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A SCHEME OVER QUASI-PRIME SPECTRUM OF MODULES

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Abstract. The notions of quasi-prime submodules and developed Zariski topology was introduced by the present authors in [1]. In this paper we use these notions to define a scheme. For an R-module M, let $X := \{Q \in q\text{Spec}(M) \mid (Q :_R M) \in \text{Spec}(R)\}$. It is proved that $(X, \mathcal{O}_X)$ is a locally ringed space. We study the morphism of locally ringed spaces induced by R-homomorphism $M \to N$, and also by ring homomorphism $R \to S$. Among other results, we show that $(X, \mathcal{O}_X)$ is a scheme by putting some suitable conditions on M.

1. INTRODUCTION

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and annihilator of M, denoted by $\text{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. If there is no ambiguity, we write $(N : M)$ (resp. $\text{Ann}(M)$) instead of $(N :_R M)$ (resp. $\text{Ann}_R(M)$). An R-module M is called *faithful* if $\text{Ann}(M) = (0)$. A proper ideal I of a ring R is said to be *quasi-prime* if for each pair of ideals A and B of R, $A \cap B \subseteq I$ yields either $A \subseteq I$ or $B \subseteq I$ (see [2], [3] and [5]). It is easy to see that every prime ideal is a quasi-prime ideal. A proper submodule N of an R-module M is called *quasi-prime* if $(N :_R M)$ is a quasi-prime ideal of R (see [1]). We define the *quasi-prime spectrum* of an R-module M to be the set of all quasi-prime submodules of M

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and denote it by $q\text{Spec}^R(M)$. If there is no ambiguity, we write only $q\text{Spec}(M)$ instead of $q\text{Spec}^R(M)$. For any $I \in q\text{Spec}(R)$, the collection of all quasi-prime submodules N of M with $(N : M) = I$ is designated by $q\text{Spec}_I(M)$. The relationship between the algebraic properties of M and the topological properties of $q\text{Spec}(M)$ is investigated in [1]. Modules whose developed Zariski topology is respectively T_0, irreducible or Noetherian have been studied by authors in [1], and several characterizations of such modules were given.

In this paper, we use the notion of quasi-prime spectrum of modules to define a scheme.

First of all, we state some preliminaries that are needed for next section. Let us M be an R-module. By $N \leq M$ we mean that N is a submodule of M. For a submodule N of M we define

$$D^M(N) \;=\; \{L \in q\text{Spec}(M) \mid (L : M) \supseteq (N : M)\}.$$

If there is no ambiguity we write $D(N)$ instead of $D^M(N)$.

Let M be an R-module. For submodules N, L and a family $\{N_i\}_{i\in I}$ of submodules of M one has

(1) $D(\mathbf{0}) = q\text{Spec}(M)$ and $D(M) = \emptyset$,
(2) $\bigcap_{i\in I} D(N_i) = D(\sum_{i\in I}(N_i : M)M)$,
(3) $D(N) \cup D(L) = D(N \cap L)$.

Now, we put

$$\zeta(M) = \{\, D(N) \mid N \leq M \,\}$$

From (1), (2) and (3) above, it is evident that for any module M there exists a topology, τ say, on $q\text{Spec}(M)$ having $\zeta(M)$ as the family of all closed sets. The topology τ is called the *developed Zariski topology* on $q\text{Spec}(M)$ (see [1]).

When $q\text{Spec}(M) \neq \emptyset$, the map $\psi : q\text{Spec}(M) \to q\text{Spec}(R/\text{Ann}(M))$ defined by $\psi(L) = (L : M)/\text{Ann}(M)$ for every $L \in q\text{Spec}(M)$, will be called the *natural map* of $q\text{Spec}(M)$. An R-module M is called *quasi-primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and $q\text{Spec}(M)$ has a surjective natural map (See [1]). For an example of quasi-primeful module see [1, Example 2.13]. Recall that a module M is said to be a *Laskerian* module, if every proper submodule of M has a primary decomposition. It is well-known that every Noetherian module is Laskerian.

For any element x of an R-module M, we define

$$c(x) := \bigcap\{A | A \text{ is an ideal of } R \text{ and } x \in AM\}.$$

An R-module M is called a *content* R-module if, for every $x \in M$, $x \in c(x)M$ ([6]). Every free module, or more generally, every projective module, is a content R-module. Content R-modules can also be characterized by that for every family $\{A_i | i \in J\}$ of ideals of R, $(\cap_{i\in J} A_i)M = \cap_{i\in J}(A_i M)$.

Remark 1.1. (See [1].) Let M be an R-module. Then M is quasi-primeful in each of the following cases:

(1) M is free;
(2) R is a PID and M is finitely generated and content;
(3) R is a Dedekind domain and M is faithfully flat and content;
(4) R is Laskerian and M is locally free;
(5) R is Laskerian and M is projective.

An R-module M is called *quasi-prime-embedding*, if the natural map

$$\psi : q\mathrm{Spec}(M) \to q\mathrm{Spec}(R/\mathrm{Ann}(M))$$

is injective. An R-module M is called a *multiplication* module if every submodule N of M is of the form IM for some ideal I of R. Every multiplication module is quasi-prime-embedding (see [1, Corollary 2.23]).

2. MAIN RESULTS

In this section we use the notion of quasi-prime spectrum of a module to define a sheaf of rings. Let M be an R-module. Here, we consider a certain subset X of $q\mathrm{Spec}(M)$ equipped with induced topology and we define a scheme over X.

Throughout the paper X denotes the subset

$$\{Q \in q\mathrm{Spec}(M) \,|\, (Q :_R M) \in \mathrm{Spec}(R)\}$$

of $q\mathrm{Spec}(M)$. We recall that, for any element r of a ring R, the set $D_r = \mathrm{Spec}(R) - V(rR)$ is open in $\mathrm{Spec}(R)$ and the family $F = \{D_r | r \in R\}$ forms a base for the Zariski topology on $\mathrm{Spec}(R)$. Each D_r, in particular, $D_1 = \mathrm{Spec}(R)$ is known to be quasi-compact. It is shown in [1, Proposition 3.17] that the set $B' = \{\Gamma_M(a) \mid a \in R\}$ forms a base for the developed Zariski topology on $q\mathrm{Spec}(M)$, where for any $a \in R$, $\Gamma_M(a) = q\mathrm{Spec}(M) - D(aM)$. For each element $a \in R$ we define $X_a = X \cap \Gamma_M(a)$.

Proposition 2.1. *For any R-module M, the set $B = \{X_a \mid a \in R\}$ forms a base for X with the induced topology.*

Proof. We may assume that $X \neq \emptyset$. Let U be any open subset in X. Then there exists an open subset G of $q\mathrm{Spec}(M)$ such that $U = X \cap G$. There exists a submodule N of M such that $G = q\mathrm{Spec}(M) - D(N)$.

Hence by [1, Proposition 3.17],

$$U = X \cap G = X \cap \Big(\bigcup_{a_i \in (N:M)} \Gamma_M(a_i) \Big) = \bigcup_{a_i \in (N:M)} (X \cap \Gamma_M(a_i)) = \bigcup_{a_i \in (N:M)} X_{a_i}.$$

□

Proposition 2.2. *Let M be an R-module. For every element $a, b \in R$,*

$$X_{ab} = X_a \cap X_b.$$

Proof. Let $Q \in X$. Then

$$\begin{aligned} Q \in X_{ab} &\Leftrightarrow Q \in X \cap \Gamma_M(ab) \\ &\Leftrightarrow (abM : M) \nsubseteq (Q : M) \in \mathrm{Spec}(R) \\ &\Leftrightarrow a \notin (Q : M) \text{ and } b \notin (Q : M) \\ &\Leftrightarrow Q \in X_a \cap X_b. \end{aligned}$$

□

Proposition 2.3. *Let M be an R-module and $a \in R$. Then $X_{a^n} = X_a$ for any positive integer n. In particular, if b is a nilpotent element of R, then $X_b = \emptyset$.*

Proof. Use Proposition 2.2. □

Recall that a sheaf $\mathcal{F}$ of rings on a topological space X consists of the Data

(a) for every open subset $U \subseteq X$, a ring $\mathcal{F}(U)$, and
(b) for every inclusion $V \subseteq U$ of open sets of X, a morphism of rings $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ with the following conditions
  (1) $\mathcal{F}(\emptyset) = 0$, where $\emptyset$ is the empty set,
  (2) $\rho_{UU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is the identity,
  (3) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$,
  (4) if U is an open set and if $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i,then $s = 0$.
  (5) if U is an open set and if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i,with the property that for each $i, j, s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i.

Let P be a point of X, one can define the stalk of $\mathcal{F}_P$ of $\mathcal{F}$ at P to be the direct limit of the $\mathcal{F}(U)$ for all open sets U containing P, via the restriction maps ρ.

Definition 2.4. Let M be an R-module. For every open subset U of X we define $\mathrm{Supp}(U) = \{(P : M) \mid P \in U\}$.

Definition 2.5. Let M be an R-module. For every open subset U of X we define $\mathcal{O}_X(U)$ to be a subring of $\prod_{\mathfrak{p}\in \mathrm{Supp}(U)} R_{\mathfrak{p}}$, as the ring of functions $s : U \to \coprod_{\mathfrak{p}\in \mathrm{Supp}(U)} R_{\mathfrak{p}}$, where $s(P) \in R_{\mathfrak{p}}$ for each $P \in U$ where $\mathfrak{p} = (P : M)$ and for each $P \in U$, there is a neighborhood V of P, contained in U, and elements $a, f \in R$, such that for each $Q \in V$, $f \notin \mathfrak{q} := (Q : M)$, and $s(Q) = a/f$ in $R_{\mathfrak{q}}$.

It is clear that for an open set U of X, $\mathcal{O}_X(U)$ is closed under sum and product. Thus $\mathcal{O}_X(U)$ is a commutative ring with identity (the identity element of $\mathcal{O}_X(U)$ is the function which sends all $P \in U$ to 1 in $R_{(P:M)}$). If $V \subseteq U$ are two open sets, the natural restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is a homomorphism of rings. It is then clear that $\mathcal{O}_X$ is a presheaf. Finally, from the local nature of the definition $\mathcal{O}_X$ is a sheaf. Hence

Lemma 2.6. *Let M be an R-module.*

(1) *For each open subset U of X, $\mathcal{O}_X(U)$ is a subring of $\prod_{\mathfrak{p}\in \mathrm{Supp}(U)} R_{\mathfrak{p}}$.*
(2) *$\mathcal{O}_X$ is a sheaf.*

In next proposition, we find the stalk of the sheaf.

Proposition 2.7. *Let M be an R-module. Then for each $P \in X$, the stalk $\mathcal{O}_{X,P}$ of the sheaf $\mathcal{O}_X$ is isomorphic to $R_{\mathfrak{p}}$, where $\mathfrak{p} := (P : M)$.*

Proof. Let $P \in X$ be a quasi-prime submodule of M such that $\mathfrak{p} = (P : M)$ and

$$m \in \mathcal{O}_{X,P} = \varinjlim_{P\in U} \mathcal{O}_X(U).$$

Then there exists a neighborhood U of P and $s \in \mathcal{O}_X(U)$ such that m is the germ of s at the point P. We define a homomorphism $\varphi : \mathcal{O}_{X,P} \to R_{\mathfrak{p}}$ by $\varphi(m) = s(P)$. This is a well-defined homomorphism. Let V be another neighborhood of P and $t \in \mathcal{O}_X(V)$ such that m is the germ of s at the point P. Then there exists an open subset $W \subseteq U \cap V$ such that $P \in W$ and $s|_W = t|_W$. Since $P \in W$, $s(P) = t(P)$. We claim that φ is an isomorphism.

Let $x \in R_{\mathfrak{p}}$. Then $x = a/f$ where $a \in R$ and $f \in R \setminus \mathfrak{p}$. Since $f \notin \mathfrak{p}$, $P \in X_f$. Now we define $s(Q) = a/f$ in $R_{\mathfrak{q}}$, where $\mathfrak{q} := (Q : M)$, for all $Q \in X_f$. Then $s \in \mathcal{O}(X_f)$. If m is the equivalent class of s in $\mathcal{O}_{X,P}$, then $\varphi(m) = x$. Hence φ is surjective.

Now, let $m \in \mathcal{O}_{X,P}$ and $\varphi(m) = 0$. Let U be an open neighborhood of P and m be the germ of $s \in \mathcal{O}_X(U)$ at P. There is an open neighborhood $V \subseteq U$ of P and elements $a, f \in R$ such that $s(Q) = a/f \in R_{\mathfrak{q}}$, where $\mathfrak{q} := (Q : M)$, for all $Q \in V$, $f \notin \mathfrak{q}$. Thus $V \subseteq X_f$. Then $0 = \varphi(m) = s(P) = a/f$ in $R_{\mathfrak{p}}$. So, there is $h \in R \setminus \mathfrak{p}$ such that $ha = 0$. By Proposition 2.2, for $Q \in X_{fh} = X_f \cap X_h$ we have $s(Q) = a/f \in R_{\mathfrak{q}}$. Since $h \notin \mathfrak{q}$, $s(Q) = \frac{a}{f} = \frac{h}{h}\frac{a}{f} = 0$. Hence $s|_{\mathcal{O}(X_{fh})} = 0$. This yields, $s = 0$ in $\mathcal{O}_X(X_{fh})$. Consequently $m = 0$. □

As a direct consequence of Proposition 2.7, we have

Corollary 2.8. *If M is an R-module, then $(X, \mathcal{O}_X)$ is a locally ringed space.*

Lemma 2.9. *Let M and M' be two R-modules and let $f : M \to M'$ be an epimorphism. If N is a quasi-prime submodule of M', then $f^{-1}(N)$ is a quasi-prime submodule of M.*

Proposition 2.10. *Let M an N be R-modules and $\phi : M \to N$ be an epimorphism. Then the map*

$$\begin{array}{rcl} \theta : q\mathrm{Spec}(N) & \to & q\mathrm{Spec}(M) \\ Q & \mapsto & \phi^{-1}(Q) \end{array}$$

is continuous. In particular, if $Y := \{Q \in q\mathrm{Spec}(N) \,|\, (Q :_R N) \in \mathrm{Spec}(R)\}$, then the map

$$\begin{array}{rcl} f = \theta|_Y : Y & \to & X \\ Q & \mapsto & \phi^{-1}(Q) \end{array}$$

is continuous.

Proof. For any $Q \in q\mathrm{Spec}(N)$ and any closed set $D^M(K)$ in $q\mathrm{Spec}(M)$, where $K \leq M$, we have

$$\begin{array}{rcl} Q \in \theta^{-1}(D^M(N)) & \Leftrightarrow & \theta(Q) = \phi^{-1}(Q) \supseteq (N : M)M \\ & \Leftrightarrow & Q \supseteq \phi((K : M)M) = (K : M)N \\ & \Leftrightarrow & Q \in D^N((K : M)N). \end{array}$$

Hence, $\theta^{-1}(D^M(N)) = D^N((K : M)N)$, so θ is continuous. The last statement follows from the first part. □

Proposition 2.11. *Let M an N be R-modules and $\phi : M \to N$ be an epimorphism and let X, Y be as in Proposition 2.10. Then ϕ induces a morphism of locally ringed spaces*

$$(f, f^\sharp) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X).$$

Proof. By Proposition 2.10, the map $f : Y \to X$ which is defined by $P \mapsto \phi^{-1}(P)$, is continuous. Let U be an open subset of X and $s \in \mathcal{O}_{\mathrm{Spec}(M)}(U)$. Suppose $P \in f^{-1}(U)$. Then $f(P) = \phi^{-1}(P) \in U$. Assume that W is an open neighborhood of $\phi^{-1}(P)$ with $W \subseteq U$ and $a, g \in R$, such that for each $Q \in W$, $g \notin \mathfrak{q} := (Q : M)$, and $s(Q) = a/g$ in $R_{\mathfrak{q}}$. Since $\phi^{-1}(P) \in W$, $P \in f^{-1}(W)$. As we mentioned, f is continuous, so $f^{-1}(W)$ is an open subset of Y. We claim that for each $Q' \in f^{-1}(W)$, $g \notin (Q' : N)$. Suppose $g \in (Q' : N)$ for some $Q' \in f^{-1}(W)$. Then $\phi^{-1}(Q') = f(Q') \in W$. Since ϕ an epimorphism, $(Q' : N) = (\phi^{-1}(Q') : M)$. So, $g \in (\phi^{-1}(Q') : M)$. This is a contradiction. Therefore, we can define

$$f^{\sharp}(U) : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$$

by $f^{\sharp}(U)(s) = s \circ f$.

Assume that $V \subseteq U$ and $P \in f^{-1}(V)$. Then according to the diagram below

$$\begin{array}{ccccc} f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{t} & R_{(P:M)} \\ \uparrow & & \uparrow & \nearrow_{t|_V} & \\ f^{-1}(V) & \xrightarrow{f} & V & & \end{array}$$

we have

$$(t \circ f)|_{f^{-1}(V)}(P) = t|_V \circ f(P). \tag{2.1}$$

Consider the diagram

(A)
$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{f^{\sharp}(U)} & \mathcal{O}_Y(f^{-1}(U)) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{f^{-1}(U)f^{-1}(V)}} \\ \mathcal{O}_X(V) & \xrightarrow{f^{\sharp}(V)} & \mathcal{O}_Y(f^{-1}(V)). \end{array}$$

Since

$$\begin{aligned} \rho'_{f^{-1}(U)f^{-1}(V)} f^{\sharp}(U)(t)(P) &= \rho'_{f^{-1}(U)f^{-1}(V)}(t \circ f)(P) \\ &= (t \circ f)|_{f^{-1}(V)}(P) \\ &= t|_V \circ f(P) \qquad \text{by equation 2.1} \\ &= \rho_{UV}(t) \circ f(P) \\ &= f^{\sharp}(V)\rho_{UV}(t)(P), \end{aligned}$$

for each $t \in \mathcal{O}_X(U)$, the diagram (A) is commutative, and it follows that

$$f^{\sharp} : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$$

is a morphism of sheaves. By Proposition 2.7, the map on stalks

$$f^{\sharp}_{P} : \mathcal{O}_{X,f(P)} \longrightarrow \mathcal{O}_{Y,P}$$

is clearly the map of local rings

$$R_{(f(P):M)} \longrightarrow R_{(P:N)}.$$

This implies that

$$(Y, \mathcal{O}_Y) \xrightarrow{(f,f^{\sharp})} (X, \mathcal{O}_X)$$

is a morphism of locally ringed spaces. □

Theorem 2.12. *Let $\Phi : R \to S$ be a ring homomorphism, N an S-module and M a quasi-primeful and quasi-prime-embedding R-module such that $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(N)$ (here, we consider N as an R-module by means of Φ). Let X, Y be as in Proposition 2.10. Then Φ induces a morphism of locally ringed spaces*

$$(Y, \mathcal{O}_Y) \xrightarrow{(h,h^{\sharp})} (X, \mathcal{O}_X).$$

Proof. Since $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(N)$, Φ induces the map $\Theta : R/\mathrm{Ann}_R(M) \to S/\mathrm{Ann}_S(N)$. It is well-known that the maps $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ by $\mathfrak{p} \mapsto \Phi^{-1}(\mathfrak{p})$ and $d : \mathrm{Spec}(S/\mathrm{Ann}_S(N)) \to \mathrm{Spec}(R/\mathrm{Ann}_R(M))$ by $\overline{\mathfrak{p}} \mapsto \Theta^{-1}(\overline{\mathfrak{p}})$ and $\psi_N : q\mathrm{Spec}(N) \to q\mathrm{Spec}(S/\mathrm{Ann}_S(N))$ with $\psi(P) = (P :_S N)/\mathrm{Ann}_S(N)$ for each $P \in q\mathrm{Spec}(N)$ are continuous maps (see [1, Proposition 3.2]). Hence, the map

$$\begin{array}{rcl} \chi_N = \psi_N|_Y : Y & \to & \mathrm{Spec}(S/\mathrm{Ann}_S(N)) \\ P & \mapsto & (P :_S N)/\mathrm{Ann}_S(N) \end{array}$$

is continuous. Also $\psi_M : q\mathrm{Spec}(M) \to q\mathrm{Spec}(R/\mathrm{Ann}_R(M))$ is homeomorphism by [1, Proposition 3.2]. Thus the map

$$\begin{array}{rcl} \chi_M = \psi_M|_X : X & \to & \mathrm{Spec}(R/\mathrm{Ann}_R(M)) \\ Q & \mapsto & (Q :_R N)/\mathrm{Ann}_R(M) \end{array}$$

is a one-to-one correspondence continuous map and χ_M^{-1} is continuous. Therefore the map

$$\begin{array}{rrcl} h : & Y & \longrightarrow & X \\ & P & \mapsto & \chi_M^{-1}\, d\, \chi_N(P) \end{array}$$

is continuous. For each $P \in Y$, we get a local homomorphism

$$\Phi_{(P:_S N)} : R_{f(P:_S N)} \longrightarrow S_{(P:_S N)}.$$

Let U be an open subset of X and let $t \in \mathcal{O}_X(U)$. Suppose that $P \in h^{-1}(U)$. Then $h(P) \in U$ and there exists a neighborhood W of

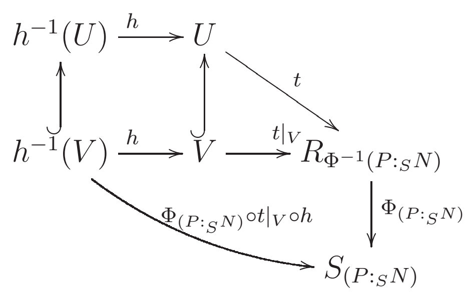
$h(P)$ with $W \subseteq U$ and elements $r, g \in R$ such that for each $Q \in W$, $g \notin (Q :_R M)$, and $t(Q) = \frac{r}{g} \in R_{(Q:_R\ M)}$. Hence $g \notin (h(P) :_R M)$. By definition of h, $(h(P) :_R\ M) = \Phi^{-1}(P :_S\ N)$. So, $\Phi(g) \notin (P :_S N)$ and $\Phi_{(P:_S N)}(\frac{r}{g})$ defines a section on $\mathcal{O}_Y(h^{-1}(W))$. Since

$$\begin{array}{ccc} R_g & \longrightarrow & S_{\Phi(g)} \\ \downarrow & & \downarrow \\ R_{\Phi^{-1}(P:_S N)} & \longrightarrow & S_{(P:_S N)} \end{array}$$

is commutative, we can define

$$h^\sharp(U) : \mathcal{O}_X(U) \longrightarrow h_*\mathcal{O}_Y(U) = \mathcal{O}_Y(h^{-1}(U))$$

by $h^\sharp(U)(t)(P) = \Phi_{(P:_S N)}(t(h(P)))$ for each $t \in \mathcal{O}_X(U)$ and $P \in h^{-1}(U)$. Assume that $V \subseteq U$ and $P \in h^{-1}(V)$. According to the diagram below



we have

$$\Phi_{(P:_S N)} \circ t|_V \circ h(P) = (\Phi_{(P:_S N)} \circ t \circ h)|_{h^{-1}(V)}(P). \tag{2.2}$$

Consider the diagram

(B)
$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{h^\sharp(U)} & \mathcal{O}_Y(h^{-1}(U)) \\ \rho_{UV}\downarrow & & \downarrow \rho'_{h^{-1}(U)h^{-1}(V)} \\ \mathcal{O}_X(V) & \xrightarrow{h^\sharp(V)} & \mathcal{O}_Y(h^{-1}(V)). \end{array}$$

Since

$$\begin{aligned} \rho'_{h^{-1}(U)h^{-1}(V)} h^\sharp(U)(t)(P) &= \rho'_{h^{-1}(U)h^{-1}(V)} \Phi_{(P:_S N)} \circ t \circ h(P) \\ &= (\Phi_{(P:_S N)} \circ t \circ h)|_{h^{-1}(V)}(P) \\ &= \Phi_{(P:_S N)} \circ t|_V \circ h(P) \qquad \text{by equation 2.2} \\ &= h^\sharp(V)(t|_V)(P) \\ &= h^\sharp(V)\rho_{UV}(t)(P), \end{aligned}$$

the diagram (B) is commutative, and it follows that

$$h^\sharp : \mathcal{O}_X \longrightarrow h_*\mathcal{O}_Y$$

is a morphism of sheaves. By Proposition 2.7, the map on stalks

$$h^\sharp_P : \mathcal{O}_{X,h(P)} \longrightarrow \mathcal{O}_{Y,P}$$

is clearly

$$R_{f(P:_S N)} \longrightarrow S_{(P:_S N)}.$$

This implies that

$$(Y, \mathcal{O}_Y) \xrightarrow{(h,h^\sharp)} (X, \mathcal{O}_X)$$

is a morphism of locally ringed spaces. □

Lemma 2.13. *Let M be a faithful and quasi-primeful R-module and let $a, b \in R$. If $X_a \subseteq X_b$, then $a \in \sqrt{Rb}$.*

Proof. Let $\mathfrak{p} \in V(Rb) := \{\mathfrak{q} \in \operatorname{Spec}(R) \,|\, \mathfrak{q} \supseteq Rb\}$. Then there exists a quasi-prime submodule Q of M such that $(Q : M) = \mathfrak{p}$. So, $Q \notin X_b$, whence $Q \notin X_a$. Therefore $a \in (aM : M) \subseteq (Q : M) = \mathfrak{p}$. Consequently, $a \in \bigcap_{\mathfrak{p} \in V(Rb)} \mathfrak{p} = \sqrt{Rb}$. □

Proposition 2.14. *Let M be a faithful and quasi-primeful R-module. For any element $f \in R$, the ring $\mathcal{O}_X(X_f)$ is isomorphic to the localized ring R_f.*

Proof. We define the map $\Theta : R_f \to \mathcal{O}_X(X_f)$ by

$$\frac{a}{f^m} \mapsto (s : Q \mapsto \frac{a}{f^m} \in R_{(Q:M)}).$$

It is easy to see Θ is a well-defined homomorphism. We are going to show that Θ is an isomorphism.

We first show that Θ is injective. If $\Theta(\frac{a}{f^n}) = \Theta(\frac{b}{f^m})$, then for every $P \in X_f$, $\frac{a}{f^n}$ and $\frac{b}{f^m}$ have the same image in $R_{\mathfrak{p}}$, where $\mathfrak{p} = (P : M)$. Thus there exists $h \in R \setminus \mathfrak{p}$ such that $h(f^m a - f^n b) = 0$ in R. Let $I = (0 :_R f^m a - f^n b)$. Then $h \in I$ and $h \notin \mathfrak{p}$, so $I \nsubseteq \mathfrak{p}$. This happen for any $P \in X_f$, so we conclude that

$$V(I) \cap \operatorname{Supp}(X_f) = \emptyset$$

hence

$$\operatorname{Supp}(X_f) \subseteq D(I) := \operatorname{Spec}(R) \setminus V(I).$$

Since M is faithful quasi-primeful,

$$D_f = \operatorname{Supp}(X_f) \subseteq D(I).$$

Therefore $f \in \sqrt{I}$ and so, $f^l \in I$ for some positive integer l. Now we have $f^l(f^m a - f^n b) = 0$ which shows that $\frac{a}{f^n} = \frac{b}{f^m}$ in $R_{\mathfrak{p}}$. Hence Θ is injective.

Let $s \in \mathcal{O}_X(X_f)$. Then we can cover X_f with open subset V_i, on which s is represented by $\frac{a_i}{g_i}$, with $g_i \notin (P : M)$ for all $P \in V_i$, in other words $V_i \subseteq X_{g_i}$. By Proposition 2.1, the open sets of the form X_h are a base for the topological space X. So, we may assume that $V_i = X_{h_i}$ for some $h_i \in R$. Since $X_{h_i} \subseteq X_{g_i}$, by Lemma 2.13, $h_i \in \sqrt{(g_i)}$. Thus $h_i^n = cg_i$ for some $n \in \mathbb{N}$ and $c \in R$. So,

$$\frac{a_i}{g_i} = \frac{ca_i}{cg_i} = \frac{ca_i}{h_i^n}.$$

We see that s is represented by $\frac{b_i}{k_i}$, $(b_i = ca_i, k_i = h_i^n)$ on X_{k_i} and (since $X_{h_i} = X_{h_i^n}$) the family X_{k_i}'s cover X_f. By [1, Proposition 3.18], that the open cover $X_f = \bigcup X_{k_i}$ has a finite subcover. Suppose, $X_f \subseteq X_{k_1} \cup \cdots \cup X_{k_n}$. For $1 \leq i, j \leq n$, $\frac{b_i}{k_i}$ and $\frac{b_j}{k_j}$ both represent s on $X_{k_i} \cap X_{k_j}$. By Proposition 2.2, $X_{k_i} \cap X_{k_j} = X_{k_i k_j}$ and by injectivity of Θ, we get $\frac{b_i}{k_i} = \frac{b_j}{k_j}$ in $R_{k_i k_j}$. Hence for some n_{ij},

$$(k_i k_j)^{n_{ij}}(k_j b_i - k_i b_j) = 0.$$

Let $m = \max\{n_{ij} | 1 \leq i, j \leq n\}$. Then

$$k_j^{m+1}(k_i^m b_i) - k_i^{m+1}(k_j^m b_j) = 0.$$

By replacing each k_i by k_i^{m+1}, and b_i by $k_i^m b_i$, we still see that s represented on X_{k_i} by $\frac{b_i}{k_i}$, and furthermore, we have $k_j b_i = k_i b_j$ for all i, j. Since $X_f \subseteq X_{k_1} \cup \cdots \cup X_{k_n}$, by [1, Proposition 3.18], we have

$$D_f = \psi(X_f) \subseteq \bigcup_{i=1}^{n} \psi(X_{k_i}) = \bigcup_{i=1}^{n} D_{k_i},$$

where ψ is the natural map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R)$. So, there are $c_1, \cdots, c_n$ in R and $t \in \mathbb{N}$, such that $f^t = \sum_i c_i k_i$. Let $a = \sum_i c_i b_i$. Then for each j we have

$$k_j a = \sum_i c_i b_i k_j = \sum_i c_i k_i b_j = b_j f^t.$$

This implies that $\frac{a}{f^t} = \frac{b_j}{k_j}$ on X_{k_j}. So $\Theta(\frac{a}{f^t}) = s$ everywhere, which shows that Θ is surjective. $\square$

Corollary 2.15. *Let M be a faithful and quasi-primeful R-module. Then $\mathcal{O}_X(X)$ is isomorphic to R.*

We recall that a scheme X is locally Noetherian if it can be covered by open affine subsets $\mathrm{Spec}(A_i)$, where each A_i is a Noetherian ring. X is Noetherian if it is locally Noetherian and quasi-compact ([4]).

Theorem 2.16. *Let M be a faithful, quasi-primeful and quasi-prime-embedding R-module. Then $(X, \mathcal{O}_X)$ is a scheme. Moreover, if R is Noetherian, then $(X, \mathcal{O}_X)$ is a Noetherian scheme.*

Proof. Let $g \in R$. Because the natural map $\psi : q\mathrm{Spec}(M) \to q\mathrm{Spec}(R)$ is continuous by [1, Proposition 3.2], the map $\psi|_{X_g} : X_g \to \psi(X_g)$ is also continuous. Since M is quasi-prime-embedding, $\psi|_{X_g}$ is a bijection. Let E be a closed subset of X_g. Then $E = X_g \cap D^M(N)$ for some submodule N of M. Hence $\psi(E) = \psi(X_g \cap D^M(N)) = \psi(X_g) \cap D^R(N : M)$ is a closed subset of $\psi(X_g)$. Therefore, $\psi|_{X_g}$ is a homeomorphism.

Suppose $X = \bigcup_{i \in I} X_{g_i}$. Since M is faithful, quasi-primeful and quasi-prime-embedding, for each $i \in I$

$$X_{g_i} \cong \psi(X_{g_i}) = \mathrm{Supp}(X_{g_i}) = D_{g_i} \cong \mathrm{Spec}(R_{g_i}).$$

Thus by Proposition 2.14, X_{g_i} is an affine scheme and this implies that $(X, \mathcal{O}_X)$ is a scheme. For the last statement, we note that since R is Noetherian, so is R_{g_i} for each $i \in I$. Hence $(X, \mathcal{O}_X)$ is a locally Noetherian scheme. By [1, Proposition 3.18], X is quasi-compact, therefore $(X, \mathcal{O}_X)$ is a Noetherian scheme. □

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