the following are equivalent:*
(i) N *is graded weakly semiprime in* M.
(ii) $(N :_R M)$ *is graded weakly semiprime in* R.
(iii) $N = QM$ *for some graded weakly semiprime ideal* Q *of* R.

Proof. Let $M = Rm_g$ for some homogeneous element $m_g \in h(M)$.

$(i) \Rightarrow (ii)$ Suppose that N is a graded weakly semiprime submodule of M. Let $a_{g'}, b_h \in h(R)$ and $k \in \mathbb{Z}^+$ such that $0 \neq a_{g'}^k b_h \in (N : M)_{g'^k h}$. Then $a_{g'}^k(b_h M) \subseteq N$ and $a_{g'}^k b_h M \neq 0$. Indeed, if $a_g^k b_h M = 0$, then $a_{g'}^k b_h \in (0 : M) = 0$, a contradiction. So $0 \neq a_{g'}^k b_h m_g \in N_{g'^k hg}$, then $a_{g'} b_h m_g \in N_{g'hg} \subseteq N$ and so $a_{g'} b_h M \subseteq N$. Hence $a_g b_h \in (N : M) \cap R_{gh} = (N : M)_{gh}$, so $(N : M)$ is graded weakly semiprime in R.

$(ii) \Rightarrow (i)$ Assume that $(N : M)$ is a graded weakly semiprime ideal of R. Let $0 \neq r_{g'}^k m_h \in N_{g'^k h}$ where $r_{g'} \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$. Then $m_h = r_{hg^{-1}} m_g$ for some $r_{g^{-1}h} \in h(R)$. So $0 \neq r_{g'}^k r_{hg^{-1}} m_g \in N$, hence $r_{g'}^k r_{hg^{-1}} \in (N : m_g) = (N : M) \cap R_{g'^k hg^{-1}} = (N : M)_{g'^k hg^{-1}}$. Thus $r_{g'} r_{hg^{-1}} \in (N : M)_{g'hg}$, since $(N : M)$ is graded weakly semiprime.

Hence $r_{g'}r_{hg^{-1}}M \subseteq N$, and so $r_{g'}m_h = r_{g'}r_{hg^{-1}}m_g \in N_{g'h}$, as required.

$(ii) \Leftrightarrow (iii)$ We choose $P = (N : M)$.

$\square$

Acknowledgments

I would like to thank the referee(s) for valuable comments and suggestions which have improved the paper.

References

1. R. Ameri, *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci., **27**(2003), 1715-1724.
2. P. Deligne, *Quantum Fields and Strings*, A course for Mathematicians, A.M.S., 1999.
3. S. Ebrahimi Atani, *On graded prime submodules*, Chiang Mai J. Sci., **33**(2006), 3-7.
4. S. Ebrahimi Atani, *On graded weakly prime submodules*, Int. Math. Forum, **2**(2006), 61-66.
5. S. Ebrahimi Atani and F. Farzalipour, *Notes on the graded prime submodules*, Int. Math. Forum, **38**(2006), 1871-1880.
6. S. Ebrahimi Atani and F. Farzalipour, *On graded secondary modules*, Turkish J. Math., **31**(2007), 371-378.
7. J. Escoriza and B. Torrecillas, *Multiplication graded rings*, Lecture Notes in Pure and Applied Mathematics, **208**(2000), 127-137.
8. J. Escoriza and B. Torrecillas, *Multiplication rings and graded rings*, Comm. in Algebra, (12) **24**(1999), 6213-6232.
9. F. Farzalipour and P. Ghiasvand, *On graded semiprime submodules*, Word Academy of Science Engineering and technology, **43**(2012), 694-697.
10. F. Farzalipour and P. Ghiasvand, *On graded semiprime and grade weakly semiprime ideals*, International Electronic J. of Algebra, **13**(2013),15-22.
11. F. Farzalipour and P. Ghiasvand, *On graded weakly semiprime submodules*, Thai J. of Math., **43**(2012), 694-697.
12. F. Farzalipour and P. Ghiasvand, *On graded weak multiplication modules*, Tamkang J. of Math., **43**(2012),171-177.
13. P. Ghiasvand and F. Farzalipour, *Some properties of graded multiplication modules*, Far East J. of Math. Sci., **34**(2009), 341-352.
14. H. A. Khashan, *Some Properties of Gr-Multiplication Ideals*, Turkish J. Math., **33**(2009), 205-213.
15. I. Kolář, P. W. Michor and J. Slovák, *Natural operations in differential geometry*, Springer-verlag, 2010.
16. K. H. Oral, U. Tekir and A. G. Ağargün, *On graded prime and primary submodules*, Turkish J. Math., **35**(2011), 159-167.
17. M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish J. Math., **28**(2004), 217-229.

F. Farzalipour
Department of Mathematics, University of Payame Noor, P.O.Box 19395-3697, Tehran, Iran.
Email: f_farzalipour@pnu.ac.ir

Journal of Algebra and related topics
Vol. 1, No 1, (2013), pp 82

زیر مدول های تقریباً نیم اول مدرج

فرخنده فرضعلی پور
دانشگاه پیام نور، تهران، ایران

چکیده
فرض کنید Gیک گروه با عنصر همانی e، Rیک حلقه جابجایی و یکدار G- مدرج و M یک R- مدول مدرج باشد. در این مقاله، مفهوم زیرمدول های تقریباً نیم اول مدرج را معرفی می‌کنیم. سپس، برخی خاصیت های اساسی زیرمدول‌های نیم اول ضعیف و زیرمدول‌های تقریباً نیم اول را مورد بررسی و مطالعه قرار داده و برخی مشخصه های آن ها را به دست می‌آوریم.

Email: f farzalipour@pnu.ac.ir