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ON A SPECIAL CLASS OF STANLEY-REISNER IDEALS

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Abstract. For an n-gon with vertices at points $1, 2, \cdots, n$, the Betti numbers of its suspension, the simplicial complex that involves two more vertices $n+1$ and $n+2$, is known. In this paper, with a constructive and simple proof, we generalize this result to find the minimal free resolution and Betti numbers of the S-module S/I where $S = K[x_1, \cdots, x_n]$ and I is the associated ideal to the generalized suspension of it in the Stanley-Reisner sense. Applications to Stanley-Reisner ideals and simplicial complexes are considered.

Introduction

Let $S = K[x_1, \cdots, x_n]$ be a polynomial ring over a field K. In [1, 2] Alwis considered the general n-gon with vertices at the points $1, 2, \cdots, n$. For its suspension, i.e., the simplicial complex that involves two more vertices ($n+1$ and $n+2$), he found the minimal free resolution and the Betti numbers of the S-module S/I where I is the associated ideal to the suspension in the Stanley-Reisner sense. In this paper, we generalize this result to sum of certain graded ideals over a graded ring. More precisely, let J_1 be an ideal of S and

$$0 \to S^{\beta_c^S} \xrightarrow{f_c} S^{\beta_{c-1}^S} \to \ldots \to S^{\beta_1^S} \xrightarrow{f_1} S^{\beta_0^S} \xrightarrow{f_0} \frac{S}{J_1} \to 0$$

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be the minimal free resolution of the S-module S/J_1. Let $x_{n+1}, \dots, x_{n+r}$ be r indeterminates over S, for some non-negative integer r, and $R = K[x_1, \dots, x_{n+r}]$. We construct the minimal free resolution of the R-module R/I where $I = J_1R + (y)$ and y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$. In other words, in Theorem 1.1 we show that the following is the minimal free resolution for R/I

$$0 \to R^{\beta_c^S} \xrightarrow{\delta_{c+1}} R^{\beta_c^S} \oplus R^{\beta_{c-1}^S} \to \dots \to R^{\beta_1^S} \oplus R^{\beta_0^S} \xrightarrow{\delta_1} R^{\beta_0^S} \to R/I \to 0.$$

By an inductive argument we may generalize the ideal (y) to $(f_1, \cdots, f_n)$ where f_i is any homogenous polynomial in $K[x_{r_i}, \cdots, x_{r_{i+1}}]$ for $r_1 < r_2 < \cdots < r_t$.

Notice that y is regular modulo the ideal J_1 and using the short exact sequence $0 \to R/J_1R \xrightarrow{*} R/J_1R \to R/I \to 0$, where the map $\xrightarrow{*}$ is simply multiplication with y which is clearly injective, the concatenation of this one-step resolution of R/I with the given resolution of S/J_1 will be obtained by mapping cone. We will also construct the minimal resolution of the R-module R/I. Furthermore we obtain the graded version of this result and compute the maps in the linear resolution explicitly in Theorem 1.1. As an application, let I be a graded ideal of S such that S/I is Cohen-Macaulay with a pure resolution where its Betti numbers are given in [3, Theorem 4.1.15]. Then in Corollary 1.7 we have the Betti numbers of ideal $J := I + (y)$ where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$.

Section 2 is devoted to further analysis of a special class of Stanley-Reisner ideals. In fact we assume that $I = (z_1, \cdots, z_t)$, where $z_i = \prod_{j=1}^{k_i} x_{i_j}$ and that each x_{i_j} occurs only once in I. Now the Betti numbers of R/I can be easily obtained from Theorem 1.1. It can also be seen from the fact that I is generated by a regular sequence and using Koszul complex. We analysis this certain family of ideals in terms of simplicial complexes. Let Δ be the simplicial complex corresponding to I. From the primary decomposition of I we see that Δ is pure of dimension $n - t - 1$. In fact it is consisting of $k_1 \cdots k_t$ facets all of dimension $n - t - 1$. Furthermore, the ideal I is perfect unmixed and R/I is a Cohen-Macaulay ring. By a result of Eagon, Reiner and a result of Terai we deduced that the regularity of R/I_{Δ^*} is $\operatorname{reg}(R/I_{\Delta^*}) = \text{proj.dim} R/I - 1$, where Δ^* is the Alexander dual of Δ. On the other hand, the regularity of R/I is $k_1 + \cdots + k_t - t$; see [4, Theorem 4.0]. Finally we provide some concrete examples to verify our results.

1. Main result; the ordinary and graded versions

We remind the reader of the concept of *mapping cone,* [5, pp. 650]. Given a morphism $\alpha : F \longrightarrow G$ of two complexes (F, φ) and (G, ψ) the mapping cone $M := M(\alpha)$ of α is the complex such that $M(\alpha)_i = F_{i-1} \oplus G_i$, with differential

$$F_i \oplus G_{i+1} \overset{\sigma_{i+1}}{\to} F_{i-1} \oplus G_i,$$

where $\sigma_{i+1} = \begin{pmatrix} -\varphi_i & \alpha_i \\ \psi_{i+1} & 0 \end{pmatrix}$, that is, on G_{i+1} the map is the differential of G, but on F_i the map is the sum of the differential of F and the given map α of complexes.

Similar to [1, Theorem 3.1] we can prove the following result. For the sake of convenience of reader we include the sketch of the proof.

Theorem 1.1. *Let J_1 be a homogenous ideal of the polynomial ring $S = K[x_1, \ldots, x_n]$. Let*

$$0 \to S^{\beta_c^S} \xrightarrow{f_c} S^{\beta_{c-1}^S} \to \ldots \to S^{\beta_1^S} \xrightarrow{f_1} S^{\beta_0^S} \xrightarrow{f_0} \frac{S}{J_1} \to 0 \tag{1.1}$$

be the minimal free resolution of the S-module S/J_1 with appropriate boundary maps. Let $x_{n+1}, \ldots, x_{n+r}$ be r indeterminate over S, for some non-negative integer r, and $R = K[x_1, \ldots, x_{n+r}]$. Then the following is the minimal free resolution of the R-module R/I where $I = J_1R + (y)$ and y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$:

$$0 \to R^{\beta_c^S} \xrightarrow{\delta_{c+1}} R^{\beta_c^S} \oplus R^{\beta_{c-1}^S} \to \ldots \to R^{\beta_1^S} \oplus R^{\beta_0^S} \xrightarrow{\delta_1} R^{\beta_0^S} \to R/I \to 0. \tag{1.2}$$

Proof. Let $J = J_1R$. Tensoring the exact sequence (1.1) with the K-module $K[x_{n+1}, \ldots, x_{n+r}]$ we exhibit a complex which is exact at all places except at degree 0. Then we consider the mapping cone of the following double complex where the two rows are the same and the vertical maps are multiplications by y:

$$\begin{tikzcd} 0 \arrow[r] & R^{\beta_c^S} \arrow[r, "\mathrm{d_c}"] \arrow[d, "\mathrm{y}"] & R^{\beta_{c-1}^S} \arrow[r] \arrow[d, "\mathrm{y}"] & \ldots \arrow[r] \arrow[d, "\mathrm{y}"] & R^{\beta_1^S} \arrow[r, "\mathrm{d_1}"] \arrow[d, "\mathrm{y}"] & R^{\beta_0^S} \arrow[r] \arrow[d, "\mathrm{y}"] & 0 \\ 0 \arrow[r] & R^{\beta_c^S} \arrow[r, "\mathrm{d_c}"] & R^{\beta_{c-1}^S} \arrow[r] & \ldots \arrow[r] & R^{\beta_1^S} \arrow[r, "\mathrm{d_1}"] & R^{\beta_0^S} \arrow[r] & 0 \end{tikzcd} \tag{1.3}$$

That is we have the following complex

$$0 \to R^{\beta_c^S} \xrightarrow{\delta_{c+1}} R^{\beta_c^S} \oplus R^{\beta_{c-1}^S} \to \ldots \to R^{\beta_1^S} \oplus R^{\beta_0^S} \xrightarrow{\delta_1} R^{\beta_0^S} \to 0 \tag{1.4}$$

where $\delta_i : R^{\beta_i^S} \oplus R^{\beta_{i-1}^S} \longrightarrow R^{\beta_{i-1}^S} \oplus R^{\beta_{i-2}^S}, i = 1, 2, \ldots, c+1$ is given by $\delta_i(p, q) = (d_i(p) + (-1)^i yq, d_{i-1}(q))$ for $i = 2, 3, \ldots, c$, and, $\delta_1(p, q) = (d_1(p) + (-1)yq, 0)$,
$\delta_{c+1}(p, q) = ((-1)^{c+1}yq, d_c(q))$. Now it is easy to see that $\delta_{i-1} \circ \delta_i = 0$. The proof is complete if we show the exactness and the minimality of (1.4). To this end note that (1.4) is exact at all places except degree 0 where its homology is R/I and the minimality is obtained directly from the minimality of (1.1). □

The following consequences are now immediate:

Corollary 1.2. *Let $S = K[x_1, \ldots, x_n]$, J_1 any homogenous ideal of S for which proj.dim$(S/J_1) = c$. Then for the ideal $I = J_1R + (y)$ of $R = S[x_1, \ldots, x_{n+r}]$ where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$. Then the i^{th} Betti number R/I, $\beta_i^R(R/I)$, is given by*

$$\beta_i^R(R/I) = \begin{cases} \beta_0^S(S/J_1), & i=0, \\ \beta_i^S(S/J_1) + \beta_{i-1}^S(S/J_1), & i=1,2,\ldots,c, \\ \beta_c^S(S/J_1), & i=c+1, \\ 0, & otherwise. \end{cases} \tag{1.5}$$

Example 1.3. *Let $S = K[x_1, \ldots, x_3]$ and $J_3 = (x_1^3, x_2x_3^3)$ be an ideal of S. Then $I_3 = (x_1^3, x_2x_3^3, x_4^2)$ is an ideal of $R = K[x_1, \ldots, x_4]$. Now (1.5) enables us to compute the Betti numbers of R/I_3 in terms of the Betti numers of S/J_3, i.e.,*

$$\begin{cases} \beta_0^R(R/I_3) = \beta_0^S(S/J_3) = 1, \\ \beta_1^R(R/I_3) = \beta_1^S(S/J_3) + \beta_0^S(S/J_3) = 2 + 1 = 3, \\ \beta_2^R(R/I_3) = \beta_2^S(S/J_3) + \beta_1^S(S/J_3) = 1 + 2 = 3, \\ \beta_3^R(R/I_3) = \beta_2^S(S/J_3) = 1. \end{cases}$$

One can note that in order to compute $\beta_i^S(S/J_3)$ for $i = 0, 1, 2$, we apply (1.5) once again to $S = K[x_1]$ and $J_1 = (x_1^3)$.

Remark 1.4. *By an inductive argument the ideal (y) in Theorem 1.1 can be extended to $(f_1, \cdots, f_t)$ where f_i is any homogenous polynomial in $K[x_{r_i}, \cdots, x_{r_{i+1}}]$ for $r_1 < r_2 < \cdots < r_t$. So, our general result as we mentioned in the abstract can be obtained from this observation.*

In the following we have the graded version of our Theorem 1.1.

Theorem 1.5. *Let $S = K[x_1, \cdots, x_n]$ be a polynomial ring over a field K, and let J be a graded ideal of S with the (minimal) graded free resolution*

$$0 \longrightarrow \oplus S(-a_{cj})^{\beta_{cj}} \longrightarrow \cdots \longrightarrow \oplus S(-a_{1j})^{\beta_{1j}} \longrightarrow S \longrightarrow S/J \longrightarrow 0, \tag{1.6}$$

then for the ring $R = K[x_1, \cdots, x_{n+r}]$ and its ideal $I := JR + (y)$, where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$ of degree $e = \sum_{i=n+1}^{n+r} \alpha_i$, the (minimal) graded free resolution of R/I is as follows:

$$\begin{aligned} 0 \longrightarrow \oplus R(-a_{cj} - e)^{\beta_{cj}} &\longrightarrow \oplus R(-a_{cj})^{\beta_{cj}} \bigoplus \oplus R(-a_{c-1\,j} - e)^{\beta_{c-1\,j}} \longrightarrow \\ &\cdots \longrightarrow \oplus R(-a_{2j})^{\beta_{2j}} \bigoplus \oplus R(-a_{1j} - e)^{\beta_{1j}} \longrightarrow \\ &\oplus R(-a_{1j})^{\beta_{1j}} \bigoplus R(-e) \longrightarrow R \longrightarrow R/I \longrightarrow 0. \end{aligned} \tag{1.7}$$

Proof. For a moment ignore the graded settings. Then by Theorem 1.1 the desired resolution is obtained provided (1.6) is a free resolution of S/J_1. Furthermore (1.7) is minimal as long as (1.6) is minimal.
It only remains to verify that in (1.7) the maps are zero maps, that is they preserve the degree. But this simple matter of checking holds due to the formulation of differentials in the new resolution:

$$\begin{aligned} &\delta_i(p, q) = (d_i(p) + (-1)^i yq, d_{i-1}(q)) \quad \text{for } i = 2, 3, \ldots, c, \text{ and,} \\ &\delta_1(p, q) = (d_1(p) + (-1)yq, 0), \\ &\delta_{c+1}(p, q) = ((-1)^{c+1} yq, d_c(q)). \end{aligned}$$

□

As an example of Theorem 1.5 we have:

Example 1.6. *Let $R = Q[x_1, \cdots, x_7]$. It is easy to see that for the ideal $I_1 := (x_1, x_2^2)$, the minimal free resolution of R/I_1 is*

$$0 \longrightarrow R(-3) \longrightarrow R(-1) \oplus R(-2) \longrightarrow R \longrightarrow R/I_1 \longrightarrow 0$$

In the following we compute the minimal free resolution of some new ideals:
(i) Let $I_2 := I_1 + (x_3^5)$. Then for R/I_2 we get

$$\begin{aligned} 0 \longrightarrow R(-8) \longrightarrow & R(-3) \oplus R(-6) \oplus R(-7) \longrightarrow \\ & R(-1) \oplus R(-2) \oplus R(-5) \longrightarrow R \longrightarrow R/I_2 \longrightarrow 0. \end{aligned}$$

(ii) For $I_3 := I_2 + (x_4^4) = (x_1, x_2^2, x_3^5, x_4^4)$, the minimal free resolution of R/I_3 is

$$\begin{aligned} 0 \longrightarrow R(-12) \longrightarrow & R(-7) \oplus R(-8) \oplus R(-10) \oplus R(-11) \longrightarrow \\ & R(-3) \oplus R(-5) \oplus R^2(-6) \oplus R(-7) \oplus R(-9) \longrightarrow \\ & R(-1) \oplus R(-2) \oplus R(-4) \oplus R(-5) \longrightarrow R \longrightarrow \\ & R/I_3 \longrightarrow 0. \end{aligned}$$

(iii) Finally for $I_4 := I_2 + (x_4x_5) = (x_1, x_2^2, x_3^5, x_4x_5)$ *we obtain*

$$\begin{aligned} 0 \longrightarrow R(-10) \longrightarrow & R(-5) \oplus R^2(-8) \oplus R(-9) \longrightarrow \\ & R^2(-3) \oplus R(-4) \oplus R(-6) \oplus R^2(-7) \longrightarrow \\ & R(-1) \oplus R^2(-2) \oplus R(-5) \longrightarrow R \longrightarrow R/I_4 \longrightarrow 0. \end{aligned}$$

By [3, Theorem 4.1.15] for a graded ideal I of a polynomial ring $S = K[x_1, \cdots, x_n]$ over a field K such that S/I is Cohen-Macaulay with a pure resolution of type $(d_1, \cdots, d_p)$ its Betti numbers are given by this formula

$$\beta_i^S(S/I) = (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{(d_j - d_i)}.$$

Now as an application of Corollary 1.2 we can compute the Betti numbers of the ideals in the following form. In fact set $R = K[x_1, \cdots, x_{n+r}]$ and $J := I + (y)$ where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$.

Corollary 1.7. *With the notations as above we have* $\beta_0^R(R/J) = \beta_0^S(S/I) = 1$,

$$\beta_i^R(R/J) = (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{(d_j - d_i)} + (-1)^i \prod_{j \neq i-1} \frac{d_j}{(d_j - d_{i-1})}$$

for $i = 1, \cdots, p$, $\beta_{p+1}^R(R/J) = \beta_p^S(S/I)$ *and* $\beta_i^R(R/J) = 0$ *for* $i > p+1$.

2. Analysis of a special class of Stanley-Reisner ideals

This section is devoted to further analysis of a special class of Stanley-Reisner ideals which is complete intersection square free monomial ideals or Stanley-Reisner ideals of complete intersection simplicial complexes. The results of this section are known, but as an application of Theorem 1.1 we will reprove them. We mention that this analysis can be done in different ways but here our aim is to avoid using complex ideas and try to use simple tools as possible so that they can be followed without much difficulty. Our special plan in the future works is to make invariants of such ideals (and new derived classes) computable by means of computer programs. We assume that $I = (z_1, \cdots, z_t)$, where $z_i = \prod_{j=1}^{k_i} x_{i_j}$ and that each x_{i_j} occurs only once in I. Now the Betti numbers of R/I can be easily obtained from Theorem 1.1. We analysis this certain family of ideals in terms of simplicial complexes in Theorem 2.2 and subsequent results.

A simplicial complex Δ over a set of vertices $V = \{v_1, \cdots, v_n\}$ is a collection of subsets of V, for which $\{v_i\} \in \Delta$ for all i and if $F \in \Delta$ then all subsets of F are also in Δ. An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called vertices and edges, respectively, and $\dim\emptyset = -1$. The maximal faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets. Let Δ be a simplicial complex on the vertex set $V = \{v_1, \cdots, v_n\}$, and K be a field. The *Stanley-Reisner* ring of the complex Δ is the graded K-algebra $K[\Delta] = K[X_1, \cdots, X_n]/I_\Delta$, where I_Δ is the ideal generated by all monomials $X_{i_1}X_{i_2}\cdots X_{i_k}$ such that $\{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\} \notin \Delta$. The dimension of a Stanley-Reisner ring can be easily determined. For a proof of the following result see [3, Theorem 5.1.4] for instance.

Theorem 2.1. *Given a simplicial complex Δ, in order to reach I_Δ we may use the primary decomposition of the Stanley-Reisner ideal of Δ*

$$I_\Delta = \bigcap_F P_F,$$

where the intersection is taken over all facets F of Δ, and P_F denotes the face ideal generated by all x_i such that $x_i \notin F$. In particular, $\dim K[\Delta] = \dim R/I_\Delta = \dim\Delta + 1$.

The simplicial complex Δ is said to be *pure* if all its facets are of the same dimension, namely $\dim\Delta$. A Cohen-Macaulay simplicial complex is pure. Our terminology and comments comes from [3, 8, 10].

Let Δ be the following simplicial complex which corresponds to the n-gon with vertices at the points $1, 2, \ldots, n$. Clearly Δ is a pure simplical complex (of dimension 1).

$$\Delta = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}, \{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}\}. \tag{2.1}$$

Let $S = K[x_1, x_2, \ldots, x_n]$, and let J_1 be the Stanley-Reisner ideal associated to Δ in (2.1), i.e., J_1= the ideal in S generated by all monomials of the form $x_{i_1}x_{i_2}\ldots x_{i_r}$, where $1 \le i_1 < i_2 < \ldots < i_r \le n$ and $\{i_1, \ldots, i_r\} \notin \Delta$. Then it easily follows that for each $n \ge 3$ we get:

$$J_1 = \begin{cases} (x_1x_2x_3), & \text{n=3;} \\ (x_1x_3, x_1x_4, \cdots, x_1x_{n-1}, x_2x_4, \cdots, x_2x_n, \cdots, x_{n-2}x_n), & \text{otherwise.} \end{cases}$$

In [2] the author showed that the ith Betti number of the S-module S/J_1, denoted by $\beta_i^S(S/J_1)$ or simply β_i^S, which is the ith Betti number of the n-gon, for $n \ge 3$ is given by

$$\beta_i^S = \begin{cases} 1, & \text{i=0,} \\ \binom{n}{i+1}\frac{i(n-i-2)}{n-1}, & \text{i=1,2,\dots,n-3,} \\ 1, & \text{i=n-2,} \\ 0, & \text{otherwise.} \end{cases}$$

As well we have

$$0 \to S^{\beta_{n-2}^S} \xrightarrow{f_{n-2}} S^{\beta_{n-3}^S} \to \dots \to S^{\beta_1^S} \xrightarrow{f_1} S^{\beta_0^S} \xrightarrow{f_0} \frac{S}{J_1} \to 0$$

is the minimal free resolution of the S-module S/J_1 with appropriate boundary maps.

As a consequence of Theorem 1.1 we compute the Betti numbers of a special class of Stanley-Reisner ideals which can be obtained from the *Koszul complex* as it is shown in the proof of the following theorem.

We analysis this certain family of ideals in terms of simplicial complexes.

Theorem 2.2. *Let Δ be a simplicial complex for which $I := I_\Delta = (z_1, \cdots, z_t)$, where $z_i = \prod_{j=1}^{k_i} x_{i_j}$ and that each x_{i_j} occurs only once in I_Δ. Then we have:*

(i) *The Betti numbers of I are given by the following formula:*

$$\beta_i^R(R/I) = \begin{cases} 1, & i=0, \\ \binom{t}{i}, & i=1,2,\dots,t-1, \\ 1, & i=t, \\ 0, & \textit{otherwise.} \end{cases}$$

(ii) *I is perfect, unmixed and also R/I is Cohen-Macaulay.*

Proof. We prove (i) by induction on t. Since for $t = 1$, I is just of the form $I = (x_1^{\alpha_1} \cdots x_s^{\alpha_s})$ for some s, where $\alpha_i \in \{0, 1\}$. So one has

$$\beta_i^R(R/I) = \begin{cases} 1, & \text{i=0,} \\ 1, & \text{i=1,} \\ 0, & \text{otherwise.} \end{cases}$$

Now let $t > 1$, and assume that the case $t - 1$ is settled. Take $S = k[x_{i_j} : i = 1, \cdots, t-1]$. Consider the ideal $J = (z_1, \cdots, z_{t-1})$ of S. Then by induction hypothesis we have

$$\beta_i^S(S/J) = \begin{cases} 1, & \text{i=0,} \\ \binom{t-1}{i}, & \text{i=1,2,\dots,t-2,} \\ 1, & \text{i=t-1,} \\ 0, & \text{otherwise.} \end{cases}$$

Formula (1.5) implies that

$$\beta_i^R(R/I) = \begin{cases} 1, & \text{i=0,} \\ \binom{t-1}{i} + \binom{t-1}{i-1} = \binom{t}{i}, & \text{i=1,2,\ldots,t-1,} \\ 1, & \text{i=t,} \\ 0, & \text{otherwise.} \end{cases}$$

For the proof (ii) we note that by Theorem 2.1 Δ is pure of dimension $n-t-1$. In fact it is consisting of $k_1 \cdots k_t$ facets all of dimension $n-t-1$. Hence,

$$\dim R/I = \dim\Delta + 1 = n - t - 1 + 1 = n - t = \dim R - t.$$

Then by [3, Theorem 2.1.2 (c)] it follows that $z_1, \cdots, z_t$ is a regular sequence on R.
Furthermore, by the Auslander-Buchsbaum formula we have

$$\operatorname{depth} R/I = \operatorname{depth} R - \operatorname{proj.dim} R/I = n - t,$$

hence the ring R/I is Cohen-Macaulay and so Δ is Cohen-Macaulay. In addition, I is perfect, i.e., we have

$$\operatorname{grade} I = \operatorname{height} I = \dim R - \dim R/I = t = \operatorname{proj.dim} R/I$$

see [3, Corollary 2.1.4]. The first equality can also be seen from the primary decomposition of I and [3, Proposition 1.2.10 (c)].
Finally let $p_1, \ldots, p_r$ be the prime ideals in the primary decomposition of I. Since I is generated by $t = \operatorname{height} I$ elements over the polynomial ring R, I is unmixed; see [7]. Hence $p_1, \ldots, p_r$ are the minimal prime ideals of I; see [3, Theorem 2.1.6]. Thus $\operatorname{Ass}(R/I) = \{p_1, \ldots, p_r\}$. □

Remark 2.3. The ideal I is generated by a regular sequence on R. Thus the Castelnuovo-Mumford regularity of R/I is $k_1 + \cdots + k_t - t$; see [4, Theorem 4.0].

Let Δ be a simplicial complex and Δ^* denote the Alexander dual of Δ, i.e., the simplicial complex

$$\Delta^* = \{F \subseteq [n] : [n] - F \notin \Delta\}$$

Corollary 2.4. *Consider the graded version of Theorem 2.2. Then the regularity of R/I_{Δ^*} is*

$$\mathit{reg}\,(R/I_{\Delta^*}) = \mathit{proj.dim} R/I - 1.$$

Proof. Using the primary decomposition of I_{Δ^*} we have

$$I_{\Delta^*} = (x_{1,1}, \ldots, x_{1,t_1}) \cap \cdots \cap (x_{s,1}, \ldots, x_{s,t_s}).$$

By a known result of Eagon and Reiner, $K[\Delta]$ is Cohen-Macaulay if and only if I_{Δ^*} has a linear resolution. Furthermore, $\operatorname{proj.dim}(K[\Delta]) =$

$\operatorname{reg}(I_{\Delta^*})$ by a result of Terai. In view of Theorem 1.5, R/I is Cohen-Macaulay and proj.dim$(K[\Delta]) = t$. Therefore, $\operatorname{reg}(I_{\Delta^*}) = t$ and so $\operatorname{reg}(R/I_{\Delta^*}) = t - 1$. $\square$

In the following we have some examples.

Example 2.5. *Let $S = K[x_1, x_2]$ and $J = (x_1x_2)$ be an ideal of S. Obviously we have $\beta_0^S(S/J) = 1, \beta_1^S(S/J) = 1$.*

Now let $I_0 = (x_1x_2, x_3x_4)$ be an ideal of $R = K[x_1, \ldots, x_4]$. Then (1.5) implies that

$$\begin{cases} \beta_0^R(R/I_0) = \beta_0^S(S/J) = 1, \\ \beta_1^R(R/I_0) = \beta_1^S(S/J) + \beta_0^S(S/J) = 1 + 1 = 2, \\ \beta_2^R(R/I_0) = \beta_1^S(S/J) = 1. \end{cases}$$

Furthermore, applying [3, Excersice 4.4.16 (b)], *it is easy to see that*

$$\begin{aligned} I_0 =& (x_1x_2, x_3x_4) = (x_1, x_3x_4) \cap (x_2, x_3x_4) \\ =& (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_2, x_4), \end{aligned}$$

Hence I_0 is the Stanley-Reisner ideal of a pure simplicial complex Δ_0 consisting of 4 facets all of dimension 1. As a result

$$\dim K[\Delta_0] = \dim R/I_0 = \dim \Delta_0 + 1 = 1 + 1 = 2. \quad \square$$

Example 2.6. *Let $S = K[x_1, \ldots, x_4]$ and $J_1 = (x_1x_3, x_2x_4)$ be an ideal of S. Then $I_1 = (x_1x_3, x_2x_4, x_5x_6)$ is an ideal of $R = K[x_1, \ldots, x_6]$ and using Example 2.5 we have*

$$\begin{cases} \beta_0^R(R/I_1) = \beta_0^S(S/J_1) = 1, \\ \beta_1^R(R/I_1) = \beta_1^S(S/J_1) + \beta_0^S(S/J_1) = 2 + 1 = 3, \\ \beta_2^R(R/I_1) = \beta_2^S(S/J_1) + \beta_1^S(S/J_1) = 1 + 2 = 3, \\ \beta_3^R(R/I_1) = \beta_2^S(S/J_1) = 1. \end{cases}$$

Furthermore, by the help of [3, Excersice 4.4.16 (b)]

$$\begin{aligned} I_1 =& (x_1x_3, x_2x_4, x_5x_6) = (x_1, x_2x_4, x_5x_6) \cap (x_3, x_2x_4, x_5x_6) \\ =& (x_1, x_2, x_5x_6) \cap (x_1, x_4, x_5x_6) \cap (x_3, x_2, x_5x_6) \cap (x_3, x_4, x_5x_6) \\ =& (x_1, x_2, x_5) \cap (x_1, x_2, x_6) \cap (x_1, x_4, x_5) \cap (x_1, x_4, x_6) \cap (x_3, x_2, x_5) \\ & \cap (x_3, x_2, x_6) \cap (x_3, x_4, x_5) \cap (x_3, x_4, x_6). \end{aligned}$$

Thus I_1 is the Stanley-Reisner ideal of a pure simplicial complex Δ_1 which consists of 8 facets all of dimension 2. One can easily see that

$$\dim K[\Delta_1] = \dim R/I_1 = \dim \Delta_1 + 1 = 2 + 1 = 3. \quad \square$$

Example 2.7. Let $S = K[x_1, \dots, x_4]$ and $J_2 = (x_1x_3, x_2x_4)$ be an ideal of S. Then $I_2 = (x_1x_3, x_2x_4, x_5x_6x_7)$ is an ideal of $R = K[x_1, \dots, x_7]$ and similar to Example 2.6 we have

$$\beta_0^R(R/I_2) = 1, \beta_1^R(R/I_2) = 3,\ \beta_2^R(R/I_2) = 3, \text{ and } \beta_3^R(R/I_2) = 1.$$

Furthermore, using [3, Excersice 4.4.16 (b)]

$$\begin{aligned} I_2 =&(x_1x_3, x_2x_4, x_5x_6x_7) = (x_1, x_2x_4, x_5x_6x_7) \cap (x_3, x_2x_4, x_5x_6x_7) \\ =&(x_1, x_2, x_5x_6x_7) \cap (x_1, x_4, x_5x_6x_7) \cap (x_3, x_2, x_5x_6x_7) \cap (x_3, x_4, x_5x_6x_7) \\ =&(x_1, x_2, x_5) \cap (x_1, x_2, x_6) \cap (x_1, x_2, x_7) \cap (x_1, x_4, x_5) \cap (x_1, x_4, x_6) \\ &\cap (x_1, x_4, x_7) \cap (x_3, x_2, x_5) \cap (x_3, x_2, x_6) \cap (x_3, x_2, x_7) \cap (x_3, x_4, x_5) \\ &\cap (x_3, x_4, x_6) \cap (x_3, x_4, x_7). \end{aligned}$$

Hence I_2 is the Stanley-Reisner ideal of a pure simplicial complex Δ_2 which consists of 12 facets all of dimension 3. One can easily see that

$$\dim K[\Delta_2] = \dim R/I_2 = \dim \Delta_2 + 1 = 3 + 1 = 4. \quad \square$$

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