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THE GENERALIZED TOTAL GRAPH OF MODULES RESPECT TO PROPER SUBMODULES OVER COMMUTATIVE RINGS

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ABSTRACT. Let M be a module over a commutative ring R and let N be a proper submodule of M. The total graph of M over R with respect to N, denoted by $T(\Gamma_N(M))$, have been introduced and studied in [2]. In this paper, A generalization of the total graph $T(\Gamma_N(M))$, denoted by $T(\Gamma_{N,I}(M))$ is presented, where I is an ideal of R. It is the graph with all elements of M as vertices, and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $m + n \in M(N, I)$, where $M(N, I) = \{m \in M : rm \in N + IM \ for \ some \ \ r \in R - I\}$. The main purpose of this paper is to extend the definitions and properties given in [2] and [12] to a more general case.

1. Introduction

Throughout of this paper R is a commutative ring with nonzero identity and M is a unitary R-module. Recently, there has been considerable attention in the work to associating graphs with algebraic structures (see [1],[5],[7] and [11]). In [6], the notion of the total graph of a commutative ring $T(\Gamma(R))$ was introduced. The vertices of this graph are all elements of R and two vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$ ($Z(R)$ is the set of zero divisors of R). The total torsion element graph of a module M over a commutative ring R denoted by $T(\Gamma(M))$ was introduced by Ebrahimi Atani and Habibi in [12], as the graph with all elements of M as vertices, and two distinct vertices

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$x, y \in M$ are adjacent if and only if $x+y \in T(M)$ ($T(M)$ is the set of torsion elements of M). Let N be a proper submodule of an R-module M and the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Also for an element $r \in R$, the submodule $\{m \in M : rm \in N\}$ will be denoted by $(N :_M r)$. In [2], Abbasi and Habibi introduced total graph of M respect to an arbitrary proper submodule N; denoted by $T(\Gamma_N(M))$. The vertex set of $T(\Gamma_N(M))$ is M and two distinct vertices $m, n \in M$ are adjacent if and only if $m+n \in M(N)$, where $M(N) = \{m \in M : rm \in N \ for\ some\ \ r \in R - (N : M)\}$. A proper submodule N of M is said to be a prime submodule if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $r \in (N : M)$. It is clear to see that if N is a prime submodule of M, then $P = (N : M)$ is a prime ideal of R and N is said to be a P-prime submodule. Now, let I be a proper ideal of R. Then $S(I)$ is the set of all elements of R that are not prime to I; i.e., $S(I) = \{a \in R : ra \in I \ for\ some\ r \in R - I\}$. It is clear that $S(P) = P$ for every prime ideal P of R. We define $M(N, I) = \{m \in M : rm \in N + IM \ for\ some\ \ r \in R - I\}$. Since $IM + N \subseteq M$, then $M(N, I)$ is not empty. $M(N, I)$ is not necessarily a submodule of M (not always closed under addition, see Example 2.2), but it is clear that if $r \in R$ and $x \in M(N, I)$, then $rx \in M(N, I)$. It is easy to see that $T(M) = M(0, 0)$ and $M(N, I) = M(N)$ for every ideal $I \subseteq (N : M)$.

In the present paper, we introduce and investigate the generalized total graph of M respect to a submodule, denoted by $T(\Gamma_{N,I}(M))$, as a (undirected) graph with all elements of M as vertices, and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $m+n \in M(N, I)$. It is easy to check that $T(\Gamma_N(M)) = T(\Gamma_{N,(N:M)}(M))$ and $T(\Gamma(M)) = T(\Gamma_{0,0}(M))$. So by this definition, we can extend the definitions and the results of graphs expressed in [2] and [12].

Let $M(\Gamma_{N,I}(M))$ be the (induced) subgraph of $T(\Gamma_{N,I}(M))$ with vertex set $M(N, I)$, and let $\overline{M}(\Gamma_{N,I}(M))$ be the (induced) subgraph $T(\Gamma_{N,I}(M))$ with vertices consisting of $M - M(N, I)$.

The study of $T(\Gamma_{N,I}(M))$ breaks naturally into two cases depending on whether or not $M(N, I)$ is a submodule of M. In the second section, we obtain some properties concerning $M(N, I)$. In the third section, we handle the case when $M(N, I)$ is a submodule of M; in forth section, we do the case when $M(N, I)$ is not a submodule of M. For every case, we characterize the girths and diameters of $T(\Gamma_{N,I}(M))$, $M(\Gamma_{N,I}(M))$ and $\overline{M}(\Gamma_{N,I}(M))$.

We begin with some notation and definitions. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we mean the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two of it's distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$. We also define $d(a, a) = 0$. The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,m}$ a star graph. For a graph Γ, the degree of a vertex v in Γ, denoted $deg(v)$, is the number of edges of Γ incident with v. For a nonnegative integer k, a graph is called k-regular if every vertex has degree k. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertices of Γ_1 is adjacent(in Γ) to some vertex of Γ_2.

2. Some Properties of $M(N, I)$

In this section we list some basic properties concerning $M(N, I)$ where N is a proper submodule of an R-module M and I is a proper ideal of R. We show that $M(N, I)$ is a union of prime submodules of M. We have the following remark by [10, 2.2 and 2.7].

Remark 2.1. Let N, L be proper submodules of an R-module M and let I, P be proper ideals of R.
(1) If $N \subseteq IM$, then $M(N, I) = M(0, I) = M(IM)$. In particular, if $N, L \subseteq IM$, then $M(N, I) = M(L, I)$.
(2) If P is a prime ideal of R and $M(N, I) \subseteq M(N, P) \neq M$, then $I \subseteq P$.
(3) If P is a prime ideal of R, then N is a P-prime submodule of M if and only if $M = M(N, P)$.
(4) If P is a prime ideal of R and $M(N, P) \neq M$, then $M(N, P)$ is a P-prime submodule of M and is the intersection of all P-prime submodules of M containing N.

The following examples show that if N is a proper submodule of an R-module M and I is a proper ideal of R, then $M(N, I)$ is not necessarily a proper submodule of M.

Example 2.2. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$, $N = 4Z \times 7Z$ and $I = 28\mathbb{Z}$. It is clear that $M(N, I)$ is not a submodule of M, since $(1, 0), (0, 1) \in M(N, I)$ but $(1, 1) \notin M(N, I)$.

Example 2.3. Let $R = Z_{12}$, $M = Z_6$.
(a) If $N = \bar{2}Z_6$ and $I = \bar{3}Z_{12}$. Then $M(N, I) = IM + N = M$.
(b) If $N = \bar{3}Z_6$ and $I = \bar{6}Z_{12}$. Then $IM = 0$ and since $\bar{3}\bar{1} \in N$ and $\bar{3} \notin I$, so $\bar{1} \in M(N, I)$. Thus $M(N, I) = M$.

Proposition 2.4. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R. If $M(N, I)$ is a proper submodule of M, then $M(N, I)$ is an $S(I)$-prime submodule of M. Moreover, $r \in S(I)$ if and only if $rm \in M(N, I)$ for every $m \in M$.*

Proof. We first show that $(M(N, I) : M) = S(I)$. Let $r \in (M(N, I) : M)$. Then $rM \subseteq M(N, I)$. Suppose that $m \in M - M(N, I)$, so $rm \in M(N, I)$ and $srm \in N + IM$ for some $s \in R - I$. Thus $rs \notin R - I$ since $m \notin M(N, I)$. Therefore $rs \in I$ and so $r \in S(I)$. Conversely, assume that $t \in S(I)$. So $tr \in I$ for some $r \in R - I$. If $m \in M$, then $r(tm) = (rt)m \in IM \subseteq IM + N$. This implies that $tm \in M(N, I)$ for every $m \in M$. Thus $t \in (M(N, I) : M)$.
Now, let $rm \in M(N, I)$ for some $r \in R$ and $m \in M$ such that $m \notin M(N, I)$. The above argument shows that $tr \in I$ for some $t \in R - I$. Therefore $r \in S(I) = (M(N, I) : M)$. The "moreover" statement follows directly from the above arguments. $\square$

Recall that if $M \neq T(M)$, then $T(M)$ is a union of prime submodules ([4, 3.3]). Now, we have the following theorem by the similar method in [4, 3.3].

Theorem 2.5. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R with $M \neq M(N, I)$. Then $M(N, I)$ is a union of prime submodules of M.*

Proof. Let $x \in M(N, I)$. Set $S_x = \{L :\ L \textit{ is a submodule of } M, x \in L \subseteq M(N, I), \textit{ and } L = \bigcup (IM + N :_M r_\lambda) \textit{ for some } \{r_\lambda\} \subseteq R\}$. Assume that $rx \in IM + N$ for some $r \in R - I$. So $x \in (IM + N :_M r)$, then $S_x \neq \emptyset$. Partially order S_x by inclusion. By Zorn's Lemma, S_x has a maximal element L_x. It suffices to show that L_x is a prime submodule.
Let $L_x = \bigcup_{\lambda \in \Lambda} (IM + N :_M r_\lambda)$ and let $rm \in L_x$ with $m \notin L_x$. If $rr_\lambda \in R - I$ for every $\lambda \in \Lambda$, then $(IM + N :_M r_\lambda) \subseteq (IM + N :_M rr_\lambda)$. Hence $L_x \subseteq L'_x = \bigcup_{\lambda \in \Lambda} (IM + N :_M rr_\lambda)$. Now, let $m_1, m_2 \in L'_x$. Then $m_i \in (IM + N :_M rr_{\lambda_i})$ for $i = 1, 2$. So $rm_i \in (IM + N :_M r_{\lambda_i}) \subseteq L_x$ and hence $rm_1 + rm_2 \in L_x$. Thus $rm_1 + rm_2 \in (IM + N :_M r_\eta)$

for some $\eta \in \Lambda$; so $m_1 + m_2 \in (IM + N :_M rr_\eta) \subseteq L'_x$. It is clear that L'_x is closed under scalar product, so L'_x is a submodule of M with $L'_x \subseteq M(N, I)$. Thus by maximality of L_x, $L_x = L'_x$. Since $rm \in L_x$, so $rm \in (IM + N :_M r_\alpha)$ for some $\alpha \in \Lambda$. Hence $m \in (IM + N :_M rr_\alpha) \subseteq L'_x = L_x$; a contradiction. So $rr_\lambda \in I$ for some $\lambda \in \Lambda$. Then $rr_\lambda M \subseteq IM$ and hence $rM \subseteq (IM + N :_M r_\lambda) \subseteq L_x$. So $M(N, I) = \bigcup_{x \in M(N,I)} L_x$ is a union of prime submodules. □

Proposition 2.6. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R with $M \neq M(N, I)$ and $M \neq T(M)$. If R is not an integral domain and $L_1 \cap L_2 = 0$ for some prime submodules $L_1, L_2 \subseteq M(N, I)$, then either $P \cap L_1 \neq 0$ or $P \cap L_2 \neq 0$ for every prime submodule P of M.*

Proof. Let L_1 be a P_1- prime submodule and L_2 be a P_2- prime submodule of M. So $P_1, P_2 \neq 0$, since R is not an integral domain. Therefore $P_1P_2M \subseteq P_1M \cap P_2M \subseteq L_1 \cap L_2 = 0$. Thus $P_1P_2M = 0 \subseteq P$. This implies that either $P_1M \subseteq P$ or $P_2M \subseteq P$, since P is a prime submodule of M. Hence either $0 \neq P_1M \subseteq P \cap L_1$ or $0 \neq P_2M \subseteq P \cap L_2$, since $M \neq T(M)$. □

Proposition 2.7. *Let N be a proper submodule of an R-module M and let P be a prime ideal of R such that $M(N, P) \neq M$. Then for every multiplicatively closed subset S of R with $S \cap P \neq \emptyset$, $S^{-1}(M(N, P)) = S^{-1}M(S^{-1}N, S^{-1}P)$.*

Proof. Assume that $m/s \in S^{-1}M(S^{-1}N, S^{-1}P)$ for some $m \in M$ and $s \in S$. So there exists $r/t \in S^{-1}R - S^{-1}P$ such that $rm/st \in (S^{-1}P)(S^{-1}M) + S^{-1}N = S^{-1}(PM + N)$. Thus $rm/st = x/s'$ for some $x \in PM + N$ and $s' \in S$. Hence $s''s'rm = s''stx$ for some $s'' \in S$. Since P is a prime ideal of R, so $s''s' \notin P$, then $rm \in M(N, P)$ by definition. So $m \in M(N, P)$ since $r \notin P$ and $M(N, P)$ is a P-prime submodule of M by [10, 2.2]. Conversely, let $m/s \in S^{-1}(M(N, P))$ for some $m \in M(N, P)$ and $s \in S$. Thus $tm \in PM + N$ for some $t \in R - P$. Then $t/1 \in S^{-1}R - S^{-1}P$ and $(t/1)(m/s) = tm/s \in S^{-1}(PM + N) = (S^{-1}P)(S^{-1}M) + S^{-1}N$. Hence $m/s \in S^{-1}M(S^{-1}N, S^{-1}P)$. □

3. The case when $M(N, I)$ is a submodule of M

In this section, we study the case when $M(N, I)$ a submodule of M (i.e when $M(N, I)$ is closed under addition). It is clear that if $M(N, I) = M$, then $T(\Gamma_{N,I}(M))$ is a complete graph. Thus, in this section we suppose that $M(N, I) \neq M$. So if $M(N, I)$ is a submodule of M, then $M(N, I)$ is actually a prime submodule of M by Proposition

2.4. We denote $M(\Gamma_{N,I}(M))$ and $\overline{M}(\Gamma_{N,I}(M))$ the (induced) subgraphs of $T(\Gamma_{N,I}(M))$ with vertices in $M(N,I)$ and $M-M(N,I)$ respectively.

Theorem 3.1. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is a submodule of M. Then:*
(1) $M(\Gamma_{N,I}(M))$ is a complete (induced) subgraph of $T(\Gamma_{N,I}(M))$ and it is disjoint from $\overline{M}(\Gamma_{N,I}(M))$.
(2) If $0 \neq IM+N \subsetneqq M(N,I)$, then $gr(M(\Gamma_{N,I}(M)))=3$.

Proof. (1) It is clear by definition that for all $m,n \in M(N,I)$, we have $m+n \in M(N,I)$; since $M(N,I)$ is a submodule of M. Thus $M(\Gamma_{N,I}(M))$ is a complete (induced) subgraph of $T(\Gamma_{N,I}(M))$. Now, suppose that $x \in M(N,I)$ and $y \in M-M(N,I)$. If x and y are adjacent, then $x+y \in M(N,I)$ which is a contradiction.
(2) Let $0 \neq x \in IM+N$ and $y \in M(N,I)-(IM+N)$. Then $0-x-y-0$ is a 3-cycle in $M(\Gamma_{N,I}(M))$. □

Theorem 3.2. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is a submodule of M.*
(1) Assume that G is an induced subgraph of $\overline{M}(\Gamma_{N,I}(M))$ and let m and m' be distinct vertices of G which are connected by a path in G. Then there exists a path in G of length at most 2 between m and m'. In particular, if $\overline{M}(\Gamma_{N,I}(M))$ is connected, then $diam(\overline{M}(\Gamma_{N,I}(M))) \leq 2$.
(2) Let m and m' be distinct elements of $\overline{M}(\Gamma_{N,I}(M))$ that are connected by a path. If m and m' are not adjacent, then $m-(-m)-m'$ and $m-(-m')-m'$ are paths of length 2 between m and m' in $\overline{M}(\Gamma_{N,I}(M))$.

Proof. (1) It suffices to show that if m_1, m_2, m_3 and m_4 are distinct vertices of subgraph G and there is a path $m_1-m_2-m_3-m_4$ from m_1 to m_4, then m_1 and m_4 are adjacent. So $m_1+m_2, m_2+m_3, m_3+m_4 \in M(N,I)$ gives $m_1+m_4=(m_1+m_2)-(m_2+m_3)+(m_3+m_4) \in M(N,I)$; since $M(N,I)$ is a submodule of M. Thus m_1 and m_4 are adjacent. So if $\overline{M}(\Gamma_{N,I}(M))$ is connected, then $diam(\overline{M}(\Gamma_{N,I}(M))) \leq 2$.
(2) Since $m+m' \notin M(N,I)$, then there exists $x \in M-M(N,I)$ such that $m-x-m'$ is a path of length 2 by part (1) above. Thus $m+x, x+m' \in M(N,I)$. Thus $m-m'=(m+x)-(x+m') \in M(N,I)$. Also $m \neq -m$ and $m' \neq -m$; since $m, m+m' \notin M(N,I)$. Thus $m-(-m)-m'$ and $m-(-m')-m'$ are paths of length 2 between m and m' in $\overline{M}(\Gamma_{N,I}(M))$. □

Theorem 3.3. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is a submodule of M. Then the following statements are equivalent:*

(1) $\overline{M}(\Gamma_{N,I}(M))$ is connected.
(2) Either $m+m' \in M(N,I)$ or $m-m' \in M(N,I)$ for all $m,m' \in M-M(N,I)$.
(3) Either $m+m' \in M(N,I)$ or $m+2m' \in M(N,I)$ for all $m,m' \in M-M(N,I)$.
In particular, either $2m \in M(N,I)$ or $3m \in M(N,I)$ (but not both) for all $m \in M-M(N,I)$.

Proof. (1) $\Rightarrow$ (2) Assume that there exist $m,m' \in M-M(N,I)$ such that $m+m' \notin M(N,I)$. If $m=m'$, then $m-m' \in M(N,I)$. Otherwise $m-(-m')-m'$ is a path from m to m' by Theorem 3.2 (2), and hence $m-m' \in M(N,I)$.
(2) $\Rightarrow$ (3) Assume that $m+m' \notin M(N,I)$ for some $m,m' \in M-M(N,I)$. Since $(m+m')-m' = m \notin M(N,I)$, so $m+2m' = (m+m')+m' \in M(N,I)$ by assumption. In particular, if $m \in M-M(N,I)$ then either $2m \in M(N,I)$ or $3m \in M(N,I)$.
(3) $\Rightarrow$ (1) Let $m,m' \in M-M(N,I)$ be distinct elements of M such that $m+m' \notin M(N,I)$. Then $m+2m' \in M(N,I)$ by assumption, so $2m' \notin M(N,I)$ since $M(N,I)$ is a submodule of M. Hence $3m' \in M(N,I)$ by hypothesis. Since $m+m' \notin M(N,I)$ and $3m' \in M(N,I)$, we conclude that $m \neq 2m'$, and so $m-2m'-m'$ is a path from m to m' in $\overline{M}(\Gamma_{N,I}(M))$ as required. $\square$

Theorem 3.4. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is a submodule of M. If $|M(N,I)| = \alpha$ and $|M/M(N,I)| = \beta$ (we allow α and β to be infinite), then:*
(1) If $2 \in S(I)$, then $\overline{M}(\Gamma_{N,I}(M))$ is a disjoint union of $\beta-1$ copies of K^{α}.
(2) If $2 \notin S(I)$, then $\overline{M}(\Gamma_{N,I}(M))$ is a disjoint union of $(\beta-1)/2$ copies of $K^{\alpha,\alpha}$.

Proof. (1) Suppose that $2 \in S(I)$ and $x \in M-M(N,I)$. So $2x \in M(N,I)$ by Proposition 2.4. Since $(x+m_1)+(x+m_2) = 2x+(m_1+m_2) \in M(N,I)$ for all $m_1,m_2 \in M(N,I)$, so each coset $x+M(N,I)$ induces a complete subgraph of $\overline{M}(\Gamma_{N,I}(M))$. Now, we show that distinct cosets form disjoint subgraphs of $\overline{M}(\Gamma_{N,I}(M))$. If $x+m_1$ and $y+m_2$ are adjacent for some $m_1,m_2 \in M-M(N,I)$ and $x,y \in M(N,I)$, then $m_1+m_2 = (x+m_1)+(y+m_2)-(x+y) \in M(N,I)$ and hence $m_1-m_2 = (m_1+m_2)-2m_1 \in M(N,I)$, by Proposition 2.4 and since $M(N,I)$ is a submodule of M. So $m_1+M(N,I) = m_2+M(N,I)$ a contradiction. Thus $\overline{M}(\Gamma_{N,I}(M))$ is a union of $\beta-1$ disjoint (induced) subgraphs $m+M(N,I)$, each of which is a K^{α}, where $\alpha = |M(N,I)| =$

$|m + M(N, I)|$.

(2) Let $m \in M - M(N, I)$ and $2 \notin S(I)$. Then no two distinct elements in $m + M(N, I)$ are adjacent. Otherwise, $(m + x) + (m + y) \in M(N, I)$ for some $x, y \in M(N, I)$. This implies that $2m \in M(N, I)$. So $2 \in S(I)$ by Proposition 2.4, a contradiction. Also, the two cosets $m + M(N, I)$ and $-m + M(N, I)$ are adjacent. So $(m + M(N, I)) \cup (-m + M(N, I))$ is a complete bipartite subgraph of $\overline{M}(\Gamma_{N,I}(M))$. If $x + m_1$ is adjacent to $y + m_2$ for some $x, y \in M - M(N, I)$ and $m_1, m_2 \in M(N, I)$, then $x + y \in M(N, I)$ and so $x + M(N, I) = -y + M(N, I)$. Thus $\overline{M}(\Gamma_{N,I}(M))$ is a union of $(\beta - 1)/2$ disjoint (induced) subgraphs $(m + M(N, I)) \cup (-m + M(N, I))$, each of which is a $K^{\alpha,\alpha}$, where $\alpha = |M(N, I)| = |m + M(N, I)|$. □

Example 3.5. Let $R = Z_{18}$, $M = R$.

(a) If $N = \bar{6}Z_{18}$ and $I = \bar{2}Z_{18}$, then $M(N, I) = IM + N = 2Z_{18}$ and $2 \in S(I) = I$ implies that $\overline{M}(\Gamma_{N,I}(M))$ is the complete graph K^9. $(\alpha = 9, \beta = 2)$

(b) If $N = \bar{6}Z_{18}$ and $I = \bar{3}Z_{18}$, then $M(N, I) = IM + N = \bar{3}Z_{18}$ and $2 \notin S(I) = I$ implies that $\overline{M}(\Gamma_{N,I}(M))$ is the complete bipartite graph $K^{6,6}$. $(\alpha = 6, \beta = 3)$

Theorem 3.6. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N, I)$ is a submodule of M. Then*

(1) $\overline{M}(\Gamma_{N,I}(M))$ is complete if and only if $|M/M(N, I)| = 2$ or $|M| = |M/M(N, I)| = 3$.

(2) $\overline{M}(\Gamma_{N,I}(M))$ is connected if and only if $|M/M(N, I)| = 2$ or $|M/M(N, I)| = 3$.

(3) $\overline{M}(\Gamma_{N,I}(M))$ (and hence $T(\Gamma_{N,I}(M))$ and $M(\Gamma_{N,I}(M))$) are totally disconnected if and only if $M(N, I) = \{0\}$ and $2 \in S(I)$.

Proof. Let $|M(N, I)| = \alpha$ and $|M/M(N, I)| = \beta$.

(1) Let $\overline{M}(\Gamma_{N,I}(M))$ be a complete graph. Then $\overline{M}(\Gamma_{N,I}(M))$ is a single graph K^α or $K^{1,1}$ by Theorem 3.4. If $2 \in S(I)$, then $\beta - 1 = 1$. Thus $\beta = 2$ and hence $|M/M(N, I)| = 2$. If $2 \notin S(I)$, then $\alpha = 1$ and $(\beta - 1)/2 = 1$. Thus $M(N, I) = N + IM = \{0\}$ and $\beta = 3$; hence $|M| = |M/M(N, I)| = 3$. Conversely, first suppose that $M/M(N, I) = \{M(N, I), x + M(N, I)\}$, where $x \notin M(N, I)$. Then $x + M(N, I) = -x + M(N, I)$ gives $2x \in M(N, I)$. Hence there exists $r \in R - I$ such that $(2r)m \in IM + N$. Since $m \notin M(N, I)$, then $2r \in I$ and hence $2 \in S(I)$. So, $\overline{M}(\Gamma_{N,I}(M))$ is a single graph K^α. Assume that $|M| = |M/M(N, I)| = 3$; If $2 \in S(I)$, then $2 \in S(I) = (M(N, I) : M)$ by Proposition 2.4. This implies that $2 \in (0 : M)$ which is a contradiction

since M is a cyclic group of order 3.
(2) Let $\overline{M}(\Gamma_{N,I}(M))$ be a connected graph. Then $\overline{M}(\Gamma_{N,I}(M))$ is a single K^{α} or $K^{\alpha,\alpha}$ by Theorem 3.4. If $2 \in S(I)$, then $\beta - 1 = 1$. So $|M/M(N,I)| = \beta = 2$. If $2 \notin S(I)$, then $(\beta - 1)/2 = 1$ gives $\beta = 3$, so $|M/M(N,I)| = \beta = 3$. Conversely, by part (1) above, we may assume that $|M/M(N,I)| = 3$. If $2 \in S(I)$, then $2 \in (M(N,I) : M)$ by Proposition 2.4. Now, suppose that $M/M(N,I) = \{M(N,I), x + M(N,I), y + M(N,I)\}$, where $x, y \in M - M(N,I)$. Since $M/M(N,I)$ is a cyclic group of order 3, we have $(x+M(N,I))+(x+M(N,I)) = y+M(N,I)$. Thus $2x - y \in M(N,I)$; hence $y \in M(N,I)$ ($2x \in M(N,I)$), a contradiction. So $2 \notin S(I)$ and $\overline{M}(\Gamma_{N,I}(M))$ is a single graph $K^{\alpha,\alpha}$ by Theorem 3.4.
(3) $\overline{M}(\Gamma_{N,I}(M))$ is totally disconnected if and only if it is a disjoint union of K^1's. By Theorem 3.4, $2 \in S(I)$ and $|M(N,I)| = 1$. $\square$

Theorem 3.7. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is a submodule of M. Then $diam(\overline{M}(\Gamma_{N,I}(M))) = 0, 1, 2$ or ∞. In particular, if $\overline{M}(\Gamma_{N,I}(M))$ is connected, then $diam(\overline{M}(\Gamma_{N,I}(M))) \leq 2$.*

Proof. Assume that $\overline{M}(\Gamma_{N,I}(M))$ is a connected subgraph of $T(\Gamma_{N,I}(M))$. Then $\overline{M}(\Gamma_{N,I}(M))$ is a singleton, a complete graph, or a complete bipartite graph by Theorem 3.4. Thus $diam(\overline{M}(\Gamma_{N,I}(M))) \leq 2$. $\square$

Now, we have the following theorem that gives a more explicit description of the diameter of $\overline{M}(\Gamma_{N,I}(M))$ by Theorem 3.4 and Theorem 3.6.

Theorem 3.8. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is a submodule of M.*
(1) $diam(\overline{M}(\Gamma_{N,I}(M))) = 0$ if and only if $M(N,I) = \{0\}$ and $|M| = 2$.
(2) $diam(\overline{M}(\Gamma_{N,I}(M))) = 1$ if and only if either $M(N,I) \neq \{0\}$ and $|M/M(N,I)| = 2$ or $M(N,I) = \{0\}$ and $|M| = 3$.
(3) $diam(\overline{M}(\Gamma_{N,I}(M))) = 2$ if and only if $M(N,I) \neq \{0\}$ and $|M/M(N,I)| = 3$.
(4) Otherwise, $diam(\overline{M}(\Gamma_{N,I}(M))) = \infty$.

Proposition 3.9. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is a submodule of M. Then $gr(\overline{M}(\Gamma_{N,I}(M))) = 3, 4$ or ∞. In particular, $gr(\overline{M}(\Gamma_{N,I}(M))) \leq 4$ if $\overline{M}(\Gamma_{N,I}(M))$ contains a cycle.*

Proof. Let $\overline{M}(\Gamma_{N,I}(M))$ contains a cycle. Since $\overline{M}(\Gamma_{N,I}(M))$ is disjoint union of either complete or complete bipartite graphs by Theorem 3.4, thus it contains either a 3-cycle or 4-cycle. So $gr(\overline{M}(\Gamma_{N,I}(M))) \leq 4$. $\square$

Theorem 3.10. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N, I)$ is a submodule of M.*
(1) (a) $gr(\overline{M}(\Gamma_{N,I}(M))) = 3$ if and only if $2 \in S(I)$ and $|M(N, I)| \geq 3$.
(b) $gr(\overline{M}(\Gamma_{N,I}(M))) = 4$ if and only if $2 \notin S(I)$ and $|M(N, I)| \geq 2$.
(c) Otherwise, $gr(\overline{M}(\Gamma_{N,I}(M))) = \infty$.
(2) (a) $gr(T(\Gamma_{N,I}(M))) = 3$ if and only if $|M(N, I)| \geq 3$.
(b) $gr(T(\Gamma_{N,I}(M))) = 4$ if and only if $2 \notin S(I)$ and $|M(N, I)| = 2$.
(c) Otherwise $gr(T(\Gamma_{N,I}(M))) = \infty$.

Proof. Apply Theorem 3.4, Proposition 3.9 and Theorem 3.1. $\square$

4. The case when $M(N, I)$ is not a submodule of M

The aim of this section is to determine when $T(\Gamma_{N,I}(M))$ is connected and we compute $diam(T(\Gamma_{N,I}(M)))$. We first show that the subgraphs $M(\Gamma_{N,I}(M))$ and $\overline{M}(\Gamma_{N,I}(M))$ are not disjoint, when $M(N, I)$ is not a submodule of M.

Theorem 4.1. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N, I)$ is not a submodule of M. Then*
(1) $M(\Gamma_{N,I}(M))$ is connected with $diam(M(\Gamma_N(M))) = 2$.
(2) Some vertex of $M(\Gamma_{N,I}(M))$ is adjacent to a vertex of $\overline{M}(\Gamma_{N,I}(M))$. In particular, the subgraphs $M(\Gamma_{N,I}(M))$ and $\overline{M}(\Gamma_{N,I}(M))$ are not disjoint.
(3) If $\overline{M}(\Gamma_{N,I}(M))$ is connected, then $T(\Gamma_{N,I}(M))$ is connected.

Proof. (1) Let $x \in M(N, I)$ be a nonzero element. Then x is adjacent to 0. So $x - 0 - x'$ is a path in $M(\Gamma_{N,I}(M))$ between any two nonzero distinct elements $x, x' \in M(N, I)$. Since $M(N, I)$ is not a submodule of M, so $|M(N, I)| \geq 3$. Thus there exist nonadjacent vertices $x, x' \in M(N, I)$. So $diam(M(\Gamma_{N,I}(M))) = 2$.
(2) Since $M(N, I)$ is not a submodule of M, so there exists nonzero elements $x, x' \in M(N, I)$ such that $x + x' \notin M(N, I)$. Then $-x \in M(N, I)$ and $x+x' \in M-M(N, I)$ are adjacent vertices in $T(\Gamma_{N,I}(M))$, since $-x + (x + x') = x' \in M(N, I)$. The "in particular" statement is clear.
(3) Since $M(\Gamma_{N,I}(M))$ and $\overline{M}(\Gamma_{N,I}(M))$ are connected and there is an edge between $M(\Gamma_{N,I}(M))$ and $\overline{M}(\Gamma_{N,I}(M))$, then there is a path from x to y for every element $x, y \in M$. Thus $T(\Gamma_{N,I}(M))$ is connected. $\square$

Theorem 4.2. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is not a submodule of M. Then $T(\Gamma_{N,I}(M))$ is connected if and only if $M =< M(N,I) >$.*

Proof. Suppose that $T(\Gamma_{N,I}(M))$ is connected, and let $m \in M$. Then there is a path $0-m_1-m_2-...-m_n-m$ from 0 to m in $T(\Gamma_{N,I}(M))$. So $m_1, m_1+m_2, ..., m_{n-1}+m_n, m_n+m \in M(N,I)$. Hence $m \in < m_1, m_1+m_2, ..., m_{n-1}+m_n, m_n+m > \subseteq < M(N,I) >$; so $M =< M(N,I) >$. Conversely, suppose that $M =< M(N,I) >$. We first show that there is a path from 0 to x in $T(\Gamma_{N,I}(M))$ for any $0 \neq x \in M$. By hypothesis, $x = m_1+m_2+...+m_n$ for some $m_1, ..., m_n \in M(N,I)$. Let $x_0 = 0$ and $x_k = (-1)^{n+k}(m_1 + ... + m_k)$ for each integer k with $0 \leq k \leq n$. Then $x_k + x_{k+1} = (-1)^{n+k+1}m_{k+1} \in M(N,I)$ for each k with $0 \leq k \leq n-1$, and thus $0 - x_1 - x_2 - ... - x_{n-1} - x_n = x$ is a path from 0 to x in $T(\Gamma_{N,I}(M))$ of length at most n. Now, let $0 \neq x, y \in M$. Then by the preceding argument, there are paths from x to 0 and 0 to y in $T(\Gamma_{N,I}(M))$. Hence there is a path from x to y in $T(\Gamma_{N,I}(M))$; so $T(\Gamma_{N,I}(M))$ is connected. □

Theorem 4.3. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is not a submodule of M. Assume that $n \geq 2$ be the least integer such that $M =< m_1, m_2, ..., m_n >$ for some $m_1, ..., m_n \in M(N,I)$ (that is, $T(\Gamma_{N,I}(M))$ is connected), then:*
(1) If n is an even integer, then diam$(T(\Gamma_{N,I}(M))) \leq n$.
(2) If n is an odd integer, then diam$(T(\Gamma_{N,I}(M))) \leq n+1$.
(3) If M is a cyclic R-module, then diam$(T(\Gamma_{N,I}(M))) \in \{n, n+1\}$.

Proof. Let x and x' be distinct elements of M. By assumption, $x = \sum_{i=1}^{n} r_i m_i$ and $x' = \sum_{i=1}^{n} r'_i m_i$ for some $r_i, r'_i \in R$.
(1) Let n be an even integer. Define $x_0 = x$, $x_n = x'$ and for each integer k with $1 \leq k \leq n-1$, $x_k = (-1)^k(\sum_{i=k+1}^{n} r_i m_i + \sum_{i=1}^{k} r'_i m_i)$. So $x_k + x_{k+1} = (-1)^k m_{k+1}(r_{k+1} - r'_{k+1}) \in M(N,I)$ for each integer k with $0 \leq k \leq n-1$. Then $x - x_1 - ... - x_{n-1} - x'$ is a path from x to x' in $T(\Gamma_{N,I}(M))$ with length at most n.
(2) Let n be an odd integer. If $x' = -x'$, then we have a path similar to the case (1) above. So we may assume that $x' \neq -x'$. If $x = -x'$, then the edge $x - x'$ exists, otherwise we define x_k similar to case (1) above for each integer k with $0 \leq k \leq n-1$, $x_n = -x'$ and $x_{n+1} = x'$. So $x_k + x_{k+1} = (-1)^k m_{k+1}(r_{k+1} - r'_{k+1}) \in M(N,I)$ for each integer k with $0 \leq k \leq n-1$ and there is a path $x - x_1 - ... - x_{n+1} (= x')$ from x to x' in $T(\Gamma_{N,I}(M))$ with length at most $n+1$.
(3) Suppose that M is a cyclic module with generator m. Let $0 - y_1 -$

$\ldots - y_{k-1} - m$ be a path from 0 to m in $T(\Gamma_{N,I}(M))$ of length k. Thus $y_1, y_1 + y_2, \ldots, y_{k-1} + m \in M(N, I)$, hence $m \in < y_1, y_1 + y_2, \ldots, y_{k-1} + m > \subseteq < M(N, I) >$. Then $k \geq n$ and the proof is complete. □

Theorem 4.4. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N, I)$ is not a submodule of M. Assume that $n \geq 2$ be the least integer such that $M =< m_1, m_2, \ldots, m_n >$ for some $m_1, \ldots, m_n \in M(N, I)$ and M be a cyclic R-module with generator m. Then*
(1) $diam(T(\Gamma_{N,I}(M))) \in \{d(0, m), d(0, m) - 1\}$.
(2) If $diam(T(\Gamma_{N,I}(M))) = n$, then $diam(\overline{M}(\Gamma_{N,I}(M))) \geq n - 2$.
(3) If $diam(T(\Gamma_{N,I}(M))) = n + 1$, then $diam(\overline{M}(\Gamma_{N,I}(M))) \geq n - 1$.

Proof. (1) This follows from Theorem 4.3.
(2) Suppose that $diam(T(\Gamma_{N,I}(M))) = n$. Since $diam(T(\Gamma_{N,I}(M))) \in \{d(0, m), d(0, m) - 1\}$ by part (1) above, so let $0 - x_1 - \ldots - x_{n-1} - m$ be a shortest path from 0 to m in $T(\Gamma_{N,I}(M))$. Then $x_1 \in M(N, I)$. If $x_i \in M(N, I)$ for some $2 \leq i \leq n-1$, then $0 - x_i - x_{i+1} - \ldots - x_{n-1} - m$ is a path from 0 to m whose length is less than n, a contradiction. So $x_i \in M - M(N, I)$ for each $2 \leq i \leq n - 1$. Hence $x_2 - \ldots - x_{n-1} - m$ is a shortest path from x_2 to m in $\overline{M}(\Gamma_{N,I}(M))$ of length $n - 2$. So $diam(\overline{M}(\Gamma_{N,I}(M))) \geq n - 2$.
(3) The proof is similar to part (2) above. □

Let N be a proper submodule of an R-module M and let I be a proper ideal of R. Recall that two submodules L and K of M are called co-maximal if $M = L + K$. Note that if proper subset $M(N, I)$ of M contains two co-maximal submodules of M, then $M(N, I)$ is not a submodule of M.

Theorem 4.5. *Let M be a finitely generated R-module and $n \geq 2$ be the least integer that $M =< m_1, m_2, \ldots, m_n >$ for some $m_1, \ldots, m_n \in M$. Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N, I)$ contains two co-maximal submodules of M. Then $T(\Gamma_{N,I}(M))$ is connected with $diam(T(\Gamma_{N,I}(M))) \leq 2n$.*

Proof. Let $L, K \subseteq M(N, I)$ be co-maximal submodules of M. Then $M = L + K$; so $m_i = x_i + y_i$ for some $x_i \in L$ and $y_i \in K$ for every $i = 1, 2, \ldots, n$. Hence $M =< x_1, \ldots, x_n, y_1, \ldots, y_n >$. Thus $T(\Gamma_{N,I}(M))$ is connected with $diam(T(\Gamma_{N,I}(M))) \leq 2n$ by Theorem 4.2 and Theorem 4.3. □

Theorem 4.6. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N, I)$ is not a submodule of M.*

(1) If $IM + N \neq \{0\}$, then $gr(M(\Gamma_{N,I}(M))) = 3$. Otherwise $gr(M(\Gamma_{N,I}(M))) \in \{3, \infty\}$.
(2) $gr(T(\Gamma_{N,I}(M))) = 3$ if and only if $gr(M(\Gamma_{N,I}(M))) = 3$.
(3) The (induced) subgraph of $M(\Gamma_{N,I}(M))$ with vertices in $N+IM$ is complete, hence $gr(M(\Gamma_{N,I}(M))) = 3$ when $|N + IM| \geq 3$.
(4) If $gr(T(\Gamma_{N,I}(M))) = 4$, then $gr(M(\Gamma_{N,I}(M))) = \infty$.
(5) If $IM + N \neq 0$ and $2 \in I$, then $gr(\overline{M}(\Gamma_{N,I}(M))) \in \{3, \infty\}$.
(6) If $2 \notin I$, then $gr(\overline{M}(\Gamma_{N,I}(M))) \in \{3, 4, \infty\}$.

Proof. (1) Suppose that $0 \neq x \in IM+N$ and $y \in M(N, I)-(IM+N)$. So $ry \in IM + N$ for some $r \in R - I$, thus $r(x + y) \in IM + N$. Hence $x+y \in M(N, I)$ and then $0-x-y-0$ is a 3-cycle in $M(\Gamma_{N,I}(M))$. Now, assume that $IM + N = \{0\}$, then $N = IM = \{0\}$. If $x + y \in M(0, I)$ for some nonzero distinct elements $x, y \in M(0, I)$, then $0 - x - y - 0$ is a 3-cycle in $M(\Gamma_{0,I}(M))$, so $gr(M(\Gamma_{0,I}(M))) = 3$. Otherwise, $x + y \in M - M(0, I)$ for all distinct elements $x, y \in M(0, I)$. Therefore, each nonzero element $x \in M(0, I)$ is adjacent to 0, and no two nonzero distinct vertices $x, y \in M(0, I)$ are adjacent. Thus $M(\Gamma_{0,I}(M))$ is a star graph with center 0 and $gr(M(\Gamma_{N,I}(M))) = \infty$.
(2) We need only show that $gr(M(\Gamma_{N,I}(M))) = 3$ when $gr(T(\Gamma_{N,I}(M)))$ $= 3$. First suppose that $2x \neq 0$ for some nonzero element $x \in M(N, I)$, then $0 - x - (-x) - 0$ is a 3-cycle in $M(N, I)$. So we may assume that $2x = 0$ for all $x \in M(N, I)$. There are elements $m, m' \in M(N, I)$ such that $m + m' \notin M(N, I)$, since $M(N, I)$ is not a submodule of M. So $2(m + m') = 0$, this implies that $2 \in I$. Let $m - m_1 - m_2 - m$ be a 3-cycle in $T(\Gamma_{N,I}(M))$. Then $m+m_1, m_1+m_2, m_2+m \in M(N, I)$. First suppose that $m + m_1 \neq 0$ and $m + m_2 \neq 0$. Since $m_1 + m_2 \in M(N, I)$; so there exists $r \in R - I$ such that $r(m_1 + m_2) \in IM + N$. Thus $r(m_1+m_2+2m) \in IM+N$ since $2 \in I$. Hence $0-(m+m_1)-(m+m_2)-0$ is a 3-cycle in $M(\Gamma_{N,I}(M))$.
Now suppose that $m + m_1 \neq 0$ and $m + m_2 = 0$, then $m_2 = -m$ and $2m \neq 0$ since m and m_2 are distinct elements. Then $0 - (m_1 + m) - (m_1 - m) - 0$ is a 3-cycle in $M(\Gamma_{N,I}(M))$ since $2 \in I$.
(3) It is clear, since $N + IM \subseteq M(N, I)$ is a submodule of M.
(4) This follows by parts (1) and (2) above.
(5) Let $\overline{M}(\Gamma_{N,I}(M))$ contains a cycle and let $0 \neq x \in IM + N$. Then there is a path $m_1 - m_2 - m_3$ in $\overline{M}(\Gamma_{N,I}(M))$. If m_1 and m_3 are adjacent vertices in $\overline{M}(\Gamma_{N,I}(M))$, then the proof is complete. So we may assume that $m_1+m_3 \notin M(N, I)$. If $m_2-m_1, m_3-m_2 \in IM+N$, then $m_3 - m_1 \in IM + N$. Since $2m_1 \in IM + N$, thus $m_1 + m_3 \in IM + N$, which is a contradiction. So, without loss of generality we may assume that $m_2 - m_1 \notin IM + N$. Hence $(x + m_1) - m_1 - m_2 - (x + m_1)$ is a

3-cycle in $\overline{M}(\Gamma_{N,I}(M))$.
(6) Assume that $\overline{M}(\Gamma_{N,I}(M))$ contains a cycle and let $0 \neq x \in IM+N$. Then there is a path $m_1 - m_2 - m_3$ in $\overline{M}(\Gamma_{N,I}(M))$. Let $m_1 + m_3 \notin M(N,I)$. Since $m_1 \neq m_3$, then either $m_1 + m_2 \neq 0$ or $m_2 + m_3 \neq 0$. We may assume that $m_1 + m_2 \neq 0$. Since $2 \notin I$, if $2m_i = 0$, then $m_i \in M(N,I)$ for some $i = 1,2,3$ which is a contradiction. Thus $m_1 - m_2 - (-m_2) - (-m_1) - m_1$ is a 4-cycle in $M(\Gamma_{N,I}(M))$. □

Recall that if $gr(T(\Gamma(M))) = 4$, then $gr(Tor(\Gamma(M))) = \infty$ if $T(M)$ is not a submodule of M [12, 3.5]. Also, if $gr(T(\Gamma_N(M))) = 4$, then $gr(M(\Gamma_N(M))) = \infty$, when $M(N)$ is not a submodule of M [2, 4.5]. Now, we provide a proof for the converse of [12, 3.5 (3)] and [2, 4.5 (4)], when R is not an integral domain and $M \neq T(M)$.

Proposition 4.7. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R such that $M(N,I)$ is not a submodule of M and let $M \neq T(M)$. If R is not an integral domain and $gr(M(\Gamma_{N,I}(M))) = \infty$, then $gr(T(\Gamma_{N,I}(M))) = 4$. Moreover, if $gr(Tor(\Gamma(M))) = \infty$, then $|M(N,I)| = 3$.*

Proof. Suppose that $gr(M(\Gamma_{N,I}(M))) = \infty$. Since $M(N,I)$ is not a submodule of M, so $M(N,I) \neq M$. Then $M(N,I) = \bigcup_{\alpha\in\Lambda} L_\alpha$, where each L_α is a prime submodule of M and $|\Lambda| \geq 2$. If $gr(M(\Gamma_{N,I}(M))) = \infty$, then $x + y \in M - M(N,I)$ for all nonzero distinct elements $x, y \in M(N,I)$. So $|L_\alpha| = 2$ for every $\alpha \in \Lambda$. Hence the intersection of any two distinct L_α's is $\{0\}$ and so $|\Lambda| = 2$ by Proposition 2.6. So $M(N,I) = L_1 \cup L_2$ for prime submodules L_1 and L_2 of M with $L_1 \cap L_2 = 0$ and $|L_1| = |L_2| = 2$. So we may assume that $L_1 = \{0, x\}$ and $L_2 = \{0, y\}$ where $2x = 2y = 0$. So $|M(N,I)| = 3$ and $x + y \notin M(N,I)$. Thus $0 - x - (x+y) - y - 0$ is a 4-cycle in $T(\Gamma_{N,I}(M))$. Then $gr(T(\Gamma_{N,I}(M))) = 4$ by Theorem **??**(2).
The "moreover" statement follows directly from the above arguments. □

Example 4.8. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$, $N = 4Z \times 7Z$ and $I = 28\mathbb{Z}$. So $M(N,I)$ is not a submodule of M by Example 2.2. Also, $|N + IM| \geq 3$, then $gr(T(\Gamma_{N,I}(M))) = gr(M(\Gamma_{N,I}(M))) = 3$ by Theorem **??**. Moreover, $(1,1) - (3,6) - (5,6) - (1,1)$ is a 3-cycle in $\overline{M}(\Gamma_{N,I}(M)))$.

Proposition 4.9. *Let N be a proper submodule of an R-module M and let I be a proper ideal of R with $|M(N,I)| = \alpha$. Let x be a vertex of $T(\Gamma_{N,I}(M))$. Then the degree of x is either α or $\alpha - 1$. In particular, if $2 \in S(I)$, then the graph $T(\Gamma_{N,I}(M))$ is a $(\alpha - 1)$-regular graph.*

Proof. If x adjacent to y, then $x+y=z\in M(N,I)$ and hence $y=z-x$ for some $z\in M(N,I)$. Now, we have two cases:
Case 1. If $2x\in M(N,I)$, then x is adjacent to $z-x$ for any $z\in M(N,I)\setminus\{2x\}$. Thus the degree of x is $\alpha-1$. In particular, if $2\in S(I)$, then $T(\Gamma_{N,I}(M))$ is a $(\alpha-1)$-regular graph by Proposition 2.4.
Case 2. Suppose that $2x\notin M(N,I)$. Then x is adjacent to $z-x$ for any $z\in M(N,I)$. Thus the degree of x is α. □

Proposition 4.10. *Let M be an R-module M and let I be a proper ideal of R such that $M(N,I)$ is not a submodule of M. If $T(\Gamma_I(R))$ is connected, then $T(\Gamma_{N,I}(M))$ is connected for every proper submodule N of M. Moreover if $diam(T(\Gamma_I(R)))=n$, then $diam(T(\Gamma_{N,I}(M)))\leq 2n+1$.*

Proof. Let $T(\Gamma_I(R))$ be connected and m,n be nonzero elements of M. Then there exists a path $s-a_1-a_2-...-a_{k-1}-1$ from s to 1 of length k from s to 1 for some nonzero element $s\in S(I)$. So $s, s+a_1, ..., a_{k-1}+1\in S(I)$. Thus $m-a_{k-1}m-...-a_1m-sm-sn-a_1n-...-a_{k-1}n-n$ is a path from m to n of length at most $2k+1$ by Proposition 2.4. The "moreover" statement follows directly from the above arguments. □

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