Journal of Algebra and Related Topics
Vol. 2, No 1, (2014), pp 55-63

CUBIC SYMMETRIC GRAPHS OF ORDERS $36p$ AND $36p^2$

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Abstract. A graph is *symmetric*, if its automorphism group is transitive on the set of its arcs. In this paper, we classify all the connected cubic symmetric graphs of order $36p$ and $36p^2$, for each prime p, of which the proof depends on the classification of finite simple groups.

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For the group-theoretic concepts and notations not defined here we refer to [16].

For a graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, the edge set, the arc set and the full automorphism group of X, respectively. Denote by $\mathbb{Z}_n$ the cyclic group of order n. For two groups M and N, $N < M$, means that N is a proper subgroup of M.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on the set of its s-arcs. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph X is said to be *s-regular*, if $\mathrm{Aut}(X)$ acts regularly on the set of its s-arcs in X. Tutte [17, 18] showed that every finite connected cubic symmetric graph is s-regular for $1 \leq s \leq 5$.

MSC(2010): Primary: 05C25; ; Secondary: 20B25.
Keywords: Symmetric graphs, s-regular graphs.
Received: 6 April 2014, Accepted: 31 May 2014.
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It follows that a connected cubic symmetric graph of order n is s-regular if and only if the order of its automorphism group is $n \cdot 3 \cdot 2^{s-1}$.

There has been a lot of interest in classification of symmetric graphs of given orders. For a prime p, Chao [4] classified symmetric graphs of order p and Cheng and Oxley [5] classified symmetric graphs of order $2p$. The classification of symmetric graphs of order $3p$ was completed by Wang and Xu [19], and latter the classification of symmetric graphs of order a product of two distinct primes was given by Praeger et al. in [14]. On the other hand, following the pioneering article of Tutte [17], cubic symmetric graphs have been extensively studied over decades by many authors and a lot of constructions and classifications of various subfamilies of cubic symmetric graphs were given. For example, the cubic symmetric graphs of orders $12p^i$ for each $i = 1, 2$ and $16p^2$, are classified in [1, 3]. Also, we note that in [2] are classified semisymmetric graphs of orders $36p$, $36p^2$, for each prime. The classification of cubic symmetric graphs of order $2pq$ was given in [20] where p and q are distinct odd primes.

In this paper, we obtain a classification of cubic symmetric graphs of orders $36p$ and $36p^2$, where p is prime. The following is the main result of this paper.

Theorem 1.1. *Let p be a prime. Let X be a cubic symmetric graph.*

(1) *If X has order $36p$, then X isomorphic to the 2-regular graphs F_{72A}, F_{108A} or the 5-regular graph F_{468A}.*
(2) *If X has order $36p^2$, then X isomorphic to the 1-regular graph F_{144A} or the 2-regular graph F_{144B}.*

2. Preliminaries

Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. For u, $v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X. The *quotient graph* X/N or X_N induced by N is defined as the graph such that the set Σ of N-orbits in $V(X)$ is the vertex set of X/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$.

A graph $\widetilde{X}$ is called a *covering* of a graph X with a projection $\wp : \widetilde{X} \to X$ if there is a surjection $\wp : V(\widetilde{X}) \to V(X)$ such that $\wp|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \wp^{-1}(v)$. The graph X is often called the *base graph*. A covering graph $\widetilde{X}$ of X with a projection $\wp$ is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition $\wp h$ of $\wp$ and h.

Proposition 2.1. [13, Theorem 9] *Let X be a connected symmetric graph of prime valency and let G be an s-regular subgroup of $Aut(X)$ for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-regular subgroup of $Aut(X_N)$, where X_N is the quotient graph of X corresponding to the orbits of N. Furthermore, X is a N-regular covering of X_N.*

Proposition 2.2. [8, Propositions 2.5] *Let X be a connected cubic symmetric graph and G be an s-regular subgroup of $Aut(X)$. Then, the stabilizer G_v of $v \in V(X)$ is isomorphic to $\mathbb{Z}_3$, S_3, $S_3 \times \mathbb{Z}_2, S_4$, or $S_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5, respectively.*

Now, we have the following obvious fact in the group theory.

Proposition 2.3. [16, pp.236] *Let G be a finite group and let p be a prime. If G has an abelian Sylow p-subgroup, then p does not divide $|G' \cap Z(G)|$.*

Proposition 2.4. [15, Theorem 8.5.3]

(1) *Let p and q be primes and let α and β be non-negative integers. Then every group of order $p^\alpha q^\beta$ is solvable.*
(2) [9, Feit-Thompson Theorem] *Every finite group of odd order is solvable.*

The following result can be obtained from [12, pp. 12-14] and [7].

Proposition 2.5. *Let p be a prime and G be a non-abelian simple group whose order divides $2^{r+1} \cdot 3^3 \cdot p^2$ for some non-negative integer $r \leqslant 5$. Then, G is isomorphic to $A_5, A_6, PSL(2,7), PSL(2,8), PSL(2,17), PSL(3,3)$ or $PSU(3,3)$ of orders $2^2 \cdot 3 \cdot 5, 2^3 \cdot 3^2 \cdot 5, 2^3 \cdot 3 \cdot 7, 2^3 \cdot 3^2 \cdot 7, 2^4 \cdot 3^2 \cdot 17, 2^4 \cdot 3^3 \cdot 13, 2^5 \cdot 3^3 \cdot 7$, respectively.*

3. Main results

In this section, we classify cubic symmetric graphs of orders $36p$ and $36p^2$, for each prime p.

At the first, we shall classify the cubic symmetric graphs of order $36p$.

Lemma 3.1. *Let p be a prime and let X be a connected cubic symmetric graph of order $36p$. Then X is isomorphic to the 2-regular graphs F_{72A}, F_{108A} or the 5-regular graph F_{468A}.*

Proof. Let X be a cubic symmetric graph of order $36p$, where p is a prime. If $p < 59$, then by [6], X is isomorphic to the cubic 2-regular graphs F_{72A}, F_{108A} or the 5-regular graph F_{468A}. So, we can assume that $p \geqslant 59$. Now, we intend to show that there is no cubic symmetric graphs of order $36p$ for $p \geqslant 59$. By way of contradiction, let X be a cubic symmetric graph of order $36p$. Throughout the proof, we let $A := Aut(X)$ and let P be a Sylow p-subgroup of A and $N_A(P)$ be the normalizer of P in A. Then by Sylow Theorem the number of Sylow p-subgroups of A is $np+1 = |A : N_A(P)|$ for some non-negative integer n. By Tutte [18], X is at most 5-regular, and hence $|A|$ is a divisor of $3{\cdot}2^4{\cdot}|V(X)| = 3{\cdot}2^4{\cdot}36p$. So $np+1 \mid 48{\cdot}36$. Furthermore, since $np+1 \geqslant 60$, we have $(n, p) = (1, 71), (1, 107), (11, 157), (1, 191), (1, 143), (1, 863)$. This implies that if $p > 863$, then P is normal in A. It is easy to check that P has more than two orbits on $V(X)$ and then by Proposition 2.1, the quotient graph X_P of X corresponding to the orbits of P is a cubic symmetric graph of order 36, which is impossible by [6]. Therefore, we can assume that $p \leqslant 863$.

Table I. Cubic symmetric graphs of order $36p$

Graph	Order	s-regular	Girth	Diameter	Bipartite
F_{72}	$36 \cdot 2 = 72$	2	6	8	Yes
F_{108}	$36 \cdot 3 = 108$	2	9	7	No
F_{468}	$36 \cdot 13 = 468$	5	12	13	Yes

Let N be a minimal normal subgroup of A. Thus $N \cong T \times T \times \ldots \times T = T^k$, where T is a non-abelian simple group. By Proposition 2.4, $|T|$ has at least three prime factors and even order. Since $|A|$ is a divisor of $2^6 \cdot 3^3 \cdot p$, $t = 1$ and N is a non-abelian simple group. Thus N has order $2^{s_1} \cdot 3^{s_2} \cdot p$, where $1 \leqslant s_1 \leqslant 6$, $1 \leqslant s_2 \leqslant 3$ and $59 \leqslant p \leqslant 863$. Nevertheless, by Proposition 2.5, there is no simple group with such orders. Therefore, N is solvable and so elementary abelian. It follows that N has more than two orbits on $V(X)$ and hence it is semiregular on $V(X)$ by Proposition 2.1. Thus $|N| \mid 36p$. Let $Q := O_p(A)$ be the maximal normal p-subgroup in A.

Suppose first that $Q = 1$. It implies four cases: $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_3, \mathbb{Z}_3^2$. We get a contradiction in each case as follows.

Case (I). $N \cong \mathbb{Z}_2$.

By Proposition 2.1, the quotient graph X_N is a cubic symmetric graph of order $18p$. Let M/N be a minimal normal subgroup of A/N. By a

similar argument as above M/N is solvable and so elementary abelian. Again, by Proposition 2.1, M/N is semiregular on the $V(X_N)$, and so $|M/N| \mid 18p$.
If $|M/N| = p$, then $|M| = 2p$. It is seen easily that the Sylow p-subgroup of M is characteristic and consequently normal in A. Then A has a normal subgroup of order p, a contradiction to $Q = 1$. Now, If $|M/N| = 2$, then $|M| = 4$. It follows that the quotient graph X_M has odd order and valency 3, which is impossible. Now, let $|M/N| = 3$, then the quotient graph X_M is a cubic symmetric graph of order $6p$. Let K/M be a minimal normal subgroup of A/M. Clearly, K/M is solvable and so elementary abelian. Again, by Proposition 2.1, K/M is semiregular on $V(X_N)$ and so $|K/M| \mid 6p$. If $|K/M| = 2$, then the quotient graph X_K is a cubic symmetric graph of odd order, a contradiction. Also, if $|K/M| = p$, $|K| = 6p$. It follows that the Sylow p-subgroup of K is normal in A, a contradiction. Now suppose that $|K/M| = 3$. Hence $|K| = 18$. It follows that X_K is a cubic symmetric graph of order $2p$. Let H/K be a minimal normal subgroup of A/K. Again, H/K is solvable and so elementary abelian. It implies that $|H/K| \mid 2p$. If $|H/K| = 2$, then $|H| = 18p$ and the quotient graph X_H is a cubic symmetric graph of odd order, a contradiction. So $|H/K| = p$. It implies that $|H| = 18p$. Since $p \geqslant 59$, the Sylow p-subgroup of H is characteristic and consequently normal in A, a contradiction. Therefore, $|M/N| = 9$ and $|M| = 18$. We get a contradiction as $|K| = 18$.

Case (II). $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
By Proposition 2.1, the quotient graph X_N is a cubic symmetric graph of order 36. But, by [6], there is no cubic symmetric graph of this order.

Case (III). $N \cong \mathbb{Z}_3$.
Thus, the quotient graph X_N is a cubic symmetric graph of order $12p$. Let L/N be a minimal normal subgroup of A/N. It is easy to see that L/N is solvable and so elementary abelian. Thus, by Proposition 2.1, it is semiregular on $V(X_N)$, which force $|T/N| \mid 12p$. Since $Q = 1$, $|T/N| = 2, 2^2$ or 3. One can see that $|L/N| = 2, 2^2$ is impossible as in Case (I). Hence, $|L/N| = 3$ and so $|L| = 9$. Then, the quotient graph X_L is a cubic symmetric graph of order $4p$. Again, we consider a minimal normal subgroup R/L of A/L and by a similar argument as Case (I), One can show that $|R| = 18, 36, 9p$. If $|R| = 18$, then this leads to a contradiction as in Case (I). Also, If $|R| = 36$, then the quotient graph X_R has an odd prime, a contradiction. Therefore, $|R| = 9p$. Since $p \geqslant 59$, the Sylow p-subgroup of R is normal in A, a contradiction.

Case (IV). $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.
So, the quotient graph X_N is a cubic symmetric graph of order $4p$. This case was rejected as Case (III).

□

Now, we consider cubic symmetric graphs of order $36p^2$, where p is prime. We have the following lemma.

Lemma 3.2. *Let p be a prime and let X be a cubic symmetric graph of order $36p^2$. Then X is isomorphic to the 1-regular graph F_{144A} or the 2-regular graph F_{144B}.*

Proof. Let X be a cubic symmetric graph of order $36p^2$. If $p < 11$, then by [6], X is isomorphic to the 1-regular graph F_{144A} or the 2-regular graph F_{144B}. Thus, in what follows, we assume that $p \geqslant 11$. To prove the lemma, it suffices to show that there is no cubic symmetric graph of order $36p^2$, for $p \geqslant 11$. Suppose to the contrary, that X is the such graph. Set $A := Aut(X)$. by Proposition 2.2, $|A| = 2^{s+1} \cdot 3^3 \cdot p^2$, where $1 \leqslant s \leqslant 5$. Let $Q := O_p(A)$ be the maximal normal p-subgroup of A.

Table II. Cubic symmetric graphs of order $36p^2$

Graph	Order	s-regular	Girth	Diameter	Bipartite
F_{144A}	$36 \cdot 2^2 = 144$	1	8	7	Yes
F_{144B}	$36 \cdot 2^2 = 144$	2	10	8	Yes

Let N be a minimal normal p-subgroup of A. Thus, $N \cong T \times T \times \ldots \times T = T^k$, where T is a non-abelian simple group. Let N be unsolvable. By Proposition 2.5, T is isomorphic to $PSL(2, 17)$ or $PSL(3, 3)$ with orders $2^4 \cdot 3^2 \cdot 17$ and $2^3 \cdot 3^3 \cdot 13$, respectively. Since 3^4 does not divide $|A|$. One has $k = 1$ and hence $p^2 \nmid |N|$. It follows that N has more than two orbits on $V(X)$. By Proposition 2.1, N is semiregular, which implies that $|N| \mid 36p^2$, a contradiction. Thus N is solvable and so elementary abelian. Again, By Proposition 2.1, N is semiregular on $V(X)$. Moreover, the quotient graph X_N of X corresponding to the orbits of N is a cubic symmetric graph with A/N as an arc-transitive subgroup of $Aut(X_N)$.

Suppose first that $Q = 1$. The semiregularity of N implies that $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_3$ or $\mathbb{Z}_3^2$. If $N \cong \mathbb{Z}_2^2$, then X_N has odd order and valency 3, a contradiction. Let $N \cong \mathbb{Z}_2$. Then the quotient graph X_N is a cubic symmetric graph of order $18p^2$. Let M/N be a minimal normal subgroup of A/N. By a similar argument as above, one can show that M/N

is solvable and so elementary abelian. Again, by Proposition 2.1, M/N is semiregular on $V(X_N)$, which implies that $M/N \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_3^2, \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. For the former by Proposition 2.1, the quotient graph X_M is a cubic graph with an odd order $9p^2$, a contradiction. Let $M/N \cong \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. It follows that M has a normal subgroup of order p or p^2, which is characteristic in M and hence is normal in A. This contradicts our assumption that $Q = 1$. Thus $M/N \cong \mathbb{Z}_3$ or $\mathbb{Z}_3^2$. It follows that the quotient graph X_M is a cubic symmetric graph of order $2p^2$ or $6p^2$. It then follows from [11, Theorem 3.2] and [10, Theorem 5.3] and its proof that, the Sylow p-subgroup $Aut(X_M)$ is normal, and also the Sylow p-subgroup A/M is normal, because $A/M \leqslant Aut(X_M)$, say by R/M. So $|R| = 6p^2$ or $18p^2$. It is easy to show that R has a characteristic subgroup of order p^2, which is normal in A, because R is normal in A, contracting to $Q = 1$. If $N \cong \mathbb{Z}_3$ or $\mathbb{Z}_3^2$, then by Proposition 2.1, the quotient graph X_N is a cubic symmetric graph of order $12p^2$ or $4p^2$. Let T/N be a minimal normal subgroup of A/N. By a similar argument in the previous case, we can get a contradiction.

Suppose now that $Q \cong \mathbb{Z}_p$. Let $C := C_A(Q)$ be the centralizer of Q in A. Also $Q \leqslant C$. because Q is abelian. Thus $p \mid |C|$. By Proposition 2.3, $p \nmid |\acute{C} \cap Z(C)|$, which implies that $\acute{C} \cap Q = 1$, where $\acute{C}$ is the derived subgroup of C. This force $p^2 \nmid |\acute{C}|$. It follows that $\acute{C}$ has more than two orbits on $V(X)$. As $\acute{C}$ is normal in A, by Proposition 2.1, it is semiregular on $V(X)$. Moreover, the quotient graph $X_{\acute{C}}$ is a cubic graph and consequently has even order. Hence $4 \nmid |\acute{C}|$ and since $p^2 \nmid |\acute{C}|$, the semiregularity of $\acute{C}$ implies $|\acute{C}| \mid 18p$. Since the Sylow p-subgroups of A are abelian, one has the property that $p^2 \mid |C|$ and so $p \mid |C/\acute{C}|$. Now let $K/\acute{C}$ be a Sylow p-subgroup of the abelian group $C/\acute{C}$. As $K/\acute{C}$ is characteristic in $C/\acute{C}$ and so is normal in $A/\acute{C}$, we have that K is normal in A. Clearly, $|K| = tp^2$, where $t \mid 18$ and since $p \geqslant 11$, K has a normal subgroup of order p^2, which is characteristic in K and hence is normal in A, contradicting to $Q \cong \mathbb{Z}_p$.

Therefore, $|Q| = p^2$. By Proposition 2.1, X_Q is a cubic symmetric graph of order 36, a contradiction. The result now follows.

□

Proof of the Theorem 1.1. We now complete the proof of the main theorem. Let X is a connected cubic symmetric graph of order $36p$ or $36p^2$, where p is a prime. Therefore, by Lemmas 3.1 and 3.2, the proof is completed.

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