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ON CONTINUOUS COHOMOLOGY OF LOCALLY COMPACT ABELIAN GROUPS AND BILINEAR MAPS

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Abstract. Let A be an abelian topological group and B a trivial topological A-module. In this paper we define the second bilinear cohomology with a trivial coefficient. We show that every abelian group can be embedded in a central extension of abelian groups with bilinear cocycle. Also we show that in the category of locally compact abelian groups a central extension with a continuous section can be embedded in the second bilinear cohomology.

1. Introduction

Let A be an abelian topological group. We say that B is a topological A-module if there is a continuous action of A on B. By a topological extension of B by A we mean a short exact sequence $0 \to B \to E \xrightarrow{\pi} A \to 0$ with a continuous section $u : A \to E$, where π is an open onto continuous homomorphism and $\pi \circ u = Id_E$ (i.e. identity on E). If B is in the center of E (i.e. $B \subseteq Z(E)$), then it is called a central extension. In general E may be nonabelian. If E is an abelian group, then the extension is called an abelian one. Extensions of topological groups admitting a continuous section are assured by the following theorem: Let G be a connected locally compact group. Then any topological group extension of G by a simply connected Lie group admits a continuous section [11, Theorem 2]. The set of extensions of B by A forms a group denoted by $Ext(A, B)$ [1]. It is known that $H^2(A, B)$, the second cohomology of B by A, is isomorphic with $Ext(A, B)$ i.e., the group of

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all extensions of B by A admitting a (global) continuous section and the given A-module structure [5]. In this paper we define the second bilinear cohomology and show that a locally compact abelian group can be embedded in a central extension with a continuous section where the associated cocycle is bilinear. In section 2, we recall some results on cohomology of topological groups which will be needed in sequel. In section 3, we define the second bilinear cohomology $H^2_{Bil}(G,B)$ and embed a locally compact abelian group in a central extension of abelian groups with a continuous section. Also we show that in the category of locally compact abelian groups if A is a discrete divisible group, then $H^2(A,B)$ and $H^2_{Bil}(A,B)$ are the same. We refer the reader to [1], [12], and [8] for basic material on the cohomology of groups and for the extension problem to [10].

2. Preliminaries

Let A be a topological group and B be an abelian topological A-module. We show the identity of a group G by e_G if it is written multiplicatively and 0_G if it is written additively. The commutator of two elements $x, y \in G$ is given by $[x, y] = xyx^{-1}y^{-1}$. If U, V are two subgroups of G, then $[U, V]$ is the subgroup generated by $[u, v]$, where $u \in U$, $v \in V$. By a factor set (cocycle), we mean a continuous map $\gamma : A \times A \to B$ such that

(1) $x, y \in A$, $\gamma(e, y) = \gamma(y, e) = e$.
(2) $x, y, z \in A$, $\gamma(x, y) + \gamma(xy, z) = \gamma(y, z) + \gamma(x, yz)$.

Any central extension $0 \to B \to E \to A \to 0$ with a continuous section $u : A \to E$ determines a factor set γ such that

$$u(x) + u(y) = u(xy) + u(x) + u(y) = u(xy) + \gamma(x, yz).$$

Conversely, any factor set $\delta : A \times A \to B$ satisfying (1) and (2) determines a central extension with a continuous section. Take the set of all (a, b), where $a \in A$, $b \in B$, with the multiplication defined by

$$(a, b).(a', b') = (aa', b + b' + \delta(a, a')).$$

We denote this group by G_δ and the direct product defines a standard topological group structure on G_δ. One gets a central extension $0 \to B \to G_\delta \xrightarrow{\pi} A \to 0$, $\pi(a, b) = a$ with a continuous section $u(a) = (a, 0)$.

A continuous map $\delta : A \times A \to B$ is bilinear if $\delta(xy, z) = \delta(x, y) + \delta(x, z)$ and $\delta(x, yz) = \delta(x, y) + \delta(x, z)$ for all $x, y, z \in A$. Note that δ satisfies (1) and (2) although the converse may not be true.

Example 2.1. Consider the extension $0 \to Z/3Z \xrightarrow{i} Z/9Z \xrightarrow{\pi} Z/3Z \to 0$ given by $i(\overline{1}) = 3$, with the section $u : Z/3Z \to Z/9Z$, $u(\overline{x})) = \overline{x}$, where $0 \leq x < 3$. Here $\overline{x}$ denotes the class of x modulo 3 or 9 depending on the context. Equipping each group with the discrete topology, we can consider this as an extension of topological groups. Then u is clearly a continuous section. Moreover,

$$\gamma(\overline{x}, \overline{y}) = \begin{cases} 0 & 0 \leq x < 3 \\ 3 & 3 \leq x + y \end{cases}$$

for all x and y, $0 \leq x, y < 3$, is the associated factor set of the extension since

$$u(\overline{x}) + u(\overline{y}) = u(\overline{(x + y)}) + \gamma(\overline{x}, \overline{y}).$$

Now γ is not bilinear since $\gamma(\overline{2}, \overline{2}) = \overline{3}$ but $\gamma(\overline{0}, \overline{0}) - \gamma(\overline{0}, \overline{1}) - \gamma(\overline{1}, \overline{0}) + \gamma(\overline{1}, \overline{1}) = 0$.

In case that the factor set γ associated with a central extension $0 \to B \to G \to A \to 0$ is bilinear, we will denote the resulting group G by $A \times_\delta B$. A continuous map $\sigma : A \times A \to B$ is called a coboundary if and only if there is a continuous map $h : A \to B$, $h(0) = 0$ and for all $x, y \in G$,

$$\sigma(x, y) = h(x) + h(y) - h(xy).$$

Let $Z^2(G, A)$ be the set of factor sets and $B^2(G, A)$ the set of all coboundaries. By the point wise addition, both are abelian groups and $B^2(G, A)$ is a subgroup of $Z^2(G, A)$. Then

$$H^2(A, B) = Z^2(A, B)/(B^2(A, B)$$

is called the second cohomology of A by B. Given a continuous map $\varphi : B \to C$, we obtain $\phi^* : H^2(A, B) \to H^2(A, C)$, given by $\phi^*[\delta] = [\phi.\delta]$. Dually, a map $\omega : G \to C$ induces a map $\omega_* : H^2(C, B) \to H^2(G, B)$, $\omega_*[\gamma] = [\gamma \circ (\omega \times \gamma)]$. Two central extensions $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ and $0 \to B \xrightarrow{i'} E \xrightarrow{\pi'} A \to 0$ are said to be equivalent if there exists an open continuous isomorphism $\phi : E \to E'$ such that the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \to & B & \xrightarrow{i} & E & \xrightarrow{\pi} & A & \to & 0 \\ & & \| & & \downarrow \phi & & \| & & \\ 0 & \to & B & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & A & \to & 0. \end{array}$$

If A and B are abelian then the set of all extensions of B by A, denoted by $Ext(A, B)$, with the Bear-sum forms an abelian group [1].

3. Cohomology of Bilinear product

In this section, we define the bilinear cohomology and will show that every abelian topological group can be embedded into a central extension of abelian groups with a continuous section in which the associated cocycle is bilinear. Note that if δ is a bilinear map from $A \times A$ to B, then it satisfies (1) and (2) although the converse is not necessarily true. In case that the factor set δ associated with a central extension $0 \to B \to E \to A \to 0$ is bilinear, we will denote the resulting group E by $A \times_\delta A$ and call this group a bilinear product of A and B.

Definition 3.1. The set of all equivalence classes of central extensions of B by A, at least one of whose representatives is a bilinear product, is denoted by $H^2_{Bil}(A, B)$. Then $H^2_{Bil}(A, B)$ is called the second bilinear cohomology of B by A.

In this section we show that for given an abelian topological group B, there is an abelian group L containing B and an embedding $\phi : B \to L$ such that the induced map $\phi^* : H^2(A, B) \to H^2(A, L)$ has image in $H^2_{Bil}(A, L)$.

Let A, B be abelian topological groups, $\alpha : A \times A \to B$ an alternating bilinear map, i.e. $\alpha(x, x) = 0$ for all $x \in G$. Let abelian group C, the bilinear map $\beta : A \times A \to C$, and the map $i_B : B \to C$ be the universal triple with the following universal property: The map $i_B : B \to C$ is such that the following diagram

(3)

$$\begin{array}{ccccc} & & C & & \\ \beta(x,y) - \beta(y,x) & \nearrow & & \nwarrow & i_B \\ A \times A & & \overset{\alpha(x,y)}{\longrightarrow} & & B \end{array}$$

is commutative.

(4) For any abelian group $\acute{C}$, any bilinear map $\acute{\beta} : A \times A \to \acute{C}$, and a map $\acute{\phi} : B \to \acute{C}$ such that

$$\begin{array}{ccccc} & & \acute{C} & & \\ \acute{\beta}(x,y) - \acute{\beta}(y,x) & \nearrow & & \nwarrow & \acute{\phi} \\ A \times A & & \overset{\alpha(x,y)}{\longrightarrow} & & B \end{array}$$

is commutative, there exists a unique $\psi : C \to \acute{C}$ such that $\acute{\phi} = \psi \circ i_B$ and $\acute{\beta} = \psi \circ \beta$.

Let $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be a central extension of abelian groups with a continuous section $u : A \to E$. Let L be an abelian group and let $\beta : E \times E \to L$ be a bilinear map. What conditions a continuous map $f : E \to L$ must satisfy such that the map $\phi : E \to A \times_\beta L$

$$(5)\ \ \phi(g) = (\pi(g), f(g)), \ \ g \in E$$

be an injective group morphism.

Lemma 3.2. *Let $0 \to B \to E \to A \to 0$ be a central extension of abelian groups with a continuous section and L and β and ϕ as in the last paragraph. A set map $f : E \to L$ makes ϕ a group morphism if and only if*

$$(6)\ \ x, y \in E, \ \ f(x + y) = f(x) + f(y) + \beta(x, y).$$

Furthermore, f makes ϕ into an injective group morphism if and only if (6) and

$$(7)\ \ f_{|\beta} : B \to L \ \ \textit{is a one to one group morphism}$$

hold.

Proof. (a): Let $y \in E$. Then

$$\begin{aligned}(\pi(x), f(x)).(\pi(y), f(y)) &= (\pi(x)\pi(y), f(x) + f(y) + \beta(x, y)) \\ &= (\pi(x + y), f(x + y)).\end{aligned}$$

So (6) holds.

(b): If x, y are not in the kernel of π, then $\pi(x) \neq \pi(y)$. So $\phi(x) \neq \phi(y)$. Hence, it is necessary and sufficient to require that f be an injective map on B. □

Remark 3.3. (i) If (6) holds, then by induction on n

$$f(nx) = nf(x) + (n(n-1))/2\beta(x, x), \ \ \textit{for all} \ \ x \in E, n \in Z.$$

(ii) If f satisfies (6), then

$$(8)\ \ f([x, y]) = \beta(x, y) - \beta(y, x), \ \ \textit{for all} \ \ x, y \in E.$$

Let $0 \to B \to E \to A \to 0$ be a central extension with a continuous section. We define a map f on B such that it satisfies (8) and will show that it can be extended to all E. Recall that a group G is divisible if $nG = G$ for every positive integer n.

Lemma 3.4. *Let E be as in Lemma 3.2, and let U and V be two subgroups of G with $[U, V] \subseteq U$. Let L be an abelian group and, $\beta : A \times A \to L$ be a bilinear map. Suppose $f_U : U \to L$, $f_V : V \to L$ are given* *with* $f_{U|U\cap V} = f_{V|U\cap V}$ *and satisfying (6). Suppose* $W =< U, V >= UV$ *be the subgroup generated by* U *and* V*. Also,* $f_W : W \to L$ *is given by*

$$f_W(U + V) = f_U(U) + f_V(V) + \beta(U, V).$$

Then, f_W *is an extension of* f_U *and* f_V *satisfying (6).*

Proof. First we show that f_W is well-defined. Suppose $u_1+v_1 = u_2+v_2$. Then $-u_2+u_1 = v_2-v_1 \in U\cap V$. So we have $f_U(-u_2+u_1) = f_V(v_2-v_1)$. Hence $f_U(u_1) - f_U(u_2) + \beta(u_2, u_2) - \beta(u_2, u_1) = f_V(u_2) - f_V(u_1) + \beta(v_1, v_1) - \beta(v_2, v_1)$. Also, $u_1 = u_2 + v_2 - v_1$ and $v_2 = u_1 - u_2 + v_1$. By definition, it is easy to show that $f_W(u_1 + v_1) - f_W(u_2 + v_2) = 0$. Now it is clear that f_W extends f_U and f_V. Assume $u_1 + v_1, u_2 + v_2 \in W$. Since $[U, V]$ is in the center of E,

$$(u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2 + [v_1, u_2]) + (v_1 + v_2)$$

with $u_1 + u_2 + [v_1, u_2] \in U$ and $v_1 + v_2 \in V$. So,

$$\begin{aligned} f_W((u_1 + v_1) + (u_2 + v_2)) &= f_W(u_1 + u_2 + [v_1, u_2]) + (v_1 + v_2) \\ &= f_U(u_1) + f_U(u_2) + \beta(v_1, u_2) + \beta(u_1, u_2) \\ &+ f_V(v_1) + f_V(v_2) + \beta(v_1, v_2) + \beta(u_1, v_1) \\ &+ \beta(u_1, v_2) + \beta(u_2, v_2). \end{aligned}$$

On the other hand,

$$\begin{aligned} &f_W(u_1 + v_1) + f_W(u_2 + v_2) + \beta(u_1 + u_2, v_1 + v_2) \\ &\qquad = f_U(u_1) + f_V(v_1) + \beta(u_1, v_1) \\ &\qquad + f_U(u_2) + f_V(v_2) + \beta(u_2, v_2) + \beta(u_1, v_2) \\ &\qquad + \beta(v_1, u_2) + \beta(v_1, v_2). \end{aligned}$$

So f_W satisfies (6). □

Let $0 \to B \to E \to A \to 0$ be a central extension, and H a proper subgroup of A with $B \subseteq H$. Let L be an abelian divisible group, $\beta : A \times A \to L$ a bilinear map, and $f : H \to L$ satisfying (6), (7), and (8). Let $g \in G - H$ and $K =< H, g >$. We want to extend f to all of K, retaining properties (6), (7), and (8).

Theorem 3.5. *Let* A*,* H*,* β*,* L*,* f *and* K *be as above. Then* f *can be extended to* K*.*

Proof. Let $H = U$ and $V =< g >$ and $f_U = f$. We define f_V. Let $I = a \in Z_+; ag \in U$. There are two cases. Case 1, $I = \emptyset$. So g is not a torsion element of G. We define $f_V(g) = 0$. Note that, by induction on n, one can show that if a function f satisfies (6), then

(9) $f(nx) = nf(x) + (n(n-1))/2\beta(x,x)$ *for all* $x \in G, n \in Z$.

Therefore, using the above statement, f_V can be extended to every power of g. Now the Lemma 3.4 gives an extension of f to all of K. Case2, $I \neq \emptyset$. Let $n_0 = min\{n, n \in I\}$. Then $V =< n_0 g >$. By (7), f_V must satisfy,

$$f_V(n_0 g) = n_0 f_V(g) + (n_0(n_0+1))/2\beta(g,g).$$

However, $n_0 g \in U$. So $f_V(n_0 g) = f_U(n_0 g)$. Hence,

$$n_0 f_V(g) = n_0 f_U(g) - (n_0(n_0+1))/2\beta(g,g).$$

Since L is divisible, we define $f_V(g)$ be the n_0-th root of $n_0 f_U(g) - (n_0(n_0+1))/2\beta(g,g)$ written by

$$f_V(g) = 1/n_0(f_U(n_0 g) - (n_0(n_0+1))/2\beta(g,g)).$$

By (9), f_V can be extended to all of V. Suppose g is a torsion element. By definition of n_0, it follows that the order of g is a multiple of n_0, say $k_0 n_0$. We show that for every $a \in Z$, $f_V(ga) = f_V((k_0 n_0 + a)g)$. By definition,

$$\begin{aligned} f_V(ag) &= af_V(g) + a(a-1)/2\beta(g,g) \\ &= a(1/n_0(f_U(n_0 g) - (n_0(n_0+1))/2\beta(g,g))) \\ &+ (a(a-1))/2\beta(g,g). \end{aligned}$$

On the other hand,

$$\begin{aligned} f_V((k_0 n_0 + a)g) &= (k_0 n_0 + a) f_V(g) + ((k_0 n_0 + a)((k_0 n_0 + a) - 1))/2\beta(g,g) \\ &= (k_0 n_0 + a)(1/n_0(f_U(n_0 g) - (n_0(n_0+1))/2\beta(g,g))) \\ &+ ((k_0 n_0 + a)((k_0 n_0 + a) - 1))/2\beta(g,g). \end{aligned}$$

Now we show that $f_{U|(U \cap V)} = f_{V|(U \cap V)}$. Since the intersection is generated by $n_0 g$, we need to check the values of f_V at $kn_0 g$ for $k \in Z$. We have $f_V(kn_0 g) = kn_0 f_V(g) + (kn_0(kn_0 - 1))/2\beta(g,g)$. By replacing f_V in terms of f_U, then $f_V(kn_0 g) = f_U(kn_0 g)$. So $f_{U|(U \cap V)} = f_{V|(U \cap V)}$. Now by Lemma 3.4, we get an extension of f to all of K.

□

Theorem 3.6. *Let* $0 \to B \to E \to A \to 0$ *be a central extension of abelian groups with a continuous section. Then there exists an abelian group* L*, a bilinear map* $\acute{\beta} : A \times A \to L$*, and a continuous map* $f : E \to L$ *satisfying (6), (7).*

Proof. Let L be the minimal divisible group containing B [4, A.15] and $j : B \to L$ an embedding. We consider the alternating map $\alpha(x, y) = [x, y]$. Then by [6], there is an abelian group C, an embedding $i_B : B \to C$, and a bilinear map $\beta : A \times A \to C$ satisfying $\beta(x, y) - \beta(y, x) = [x, y]$ for all x and y in G, with the triple (C, i_B, β) being universal. Since L is divisible, there is a map $\chi : C \to L$ such that $\chi \circ i_B = j$ [4, A.7]. Let $\acute{\beta} = \chi \circ \beta$ and $j : B \to L$, $f = j$. Then, f satisfies (7). Also

$$\begin{aligned} f([x, y]) = j([x, y]) = j(\alpha(x, y)) &= \chi(i_B(\alpha(x, y))) \\ &= \chi(\beta(x, y) - \beta(y, x)) \\ &= \acute{\beta}(x, y) - \acute{\beta}(y, x). \end{aligned}$$

So f satisfies (7) and (8). We consider the set of pairs (H, f_H), where H is a subgroup of G containing B, and $f_H : H \to L$ is a continuous function satisfying (6), (7), and (8). We partially order this set by $(H, f_H) \leq (\acute{H}, \acute{f}_H)$ if and only if $H \subseteq \acute{H}$ and $(f_{\acute{H}})_{|_H} = f_H$.

Note that the existence of f shows that the above set is non-empty for the given L. By Theorem 3.5, and applying *Zorn's* Lemma to this set of pairs, a maximal element in the set must have the first coordinate equal to E. □

Note that L in Theorem 3.6 only depends on B.

Corollary 3.7. *Let $0 \to B \to E \to A \to 0$ be a central extension with a continuous section. Then there exists an abelian group L containing B, a bilinear map $\beta : A \times A \to L$, and an embedding map $\phi : E \to A \times_\beta L$ such that the following diagram*

$$\begin{array}{ccccccccc} 0 & \to & B & \xrightarrow{i} & E & \xrightarrow{\pi} & A & \to & 0 \\ & & j \downarrow & & \downarrow \phi & & \| & & \\ 0 & \to & B & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & A & \to & 0 \end{array}$$

is commutative and has the exact rows.

Proof. Let f and β be as in Theorem 3.6 and $\phi(g) = (\pi(g), f(g))$. So the square on the right commutes. Now by the construction, $\phi(i(b)) = (1, b) = j(\acute{i}(b))$. So the diagram commutes. □

Theorem 3.8. *Let $0 \to B \to E \to A \to 0$ be a central extension of topological groups with a continuous section $u : A \to G$ and let γ in $H^2(A, B)$ be a factor set representing this extension. Then there exists an abelian group L and an embedding $j : B \to L$ such that*

$$j^*(\gamma) \in H^2_{Bil}(A, B).$$

Moreover, the group L and j depend only on B.

Proof. Let L, j and $f : E \to L$ be as in the Theorem 3.6. Let $h : A \to L$ be the continuous map $h(x) = f(u(x))$ for every $x \in A$. Then $h(e) = f(u(e)) = f(e) = 0$. Now we show that $j \circ \gamma$ is equal to the β bilinear product obtained in Theorem 3.6.

$$\begin{aligned} h(x) + h(y) - h(xy) &= f(u(x)) + f(u(y)) - f(u(xy)) \\ &= f(u(x)) + f(u(y)) - f(u(x) + u(y) - \gamma(x, y) \\ &= f(u(x)) + f(u(y)) - f(u(x) + u(y)) - j(\gamma(x, y)) \\ &= f(u(x)) + f(u(y)) - f(u(x)) \\ &- f(u(y)) - \gamma(u(x). \end{aligned}$$

Since $u(x)$ is the image of x, we have

$$\begin{aligned} u(y)) + j(\gamma(x, y) &= j(\gamma(x, y)) - \gamma(u(x), u(y)) \\ &= j(\gamma(x, y)) - \gamma(x, y). \end{aligned}$$

So, $[\beta] = [j \circ \gamma] \in H^2(A, L)$.

□

In fact we have shown the following.

Theorem 3.9. *Let B be an abelian group. Then there exists an abelian group L, and an embedding $j : B \to L$ such that $j^*(H^2(A, B)) \subseteq H^2_{Bil}(A, L)$ for every abelian group A.*

Let $\pounds$ denote the category of locally compact abelian groups with continuous homomorphisms as morphisms.

Theorem 3.10. *Let $A, B \in \pounds$, A discrete, and B be a trivial A-module. Then there exists a locally compact abelian group L and an embedding $j : B \to L$ such that*

$$j^* : H^2(A, B) \to H^2(A, L).$$

has image contained in $H^2_{Bil}(A, L)$, and kernel equal to $Ext_{ab}(A, B)$ (that is, the group of equivalence classes of the abelian extensions of B by A with continuous sections).

Proof. Let L be the minimal divisible group containing B and $j : B \to L$ an inclusion as in Theorem 3.7. By [4, 4.18 h], L is a locally compact abelian group containing B as an open subgroup. By Theorem 3.9,

$$j^*(H^2(A, B)) \subseteq H^2_{Bil}(A, L)$$

Let $0 \to B \overset{j}{\to} E \overset{p}{\to} A \to 0$ be a central extension. Since L/B is discrete [9], then the extension has a continuous section. Let $A, B \in \pounds$ and B be a trivial A-module. By [6] there is long exact sequence

$$0 \to Hom(A,B) \overset{j^*}{\to} Hom(A,L) \overset{p^*}{\to} Hom(A,L/B)$$
$$\overset{\delta_1}{\to} H^2(A,B) \overset{j^*}{\to} H^2(A,L) \overset{p^*}{\to} \cdots .$$

By [5], $H^2(A,B)$ is isomorphic with $Ext(A,B)$, the group of all extensions of B by A having continuous sections. So

$$0 \to Hom(A,B) \overset{j^*}{\to} Hom(A,L) \overset{p^*}{\to} Hom(A,L/B)$$
$$\overset{\delta_1}{\to} Ext(A,B) \overset{j^*}{\to} Ext(A,L) \overset{p^*}{\to} \cdots .$$

By [3, Corollary 2.10], there is an exact sequence

$$0 \to Hom(A,B) \overset{j^*}{\to} Hom(A,L)$$
$$\overset{p^*}{\to} Hom(A,L/B) \overset{\delta_1}{\to} Ext_{ab}(A,B)$$
$$\overset{j^*}{\to} Ext_{ab}(A,L) \overset{p^*}{\to} Ext_{ab}(A,l/B) \to 0.$$

Since L is divisible, $Ext_{ab}(A,L) = 0$ [2]. Hence, δ_1 is a map onto $Ext_{ab}(A,B)$ which is a subgroup of $H^2(A,B)$. So, the kernel of $j^* : H^2(A,B) \to H^2(A,L)$ is $Ext_{ab}(A,B)$.

□

Corollary 3.11. *Let A be a discrete locally compact abelian group and B a divisible group in $\pounds$ and a trivial A-module. Then every central extension of B by A is equivalent to a bilinear product of B by A. In other word,*

$$H^2(A,B)) = H^2_{Bil}(A,L).$$

Proof. In Theorem 3.10, let $L = B$ and $j : B \to L$ be the identity map, $j = Id_B$. Then, by Theorem 3.10, $Id^* : H^2(A,B) \to H^2(A,B)$ has the image in $H^2_{Bil}(A,L)$ with the trivial kernel. Hence,

$$H^2(A,B) = H^2_{Bil}(A,B).$$

□

Remark 3.12. If A is a free abelian topological group (in Makov sense) [7] and $B \in \pounds$, then $H^2(A,B) = 0$ by [5]. Since A is free, any central extension of bilinear product splits. Hence, $H^2_{Bil}(A,B) = 0$. Therefore,

$$H^2(A,B)) = H^2_{Bil}(A,L).$$

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