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TRIPLE FACTORIZATION OF NON-ABELIAN GROUPS BY TWO MAXIMAL SUBGROUPS

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Abstract. The triple factorization of a group G has been studied recently showing that $G = ABA$ for some proper subgroups A and B of G, the definition of rank-two geometry and rank-two coset geometry which is closely related to the triple factorization was defined and calculated for abelian groups. In this paper we study two infinite classes of non-abelian finite groups D_{2n} and $PSL(2, 2^n)$ for their triple factorizations by finding certain suitable maximal subgroups, which these subgroups are define with original generators of these groups. The related rank-two coset geometries motivate us to define the rank-two coset geometry graphs which could be of intrinsic tool on the study of triple factorization of non-abelian groups.

1. Introduction

The factorization of a finite group G as the inner product $G = ABA$ where, A and B are proper subgroups of G, defined and studied by Gorenstein ([8]) in 1962 and the notation $T = (G, A, B)$ is used for a triple factorization of the group G. This was based on the article [9] where a class of Frobenius groups is studied. In 1990 the factorization of finite simple groups and their automorphism groups were studied in [11]. Also the geometric ABA groups have been studied by Higman ([10]) in 1961. Following [1] and using the notations of the articles [3, 4, 15] we recall the notions of incidence geometry, rank-two

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geometry, flag and rank-two coset geometry. These are related to the triple factorization of a group and product of groups ([2]). The aim of this paper is to study the rank-two coset geometry by defining a graph, which is named a rank-two coset geometry graph. The notation $\Gamma(G, A, B)$ will be used for this graph, where $G = ABA$. Our computational results based on the study of two classes of non-abelian groups D_{2n} (the dihedral group of order $2n$) and the projective special linear groups $PSL(2, 2^n)$, $(n \geq 3)$. The nice and very interesting presentation of projective special linear groups may be found in ([5, 6, 7]) and the related references.

It is necessary to recall that for studying the triple factorization of groups the important tools come from permutation group theory and we recall some of them which will be useful in our proofs. The set of all permutations of a set Ω is the symmetric group on Ω, denoted by $Sym(\Omega)$, and a subgroup of $Sym(\Omega)$ is called a permutation group on Ω. If a group G acts on Ω we denote the induced permutation group of G by G^{Ω}, a subgroup of $Sym(\Omega)$. We say that G is transitive on Ω if for all $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$. For a transitive group G on the set Ω, a nonempty subset Δ of Ω is called a block for G if for each $g \in G$, either $\Delta^g = \Delta$, or $\Delta^g \cap \Delta = \emptyset$; in this case the set $\Sigma = \{\Delta^g | g \in G\}$ is said to be a block system for G. The group G induces a transitive permutation group G^{Σ} on Σ, and the set stabilizer G_{Δ} induces a transitive permutation group G_{Δ}^{Δ} on Δ. If the only blocks for G are the singleton subsets or the whole of Ω we say that G is primitive, and otherwise G is imprimitive.

Definition 1.1. A triple factorization $T = (G, A, B)$ of a finite group G is called degenerate if $G = AB$ or $G = BA$. Otherwise, $T = (G, A, B)$ is called a non-degenerate triple factorization. A group with a triple factorization $T = (G, A, B)$, is sometimes called an ABA-group.

Definition 1.2. Let P and L be the sets of right cosets of the proper subgroups A and B of a finite group G, respectively. The property $*$ between the elements of P and L which is named a "non-empty intersection relation" is defined as follows:

$$Ax * By \Longleftrightarrow Ax \cap By \neq \emptyset$$

Then $(\Omega = P \cup L, *)$ is called a rank-two coset geometry and will be denoted by $Cos(G, A, B)$.

In a rank-two coset geometry, if the property $*$ holds between two members $Ax \in P$ and $By \in L$, then we say that these members are incident, and in this case the pair (Ax, By) is called a flag of rank-two coset geometry.

Definition 1.3. The rank-two coset geometry graph of a finite non-abelian group G will be denoted by $\Gamma(G, A, B)$, is an undirected graph with the vertex set $P \cup L$ and two points Ax and By are adjacent if and only if $Ax \cap By \neq \emptyset$ where, $G = ABA$.

Our main results are the following theorems:

Theorem A. Let $G = D_{2n} = \langle a, b | a^n = b^2 = (ab)^2 = 1 \rangle$ be the dihedral group of order $2n$. Then,

(1) For $n = 3k$, $(k = 1, 2, ...)$, there are at least two proper dihedral subgroups B and C of G such that $G = BCB$ (non-degenerate triple factorization).
(2) For $n = 2^k$, $(k = 1, 2, ...)$, there is no non-degenerate triple factorization for G.
(3) For the prime values of $n \geq 5$, there is no non-degenerate triple factorization for G.
(4) The graph associated to a triple factorization $T = (G, A, B)$ of G, $(\Gamma(G, A, B))$ is bipartite graph if and only if the factorization is degenerate.

Theorem B. Let $G = PSL(2, 2^n)$, $(n \geq 3)$ be the projective special linear group. Then, there are two distinct maximal subgroups A and B of G such that $G = ABA$. Moreover, $\Gamma(G, A, B) \simeq K_{r,s}$, the bipartite graph.

2. The dihedral Groups D_{2n}, $n \geq 3$

The well-known presentation for the dihedral group of order $2n$ is $D_{2n} = \langle a, b | a^n = b^2 = (ab)^2 = 1 \rangle$. The nature and the number of subgroups of D_{2n} are of interest to know and we collect all of these subgroups and their properties in the following preliminary lemma.

Lemma 2.1. *Every subgroup of D_{2n} $(n \geq 3)$, is cyclic or a dihedral group such that:*

(i) *the cyclic subgroups are $< a^d >$, where $d|n$ and $|D_{2n} :< a^d > | = 2d$,*
(ii) *the dihedral subgroups are $< a^d, a^i b >$, where $d|n$, and $0 \leq i \leq d - 1$, and $|D_{2n} :< a^d, a^i b > | = d$,*

(iii) *let n be odd and $m|2n$. For odd values of m there are m subgroups of index m in D_{2n}. However, if m is even there is exactly one subgroup of index m,*
(iv) *let n be even and $m|2n$. For odd values of m there are m subgroups of index m. If m is even and doesn't divide n, there is only one subgroup of index m. Finally, if m is even and $m|n$, there are exactly $m+1$ subgroups of index m.*

There are also certain obvious relations in D_{2n}. Indeed, for every integer $i = 1, 2, ..., n$, the following relations hold in D_{2n}:
$ba^ib = a^{-i}$, $a^iba^{-i} = a^{2i}b$, $(a^ib)b(a^ib)^{-1} = a^{2i}b$, $a^iba^{-i} = b$.

The following lemma gives a necessary and sufficient condition for a triple $T = (G, A, B)$ to be a triple factorization of finite group G in terms of the two proper and distinct subgroups A and B (for a proof one may see [1]).

Lemma 2.2. *Let A and B be two proper subgroups of a group G, and consider the right coset action of G on $\Omega_A = \{Ag|g \in G\}$. Set $\alpha = A \in \Omega_A$. Then $T = (G, A, B)$ is a triple factorization if and only if the B-orbit α^B intersects nontrivially each G_α-orbit in Ω_A.*

It is necessary to recall that, using the permutation notations the Lemma 2.2 may be reduced to:

Lemma 2.3. *Let A and B be two proper subgroups of a group G and consider the right coset action of G on $\Omega_A = \{Ag|g \in G\}$. Set $\alpha = A \in \Omega_A$. Then, $T = (G, A, B)$ is a triple factorization if and only if for all $g \in G$ there exists elements $b \in B$, $a \in A$ such that $Ab = Aga$.*

Lemma 2.4. *For any two proper and distinct subgroups A and B of D_{2n} if $T = (D_{2n}, A, B)$ is a degenerate (non-degenerate) triple factorization for D_{2n} then $T = (D_{2n}, B, A)$ is also a degenerate (non-degenerate) triple factorization for D_{2n}. Moreover, $D_{2n} = ABA = BAB$.*

Proof. The proof is easy by using Lemma 2.3 and the relations of D_{2n}. □

Proof of Theorem A.

(1) For $n = 3k$, $(k = 1, 2, 3, ...)$, $D_{2n} = \langle a, b|a^{3k} = b^2 = (ab)^2 = 1\rangle$ and its dihedral subgroups are in the form $< a^d, a^ib >$ where, $d \geq 3$, $d|n = 3k$ and $0 \leq i \leq d-1$. Now if $B =< a^r, a^ib >$ and $C =< a^s, a^jb >$ be two distinct dihedral subgroup of $D_{2n} =$

$D_{2(3k)}$ such that $|B||C||B| \geq 2n$, then for some i, j, l, m, n that, $0 \leq i, j, l \leq n-1$ and $0 \leq m, n \leq 1$, there exist elements $x = a^i b^m \in B$, $y = a^j b^n \in C$ and $g = a^l \in D_{2n}$ such that $By = Bgx$. So, by Lemma 2.3, $T = (D_{2n}, A, B)$ is a triple factorization of D_{2n}, and by using the relations $ba^ib = a^{-i}$, $a^iba^{-i} = a^{2i}b$, $(a^ib)b(a^ib)^{-1} = a^{2i}b$ and $a^iba^{-i} = b$, $(0 \leq i \leq n-1)$ of D_{2n} we get that for every $0 \leq r, s, l \leq n-1$ and $0 \leq \alpha, \beta, \gamma \leq 1$, the word $a^r b^\alpha a^s b^\beta a^l b^\gamma$ of BCB is one of the elements of D_{2n}. So, this triple factorization is non-degenerate and $D_{2n} = BCB = CBC$.

(2) For $n = 2^k$, $(k = 1, 2, 3, \ldots)$, by Lemma 2.1, the number of non-trivial cyclic and dihedral subgroups of D_{2n} is k and $2^{k+1}-2$, respectively. In the case $k = 1$, the non-trivial cyclic subgroup of D_4 is $A =< a >= \{1, a\}$ and the nontrivial dihedral subgroups are $B =< a^2, a^0b >=< 1, b >= \{1, b\}$ and $C =< a^2, a^1b >=< 1, ab >= \{1, ab\}$, such that by using the relations of D_{2n} we get, $AB = BA = AC = CA = BC = CB = D_4$. And for every $k \geq 2$, it is easy to see that for the cyclic subgroup $A =< a^1 >$ and for any two distinct nontrivial dihedral subgroups B and C satisfying $B \nsubseteq C$, $C \nsubseteq B$ and $|B||C||B| \geq 2n$ we get $AB = BA = AC = CA = BC = CB = D_{2n}$. Hence, the triples (D_{2n}, A, B), (D_{2n}, A, C) and (D_{2n}, B, C) are degenerate triple factorizations.

(3) For the prime values of $n \geq 5$, the number of nontrivial cyclic and dihedral subgroups of D_{2n} are 1 and n, respectively, where $A =< a >$ is the only nontrivial cyclic subgroup and for every i $(i = 0, 1, \ldots, n-1)$, $B_i =< a^n, a^ib >$ is a nontrivial dihedral subgroup. By using the relations of D_{2n} one may see that for every $1 \leq i, j \leq n-1$, $AB_iA = AB_i = D_{2n}$ but $B_iB_jB_i \neq D_{2n}$. Thus, in this case there is no non-degenerate triple factorization for D_{2n}.

(4) By (2) and (3), $T = (D_{2n}, A, B_i)$ is a degenerate triple factorization of D_{2n} where, $A =< a >$ is the only cyclic subgroup of D_{2n} of index 2 and $B_i =< a^n, a^ib >$, $(i = 0, 1, \ldots, n-1)$ is a dihedral subgroup of index n, where $n \geq 5$ is a prime and the set of distinct right cosets of A and B_i are $\{A, Ab\}$ and $\{B_i, B_ia, B_ia^2, \ldots, B_ia^{n-1}\}$, respectively. By using the relations of D_{2n} we get that for every $0 \leq i, k \leq n-1$, $A \cap B_ia^k$ and $Ab \cap B_ia^k$ are not empty. So by the definition of rank-two coset geometry, for every i, $(i = 0, 1, \ldots, n-1)$, each coset of A is adjacent to all cosets of B_i. Therefore, $\Gamma(D_{2n}, A, B_i) = K_{2,n-1}$, the complete bipartite graph. By the same method one may

see that if $T = (D_{2n}, A, B)$ is a degenerate triple factorization for two distinct subgroups A and B, then $\Gamma(D_{2n}, A, B) = K_{r,s}$ where, r and s are the indices of the subgroups A and B, respectively. For the inverse case, let $\Gamma(D_{2n}, B, C) = K_{p,q}$. then by definition of rank-two geometry graph p and q are the orders of two distinct proper subgroups $B =< a^r, a^i b >$ and $C =< a^s, a^j b >$, where $|D_{2n} : B| = r$, $|D_{2n} : C| = s$ and the set of right cosets of B and C are $\{B, Ba, Ba^2, ..., Ba^{r-1}\}$ and $\{C, Ca, Ca^2, ..., Ca^{s-1}\}$, respectively. Now by considering the elements of subgroups B, C and D_{2n} one may see that $D_{2n} = BC$ and the triple factorization is degenerate. □

3. The Groups $PSL(2, 2^n)$, $n \geq 3$

The projective special linear group $PSL(2, F)$ is the quotient of the special linear group $SL(2, F)$ by its center. When $F = GF(2^n)$ we known that $|SL(2, 2^n)| = |PSL(2, 2^n)| = 2^n(2^{2n} - 1)$.

To study of the triple factorization of $PSL(2, 2^n)$, $n \geq 3$ by two maximal subgroup, we use the Sinkov's ([13]) presentation for $PSL(2, 2^n)$ with three generators and $n + 5$ relations as follows:

$$\langle x, y, z | x^l = y^2 = z^3 = (xz)^2 = (yz)^2 = [x^i, y]^2 = R = 1, i = 1, ..., n-1 \rangle$$

where, $l = 2^n - 1$, $R = x^{-n} yxy^{a_{n-1}} xy^{a_{n-2}} ... xy^{a_0}$ and a_0, a_1, ..., a_{n-1} are the coefficients of an irreducible polynomial of degree n, on the field $GF(2)$ which vanishes at least for a primitive element of $GF(2^n)$. Note that the interesting and efficient presentations for the cases $n = 3, 4, 5$ have been given in [6].

Although, Gorenstein ([8]) and Dickson ([7]) have studied the maximal subgroups of $PSL(2, 2^n)$ in the special case, but we are inerested to identify them in terms of the original generators of $PSL(2, 2^n)$, to find the maximal subgroups in terms of the original generators x, y and z. First of all we adopt the Dickson's results for $q = 2^n$ as saying that, there are five kinds of maximal subgroups for $PSL(2, q)$ as follows:

Type(1): $E_q \ltimes Z_{q-1}$ of order $q(q - 1)$ (semidirect product of the elementary abelian group of order q by the cyclic group of order $q - 1$).

Type(2): $D_{2(q+1)}$, the dihedral group of order $2(q + 1)$.

Type(3): $D_{2(q-1)}$, the dihedral group of order $2(q - 1)$.

Type(4): A_5, the alternating group of order 60, when $q = 4^r$, (r is a prime).

Type(5): $PSL(2, q')$, the projective group of order $q'((q')^2 - 1)$, when $q' > 2$, $q = (q')^m$, (m is a prime) or $q' = 2$ and $q = (q')^2$.

First of all we derive certain useful information on the above given presentation of $PSL(2, 2^n)$.

Lemma 3.1. *There are exactly $P_2(n)$ presentations for the group $PSL(2, 2^n)$, $(n \geq 3)$, where $P_2(n) = \frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})2^d$ and μ is the Mobius function.*

Proof. In the relation $x^n = yxy^{a_{n-1}}xy^{a_{n-2}}...xy^{a_0}$ of Sinkov's presentation, every choice of a_0, a_1,...,a_{n-1}, yields an irreducible polynomial over $GF(2)$ of degree n. On the other hand by the elementary results of [12], the number of such polynomials is $P_2(n) = \frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})2^d$, where μ is the Mobius function. More precisely, the named polynomial should be in the form $m(x) = x^n + \sum_{1\leq i\leq n-1} a_i x^i$, where, for at least a primitive α of $GF(2^n)$, $m(\alpha) = 0$. So, the number of distinct presentations for $PSL(2, 2^n)$ is $P_2(n)$. □

Lemma 3.2. *For every integer $n \geq 3$, the last relation of the presentation of $PSL(2, 2^n)$ will be reduced to $x^n = yx^{n-1}yxy$ or $x^n = yx^{n-2}yx^2y$, if n is even either is odd.*

Proof. For $n = 3$, $P_2(3) = 2$ (the number of irreducible polynomials of degree 3 over $GF(2)$) and one of these polynomials is the trinomial $m(x) = x^3 + x^2 + 1$. For $n = 4$, $P_2(4) = 3$ and one of these polynomials is the trinomial $m(x) = x^4 + x + 1$. On the other hand by using the results of [14] we deduce that, for every integer $n \geq 3$, at least one of the irreducible polynomials of degree n is a trinomial, and this trinomial is in the form $m(x) = x^n + x^2 + 1$ or $m(x) = x^n + x + 1$ when n is odd either n is even, respectively. Now, by considering the coefficients of this trinomials we see that the relation $x^n = yxy^{a_{n-1}}xy^{a_{n-2}}...xy^{a_0}$ for even values of n is equal to $x^n = yx^{n-1}yxy$ and for the odd values of n is equal to $x^n = yx^{n-2}yx^2y$. □

Lemma 3.3. *Let $n \geq 3$. By considering the types of maximal subgroups of $G = PSL(2, 2^n)$, if the subgroup H is of type $E_{2^n} \ltimes Z_{2^n-1}$ and the subgroup K is of type $D_{2(2^n+1)}$ or $D_{2(2^n-1)}$ then, there exist elements $h \in H$, $k \in K$ and $g \in G$ such that $Hgh = Hk$.*

Proof. For every integer $n \geq 3$, consider the maximal subgroups $H = E_{2^n} \ltimes Z_{2^n-1}$ and $K = D_{2(2^n+1)}$. For every elements $g \in G$, $h \in H$ and

$k \in K$ if $Hgh \neq Hk$, then $Hghk^{-1} \neq H$. Indeed, for every elements g, h and k from G, H and K, the element ghk^{-1} doesn't belong to H, which is a contraction, because for three elements h, k and $g' = hkh^{-1}$ from H, K and G, $g'hk^{-1} = (hkh^{-1})hk^{-1} = h \in H$. So, there exist elements $h \in H$, $k \in K$ and $g \in G$ such that $Hgh = Hk$. $\square$

Proof of Theorem B.

For every integer $n \geq 3$, the subgroups $A = E_{2^n} \ltimes Z_{2^n-1}$, $B = D_{2(2^n+1)}$ and $C = D_{2(2^n-1)}$ are three maximal subgroup of $G = PSL(2, 2^n)$ where, the indices of these subgroups are $2^n + 1$, $\frac{2^n(2^n-1)}{2}$ and $\frac{2^n(2^n+1)}{2}$, respectively. For these subgroups we give generators in terms of the original generators x, y and z as follows:

$A = E_{2^n} \ltimes Z_{2^n-1} \simeq < [x, y], [x^2, y], ..., [x^{n-1}, y], y, x >, \quad n \geq 3,$
$B = D_{2(2^n+1)} \simeq < xz, x^{-n}yx^{n+1}z >, \quad n \geq 4,$
$(B = D_{2(2^3+1)} \simeq < xz, x^{-1}yx^2z >),$
$C = D_{2(2^n-1)} \simeq < xz, y^{-1}xyz >, \quad n \geq 3,$

where, $A \cap B$ and $A \cap C$ are non-empty, and the equation $Aga = Ab$ holds for at least three nontrivial elements $g \in PSL(2, 2^n)$, $a \in A$ and $b \in B$. Thus by Lemmas 2.3 and 3.3, the triple $(PSL(2, 2^n), A, B)$ is a triple factorization for the group $PSL(2, 2^n)$, i.e.; $PSL(2, 2^n) = ABA$.

Moreover, if the triple $(PSL(2, 2^n), A, B)$ is a triple factorization of $PSL(2, 2^n)$ on two maximal subgroups $A = E_{2^n} \ltimes Z_{2^n-1}$ and $B = D_{2(2^n+1)}$ then, by Higman-Mclaughlin ([10]), every triple factorization $PSL(2, 2^n) = ABA$ gives a G-flag-transitive rank-two coset geometry $(P \cup L, *) = Cos(PSL(2, 2^n), A, B)$ where, P and L are the set of right cosets of maximal subgroups A and B, and G acts transitively on the elements of P and L. So G acts on the flags of rank-two coset geometry as follows:

$(Ax, By)^g = (Axg, Byg)$, $g \in G$, $Ax \in P$, $By \in L$.

Thus, for this rank-two coset geometry we can define a rank-two coset geometry graph, and $\Gamma(PSL(2, 2^n), A, B) = K_{r,s}$, where $r = 2^n + 1$ and $s = \frac{2^n(2^n-1)}{2}$ are the indices of the maximal subgroups A and B, respectively. $\square$

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