Journal of Algebra and Related Topics
Vol. 2, No 2, (2014), pp 43-61

FIRST NON-ABELIAN COHOMOLOGY OF TOPOLOGICAL GROUPS II

H. SAHLEH * AND H. E. KOSHKOSHI

Abstract. In this paper we introduce a new definition of the first non-abelian cohomology of topological groups. We relate the cohomology of a normal subgroup N of a topological group G and the quotient G/N to the cohomology of G. We get the inflation-restriction exact sequence. Also, we obtain a seven-term exact cohomology sequence up to dimension 2. We give an interpretation of the first non-abelian cohomology of a topological group by the notion of a principle homogeneous space.

1. Introduction

The first non-abelian cohomology of group G with coefficients in a crossed G-module (A, μ) was (algebraically) introduced by Guin [1, 2]. The Guin's approach extended by H. Inassaridze to any dimension of non-abelian cohomology of G with coefficients in a (partially) crossed topological $G-R$-bimodule (A, μ) (see [4, 5, 6]). We generalize the Inassaridze's approach to define the first cohomology of non-abelain cohomology of topological groups. We continue to study non-abelian cohomology of topological groups (see [7, 8]).

In this paper, topological groups are not necessarily abelian, unless otherwise specified. Let G and A be topological groups. It is said that A is a topological G-module, whenever G acts continuously on the left of A. For all $g \in G$ and $a \in A$ we denote the action of g on a by ${}^{g}a$. If

MSC(2010): Primary: 22A05, 20J06; Secondary: 18G50
Keywords: Non-abelian cohomology of topological groups, cocompatible triple, partially crossed topological bimodule, principle homogeneous space.
Received: 9 August 2014, Accepted: 29 November 2014.
*Corresponding author.

A is a topological G-module, $H^1(G, A)$ denotes the first cohomology of G with coefficients in A [8]. The group isomorphism is denoted by $\cong$. The center and the commutator subgroup of a topological group G is denoted by $Z(G)$ and $[G, G]$, respectively. If the topological groups G and R act continuously on a topological group A, then the notation ${}^{gr}a$ means ${}^{g}({}^{r}a)$, $g \in G$, $r \in R$, $a \in A$. We assume that every topological group acts on itself by conjugation. The coboundary map δ_G^n is as in [3, Definition (3.1)]. If G and H are topological groups, then $\mathbf{1} : G \to H$ denotes the trivial homomorphism. If $f : G \to H$ is a map, then in what follows $f^{-1} : G \to H$ denotes a map given by $f^{-1}(g) = f(g)^{-1}$. Note that if P is a principle homogeneous space over a partially crossed topological $G-R$-bimodule (A, μ), then $e_p^{-1} : P \to A$ is the inverse map of the homeomorphism $e_p : A \to P$, $p \in P$ (see section 5).

In section 2, using the notion of partially crossed topological $G-R$-bimodule (A, μ) we define $Der_c(G, (A, \mu))$ as in [7] and the first non-abelian cohomology, $H^1(G, (A, \mu))$, of G with coefficients in (A, μ) as a quotient of $Der_c(G, (A, \mu))$ (see Definition 2.9). Every topological G-module A realises a (partially) crossed topological $G - A/Z(A)$-bimodule (A, π_A). So, we may define the first cohomology, $\bar{H}^1(G, A)$, of G with coefficients in A as follows:

$$\bar{H}^1(G, A) = H^1(G, (A, \pi_A))$$

In general this definition of the first non-abelian cohomology differs from the first pointed set cohomology, $H^1(G, A)$, which is defined in [8]. We show that there is a natural injection $\zeta : H^1(G, (A, \mu)) \to H^1(G, A)$. Thus, there is an injection $\bar{H}^1(G, A) \to H^1(G, A)$. As a result, we may identify $\bar{H}^1(G, A)$ with $H^1(G, A)$, whenever $H^1(G, A/Z(A))$ is null. In particular, if A is abelian, then $\bar{H}^1(G, A) \cong H^1(G, A)$.

In section 3, we introduce a new concept called *cocompatible triple* to study the change of groups. Using the cocompatible triple we define the inflation and the restriction maps and will show that for any normal subgroup N of G, there is an exact sequence

$$1 \longrightarrow H^1(G/N, (A^N, \mu^N)) \xrightarrow{Inf^1} H^1(G, (A, \mu)) \xrightarrow{Res^1} H^1(N, (A, \mu))^{G/N}$$

In section 4, we show that for every proper extension, $1 \to (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \to 1$, with continuous sections there exists a seven-term exact sequence,

$$0 \to H^0(G, A) \xrightarrow{\iota^0} H^0(G, B) \xrightarrow{\pi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A) \xrightarrow{\iota^1}$$
$$H^1(G, (B, \mu)) \xrightarrow{\pi^1} H^1(G, (C, \lambda)) \xrightarrow{\delta^1} H^2(G, A).$$

In section 5, we define a principle homogeneous space (or topological torsor) over a partially crossed topological $G-R$-bimodule (A, μ) and we show that the first non-abelian cohomology of G with coefficients in (A, μ) classifies the set of all principle homogeneous spaces over (A, μ). We denote by $\mathcal{P}(A, \mu)$ the set of all classes of principle homogeneous spaces over (A, μ). Thus, naturally there exists a group product on $\mathcal{P}(A, \mu)$, whenever (A, μ) satisfying conditions of Theorem 2.23. As a result, if (A, μ) is a topological crossed G-module, then $\mathcal{P}(A, \mu)$ is a group (not necessarily abelian).

2. Partially Crossed Topological Bimodules and the First Non-abelian Cohomology.

In this section, we introduce a partially crossed topological $G-R$-bimodule (A, μ) to define the first non-abelian cohomology, $H^1(G, (A, \mu))$, of G with coefficients in (A, μ).

Definition 2.1. A precrossed topological R-module (A, μ) consists of a topological R-module A and a continuous homomorphism $\mu : A \to R$ such that

$$\mu({}^r a) = {}^r\mu(a),\ r \in R, a \in A.$$

If in addition we have

$${}^{\mu(a)}b = {}^a b,\ a, b \in A,$$

then (A, μ) is called a crossed topological R-module.

Definition 2.2. A precrossed topological R-module (A, μ) is said to be a partially crossed topological R-module, whenever it satisfies the following equality

$${}^{\mu(a)}b = {}^a b,$$

for all $b \in A$ and for all $a \in A$ such that $\mu(a) \in [R, R]$.

It is clear that every crossed topological R-module is a partially crossed topological R-module.

Definition 2.3. Let G, R and A be topological groups. Precrossed topological R-module (A, μ) is said to be a precrossed topological $G-R$-bimodule, whenever

(1) G continuously acts on R and A;
(2) $\mu : A \to R$ is a G-homomorphism;
(3) ${}^{({}^g r)}a = {}^{grg^{-1}}a$ for all $g \in G$, $r \in R$ and $a \in A$.

Definition 2.4. A precrossed topological $G-R$-bimodule (A, μ) is said to be a (partially) crossed topological $G-R$-bimodule, if (A, μ) is a (partially) crossed topological R-module.

Example 2.5. (1) Let A be an arbitrary topological G-module. Obviously, $Z(A)$ is a topological G-module. Now, we define an action of $R = A/Z(A)$ on A and an action of G on R by:

$$^{aZ(A)}b = {}^{a}b, \forall a, b \in A, \quad {}^{g}(aZ(A)) = {}^{g}aZ(A), \forall g \in G, a \in A. \tag{2.1}$$

Let $\pi_A : A \to R$ be the canonical homomorphism. It is easy to see that under (2.1) the pair (A, π_A) is a crossed topological $G - R$-bimodule.

(2) By part (1), for any topological group G the pair (G, π_G) is a crossed topological $G - G/Z(G)$-bimodule.

Notice 2.6. It is obvious that any precrossed (crossed or partially crossed) topological R-module is naturally viewed as a precrossed (crossed or partially crossed) topological $R - R$-bimodule.

Definition 2.7. A morphism $f : (A, \mu) \to (B, \nu)$ of precrossed (crossed) topological $G - R$-bimodule is a continuous homomorphism of topological groups $f : A \to B$ such that

(1) $f({}^{r}a) = {}^{r}f(a)$, $r \in R$, $a \in A$;
(2) $f({}^{g}a) = {}^{g}f(a)$, $g \in G$, $a \in A$;
(3) $\mu = \nu \circ f$.

Definition 2.8. Let (A, μ) be a partially crossed topological $G - R$-bimodule. Denote by $Der(G, (A, \mu))$ the set of all pairs (α, r) where α is a crossed homomorphism from G into A, i.e., $\alpha(gh) = \alpha(g){}^{g}\alpha(h), \forall g, h \in G$; and r is an element of R such that

$$\mu \circ \alpha(g) = r{}^{g}r^{-1}, \forall g \in G.$$

Let $Der_c(G, (A, \mu))$ be a subset of $Der(G, (A, \mu))$ which is defined as follows:

$$Der_c(G, (A, \mu)) = \{(\alpha, r) \in Der(G, (A, \mu)) | \alpha \text{ is continuous}\}.$$

There is a group product $\star$ in $Der(G, (A, \mu))$ by $(\alpha, r) \star (\beta, s) = (\alpha * \beta, rs)$, where $\alpha * \beta(g) = {}^{r}\beta(g)\alpha(g), \forall g \in G$ (see [4, Proposition 1.5]). It is easy to see that $Der_c(G, (A, \mu))$ is a subgroup of $Der(G, (A, \mu))$.

Let R be a topological G-module, then we define

$$H^0(G, R) = \{r | {}^{g}r = r, \forall g \in G\}.$$

Let (A, μ) be a partially crossed topological $G - R$-bimodule. H. Inassaridze [4] introduced the equivalence relation $\sim$ on the group $Der(G, (A, \mu))$ as $(\alpha, r) \sim (\beta, s)$ whenever

$$\exists\ a \in A \wedge (\forall g \in G \Rightarrow \beta(g) = a^{-1}\alpha(g){}^{g}a \tag{2.2}$$

and

$$s = \mu(a)^{-1}r \ mod\ H^0(G, R) \tag{2.3}$$

Let $\sim'$ be the restriction of $\sim$ to $Der_c(G,(A,\mu))$. Therefore, $\sim'$ is an equivalence relation. In other word, $(\alpha,r) \sim' (\beta,s)$ if and only if $(\alpha,r) \sim (\beta,s)$, whenever $(\alpha,r),(\beta,s) \in Der_c(G,(A,\mu))$.

Let A be a topological G-module. We denote by $Inn(G,A)$ the set of all inner crossed homomorphisms from G into A, in other words:

$$Inn(G,A) = \{Inn(a)|a \in A, Inn(a)(g) = a^g a^{-1}\}$$

Similarly, if (A,μ) is a partially crossed topological $G-R$-bimodule, then put

$$Inn(G,(A,\mu)) = \{(Inn(a),\mu(a)z)|a \in A, z \in H^0(G,R)\}.$$

Definition 2.9. Let (A,μ) be a partially crossed topological $G-R$-bimodule. The quotient set $Der_c(G,(A,\mu))/\sim'$ will be called the first cohomology of G with the coefficients in (A,μ) and is denoted by $H^1(G,(A,\mu))$.

Fact 2.10. *Let (A,μ) is a partially crossed topological $G-R$-bimodule. Then*

(1) ${}^{zg}a = {}^{gz}a$, $\forall a \in A$, $g \in G$, $z \in H^0(G,R)$.
(2) $\alpha(g)^{gr}a = {}^{rg}a\alpha(g)$, $\forall g \in G, a \in A, (\alpha,r) \in Der_c(G,(A,\mu))$.

Proof. See [4, p. 499] and [5, p. 315]. □

Definition 2.11. Let A be a topological G-module. Denote by $\bar{H}^1(G,A)$ the first cohomology of G with coefficients in A and define it as follows:

$$\bar{H}^1(G,A) = H^1(G,(A,\pi_A)).$$

Let (A,μ) be a partially crossed topological $G-R$-bimodule, then there is the following natural map

$$\zeta : H^1(G,(A,\mu)) \to H^1(G,A)$$
$$[(\alpha,r)] \mapsto [\alpha].$$

Theorem 2.12. *Let (A,μ) be a partially crossed topological $G-R$-bimodule. Then*

(i) *$H^1(G,(A,\mu))$ can be naturally embedded in $H^1(G,A)$.*
(ii) *There is a bijection between $H^1(G,(A,\mu))$ and $H^1(G,A)$ if and only if the induced map $\mu^1 : H^1(G,A) \to H^1(G,R)$ is trivial (that is, $\mu^1 = \mathbf{1}$).*

Proof. Let ζ be the natural map mentioned above. (i). Suppose that $\zeta[(\alpha,r)]) = \zeta([(\beta,s)])$. Hence there is $a \in A$ such that $\beta(g) = a^{-1}\alpha(g)^g a$, for all $g \in G$. Hence, $\mu\beta(g) = \mu(a)^{-1}\mu\alpha(g)^g\mu(a)$. Thus, $s^g s^{-1} = \mu(a)^{-1} r^g r^{-1}\, {}^g\mu(a)$, for all $g \in G$. Therefore for all $g \in G$,

${}^{g}(r^{-1}\mu(a)s) = r^{-1}\mu(a)s$, i.e., $r^{-1}\mu(a)s \in H^0(G,R)$. This shows that ζ is one to one.

(ii). Let $\alpha \in Der_c(G,A)$, then $\mu\alpha \in Der_c(G,R)$. If $\mu^1 = \mathbf{1}$, then there is an $r \in R$ such that for all $g \in G$, $\mu\alpha(g) = r^g r^{-1}$. Thus $\zeta([(\alpha,r)]) = [\alpha]$. So, ζ is onto. Conversely, if ζ is onto. Let $\alpha \in Der_c(G,A)$, then there is $(\beta,r) \in Der_c(G,(A,\mu))$ such that $\zeta([(\beta,r)]) = [\alpha]$. Thus, there is $a \in A$ such that $\alpha(g) = a^{-1}\beta(g)^g a$, whence $\mu^1([\alpha]) = \mu^1([\beta]) = [\mu\beta] = [Inn(r)] = 1$, and this finishes the proof. □

Corollary 2.13. *Let (A,μ) be a partially crossed topological $G-R$-bimodule and $H^1(G,R) = 0$. Then, there is a bijection between $H^1(G,A)$ and $H^1(G,(A,\mu))$.*

Proof. The proof is immediate. □

Theorem 2.12 implies immediately the following two corollaries:

Corollary 2.14. *Let A be a topological G-module. There is a bijection between $\bar{H}^1(G,A)$ and $H^1(G,A)$ if and only if $\pi^1_A : H^1(G,A) \to H^1(G,A/Z(A))$ is null.*

Corollary 2.15. *If A is a topological G-module and $H^1(G,A/Z(A)) = 0$, then there is a bijection between $\bar{H}^1(G,A)$ and $H^1(G,A)$. In particular if A is abelian, then $\bar{H}^1(G,A) \cong H^1(G,A)$.*

According to part (i) of the proof of Theorem 2.12, we have the next remark.

Remark 2.16. Let (A,μ) be a partially crossed (topological) $G-R$-bimodule, and let $(\alpha,r),(\beta,s) \in Der(G,(A,\mu))$. Then

$$(\alpha,r) \sim (\beta,s) \Leftrightarrow \exists a \in A \wedge (\forall g \in G \Rightarrow \beta(g) = a^{-1}\alpha(g)^g a).$$

In other words, the condition (2.2) implies the condition (2.3). Thus, we can delete the condition (2.3) of the equivalence relation $\sim$.

Definition 2.17. Let (A,μ) be a partially crossed topological $G-R$-bimodule. The quotient set $Der_c(G,(A,\mu))/\sim'$ will be called the first cohomology of G with the coefficients in (A,μ) and is denoted by $H^1(G,(A,\mu))$.

Theorem 2.18. *Let (A,μ) be a crossed topological $G-R$-bimodule. Then, $Inn(G,(A,\mu))$ is a subgroup of $Der_c(G,(A,\mu))$.*

Proof. Suppose that $(Inn(a),\mu(a)z)),(Inn(a'),\mu(a')z') \in Inn(G,(A,\mu))$. We get $(Inn(a),\mu(a)z))(Inn(a'),\mu(a')z') = (Inn(a^z a'),\mu(a^z a')zz')$, since for any $g \in G$

$$\begin{aligned}
{}^{\mu(a)z}Inn(a')Inn(a)(g) &= {}^{\mu(a)z}Inn(a')(g)Inn(a)(g) \\
&= a^z Inn(a')(g)a^{-1}Inn(a)(g) = a^z(a'^g a'^{-1})a^{-1}(a^g a^{-1}) \\
&= a^z a'^{zg} a'^{-1g} a^{-1} = a^z a'^{gz} a'^{-1g} a^{-1} = a^z a'^g({}^z a'^{-1} a^{-1}) \\
&= a^z a'^g (a^z a')^{-1} = Inn(a^z a')(g),
\end{aligned}$$

and also, $\mu(a)z\mu(a')z' = \mu(a)z\mu(a')z^{-1}zz' = \mu(a)^z\mu(a')zz' = \mu(a^z a')zz'$. Thus, $Inn(G,(A,\mu))$ is closed under multiplication.

In addition, We have $(Inn(a),\mu(a)z)^{-1} = (Inn({}^{z^{-1}}a^{-1}),\mu({}^{z^{-1}}a^{-1})z^{-1})$, since for any $g \in G$

$$\begin{aligned}
&{}^{(\mu(a)z)^{-1}}(Inn(a)(g))^{-1} = {}^{z^{-1}\mu(a^{-1})}(a^g a^{-1})^{-1} = {}^{z^{-1}\mu(a^{-1})}({}^g a a^{-1}) = \\
&{}^{z^{-1}}(a^{-1g} a a^{-1} a) = {}^{z^{-1}}(a^{-1g} a) = {}^{z^{-1}}a^{-1^{z^{-1}g}} a = {}^{z^{-1}}a^{-1gz^{-1}} a = \\
&{}^{z^{-1}}a^{-1g}({}^{z^{-1}}a) = Inn({}^{z^{-1}}a^{-1})(g),
\end{aligned}$$

and also, $(\mu(a)z)^{-1} = z^{-1}\mu(a^{-1}) = \mu({}^{z^{-1}}a^{-1})z^{-1}$. Therefore, $Inn(G,(A,\mu))$ is closed under inversion. So, $Inn(G,(A,\mu))$ is a subgroup of $Der_c(G,(A,\mu))$. $\square$

Remark 2.19. Note that $Inn(G,(A,\mu))$ is not necessarily a normal subgroup of $Der_c(G,(A,\mu))$. If $H^1(G,(A,\mu))$ is a group, then $Inn(G,(A,\mu))$ is a normal subgroup of $Der_c(G,(A,\mu))$ and

$$H^1(G,(A,\mu)) \cong Der_c(G,(A,\mu))/Inn(G,(A,\mu)).$$

Remark 2.20. Let (A,μ) be a crossed topological $G-R$-bimodule. Then, $Inn(G,(A,\mu))$ is a normal subgroup of $Der_c(G,(A,\mu))$ if and only if ${}^{({}^r z)}\alpha^{-1}\alpha \in Inn(G,A)$ for all $(\alpha,r) \in Der_c(G,(A,\mu)), z \in H^0(G,R)$.

Proof. Suppose that $(\alpha,r) \in Der_c(G,(A,\mu))$ and $(Inn(a),\mu(a)z) \in Inn(G,(A,\mu))$. We have

$$(\alpha,r)(Inn(a),\mu(a)z)(\alpha,r)^{-1} = ({}^{r\mu(a)zr^{-1}}\alpha^{-1\,r}Inn(a)\alpha, r\mu(a)zr^{-1})$$

Thus,

$$\begin{aligned}
({}^{r\mu(a)zr^{-1}}\alpha^{-1\,r}Inn(a)\alpha)(g) &= {}^{\mu({}^r a)({}^r z)}\alpha^{-1}(g)^r Inn(a)(g)\alpha(g) \\
&= {}^r a^{({}^r z)}\alpha^{-1}(g)^{rg}a^{-1}\alpha(g) = {}^r a^{({}^r z)}\alpha^{-1}(g)\alpha(g)^{gr}a^{-1}.
\end{aligned}$$

The last equality is obtained by part (2) of Fact 2.10. Hence, we have ${}^{r\mu(a)zr^{-1}}\alpha^{-1\,r}Inn(a)\alpha \sim' {}^{({}^r z)}\alpha^{-1}\alpha$. This completes the proof. $\square$

As a consequence of Remark 2.20 we get the following corollary.

Corollary 2.21. *Let (A,μ) be a crossed topological $G-R$-bimodule and A a trivial R-module. Then $Inn(G,(A,\mu))$ is a normal subgroup of $Der_c(G,(A,\mu))$.*

Let (A, μ) be a partially crossed topological $G-R$-bimodule. On can see there is a natural action of $H^0(G, R)$ on $H^1(G, (A, \mu))$ as follows:

$$^{z}[(\alpha, r)] = [(^{z}\alpha, ^{z}r)],\ z \in H^0(G, R), [(\alpha, r)] \in H^1(G, (A, \mu)),$$

where $(^{z}\alpha)(g) = {}^{z}\alpha(g)$, for all $g \in G$. Note that by part (1) of Fact 2.10, $^{z}\alpha$ is a crossed homomorphism.

Lemma 2.22. *Let (A, μ) be a partially crossed topological $G-R$-bimodule. If $Der(G, (A, \mu))/\sim$ is a group, then $H^1(G, (A, \mu))$ is isomorphic to a subgroup of $Der(G, (A, \mu))/\sim$.*

Proof. Clearly, the natural map $\jmath : H^1(G, (A, \mu)) \to Der(G, (A, \mu))/\sim$, $[(\alpha, r)] \mapsto cls(\alpha, r)$ is injective. The equivalence relation $\sim'$ is congruence, since by assumption, $\sim$ is congruence. Thus, $\jmath$ is a homomorphism. This completes the proof. □

Indeed in Lemma 2.22, $Der(G, (A, \mu))/\sim$ is the first non-abelian cohomology of G with coefficients in (A, μ) [4, 6].

Theorem 2.23. *Let (A, μ) be a partially crossed topological $G-R$-bimodule satisfying the following conditions*

(i) *$H^0(G, R)$ is a normal subgroup of R;*
(ii) *for every $c \in H^0(G, R)$ and $(\alpha, r) \in Der_c(G, (A, \mu))$, there exists $a \in Ker\mu$ and $^{c}\alpha(g) = a^{-1}\alpha(g)^{g}a$, $\forall g \in G$.*

Then, $Der_c(G, (A, \mu))$ induces a group structure on $H^1(G, (A, \mu))$.

Proof. By [4, Theorem 2.1], the quotient set $Der(G, (A, \mu))/\sim$ is a group. Thus, by Lemma 2.22, $H^1(G, (A, \mu))$ is a group. □

Let A and B be topological G-modules and let $\mu : A \to B$ be a continuous G-homomorphism. We say that μ is a G-retraction whenever there is a continuous G-homomorphism $\rho : B \to A$ such that $\mu\rho = Id_B$. For example, (G, Id_G) is a crossed topological G-module and clearly, $Id_G : G \to G$ is a G-retraction.

Theorem 2.24. *Let (A, μ) be a partially crossed topological $G-R$-bimodule and suppose that $\mu : A \to R$ is a G-retraction. Then, the following is an exact sequence.*

$$1 \to H^1(G, (A, \mu)) \xrightarrow{\zeta} H^1(G, A) \xrightarrow{\mu^1} H^1(G, R) \to 1$$

Proof. By part (i) of Theorem 2.12, ζ is one to one. If $(\alpha, r) \in Der_c(G, (A, \mu))$, then $\mu^1\zeta([(\alpha, r)]) = \mu^1([\alpha]) = [\mu \circ \alpha] = [Inn(r)] = 1$. Thus $Im\zeta \subset Ker\mu^1$. Vice versa, if $[\alpha] \in Ker\mu^1$, then $\mu\alpha$ is cohomologous to $\mathbf{1}$. Hence, there is $r \in R$ such that $\mu\alpha(g) = r^{g}r^{-1}$, for all $g \in G$. So, $(\alpha, r) \in Der_c(G, (A, \mu))$ and $\zeta([(\alpha, r)]) = [\alpha]$. Therefore, $Ker\mu^1 \subset Im\zeta$. Finally, we show that μ^1 is onto. Suppose that $\alpha \in Der_c(G, R)$. Set $\beta = \rho\alpha$. Obviously, $\beta \in Der_c(G, A)$ and $\mu^1([\beta]) = [\alpha]$. $\square$

Theorem 2.25. *Let (A, μ) be a partially crossed topological $G-R$-bimodule satisfying the following conditions:*

1. *A and R are abelian;*
2. *for any $r \in R$ and $(\alpha, s) \in Der_c(G, (A, \mu))$ there exists $a \in Ker\mu$ and ${}^r\alpha(g) = a^{-1}\alpha(g)^g a$;*
3. *$\mu : A \to R$ is a G-retraction.*

Then, $H^1(G, A) \cong H^1(G, (A, \mu)) \oplus H^1(G, R)$.

Proof. The condition (1) implies that $H^1(G, A)$ and $H^1(G, R)$ are abelian groups. By Theorem 2.23, $H^1(G, (A, \mu))$ is a group. The conditions (1) and (2) imply that the map ζ is a homomorphism, since

$$\zeta([(\alpha, r)[(\beta, s)]) = \zeta([\alpha^r\beta], rs)]) = [\alpha^r\beta] = [\alpha][{}^r\beta] = [\alpha][\beta] = \zeta([(\alpha, r)])\zeta([\beta, s)]).$$

Hence, by Theorem 2.24, $1 \to H^1(G, (A, \mu)) \xrightarrow{\zeta} H^1(G, A) \xrightarrow{\mu^1} H^1(G, R) \to 1$ is an exact sequence (of groups and homomorphisms). By (3), there is a continuous G-homomorphism ρ such that $\mu\rho = Id_R$. Thus, $\mu^1\rho^1 = (\mu\rho)^1 = (Id_R)^1 = Id_{H^1(G,R)}$. This completes the proof. $\square$

As an immediate result of Theorem 2.25, we have:

Corollary 2.26. *Let (A, μ) be a partially crossed topological G-module satisfying the following conditions:*

1. *G and A are abelian;*
2. *$\mu : A \to G$ is a G-retraction.*

Then, $H^1(G, A) \cong H^1(G, (A, \mu)) \oplus Hom_c(G, G)$.

Note that if A is an abelian topological G-module, then $(\alpha, r) \sim' (\alpha, 1)$, for every $(\alpha, r) \in Der_c(G, (A, \mathbf{1}))$. Therefore, we have the next theorem.

Theorem 2.27. *Let G be a topological group. Then,*

$$\tau_A : H^1(G, A) \to H^1(G, (A, \mathbf{1})),\ \tau_A([\alpha]) = [(\alpha, 1)]$$

is a natural isomorphism in the category of abelian topological G-modules.

Proof. The proof is an standard argument. $\square$

3. Change of Groups for the First Cohomology.

We introduce a notion called cocompatible triple and we get inflation-restriction exact sequence for the first non-abelian cohomology groups.

Definition 3.1. Let (A,μ) be a partially crossed topological $G-R$-bimodule and (A',μ') a partially crossed topological $G'-R'$-bimodule. Suppose that $\phi : G' \to G$, $\varphi : R \to R'$ and $\psi : A \to A'$ are continuous homomorphisms. The triple (ϕ,φ,ψ) is called a cocompatible triple whenever the following conditions hold:

(1) ${}^{g'}\varphi(r) = \varphi({}^{\phi(g')}r)$, $\forall g' \in G', r \in R$;
(2) ${}^{g'}\psi(a) = \psi({}^{\phi(g')}a)$, $\forall g' \in G', a \in A$.

Example 3.2. If (A,μ) is a partially crossed topological $G-R$-bimodule and N a subgroup of G. Then, (A,μ) is a partially crossed $N-R$-bimodule. The triple $(\imath, Id_R, Id_A)$ is a cocompatible triple, where $\imath : N \to G$ is the inclusion map and Id_R and Id_A are the identity maps.

Example 3.3. If N is a normal subgroup of G and $\mu^N : A^N \to R^N$ is the restriction of $\mu : A \to R$. Clearly (A^N,μ^N) is a partially crossed topological $G/N-R^N$-bimodule. The triple $(\pi,\imath,\jmath)$ is a cocompatible triple, where $\pi : G \to G/N$ is the natural map, $\imath : R^N \to R$ and $\jmath : A^N \to A$ are the inclusion maps.

Note that a cocompatible triple (ϕ,φ,ψ) induces a natural map as follows:

$$Der_c(G,(A,\mu)) \to Der_c(G',(A',\mu')),\ (\alpha,r) \mapsto (\psi\circ\alpha\circ\phi, \varphi(r))$$

which induces naturally the map:

$$(\phi,\varphi,\psi)^1 : H^1(G,(A,\mu)) \to H^1(G',(A',\mu')),$$
$$[(\alpha,r)] \mapsto [(\psi\circ\alpha\circ\phi,\varphi(r))].$$

Definition 3.4. Let (A,μ) be a partially crossed topological $G-R$-bimodule and N a subgroup of G. The induced map $(\imath, Id_R, Id_A)^1$ is called the restriction map and it is denoted by $Res^1 : H^1(G,(A,\mu)) \to H^1(N,(A,\mu))$.

Definition 3.5. Let (A,μ) be a partially crossed topological $G-R$-bimodule and N a normal subgroup of G. The induced map $(\pi,\imath,\jmath)^1$ is called the inflation map and it is denoted by $Inf^1 : H^1(G/N,(A^N,\mu^N)) \to H^1(G,(A,\mu))$.

Lemma 3.6. *Let (A,μ) be a partially crossed topological $G-R$-bimodule, and N a normal subgroup of G. Then,*

(i) *$H^1(N,(A,\mu))$ is a G/N-set. Moreover, if $H^1(N,(A,\mu))$ is a group, then $H^1(N,(A,\mu))$ is a G/N-module.*

(ii) $ImRes^1 \subset H^1(N,(A,\mu))^{G/N}$.

Proof. (i) Since N is a normal subgroup of G, then, there is an action of G on $Der_c(N,(A,\mu))$ as follows:
For every $g \in G$ we define ${}^g(\alpha, r) = (\tilde{\alpha}, {}^g r)$ with $\tilde{\alpha}(n) = {}^g\alpha({}^{g^{-1}}n), n \in N$.

In fact, $\tilde{\alpha}$ is continuous and we have:

$$\tilde{\alpha}(mn) = {}^g\alpha({}^{g^{-1}}(mn)) = {}^g\alpha({}^{g^{-1}}m{}^{g^{-1}}n) = {}^g\alpha({}^{g^{-1}}m)\text{-}\\ {}^{mg}\alpha({}^{g^{-1}}n) = \tilde{\alpha}(m){}^m\tilde{\alpha}(n),$$

whence, $\tilde{\alpha} \in Der_c(N, A)$. Also it is easy to see that $\mu(\tilde{\alpha}(n)) = ({}^g r)^n({}^g r^{-1})$, for every $n \in N$. Hence, ${}^g(\alpha,r) \in Der_c(N,(A,\mu))$. It is clear that ${}^{gh}(\alpha,r) = {}^g({}^h(\alpha,r))$. It is easy to verify that ${}^g((\alpha,r)(\beta,s)) = {}^g(\alpha,r){}^g(\beta,s)$. Consequently $Der_c(N,(A,\mu))$ is a G-module. Now suppose that $(\alpha,r) \sim (\beta,s)$. Then, there is an $a \in A$ such that $\beta(n) = a^{-1}\alpha(n)^n a, \forall n \in N$. Thus, for every $g \in G$, $n \in N$,

$${}^g\beta({}^{g^{-1}}n) = {}^g a^{-1}({}^g\alpha({}^{g^{-1}}n)){}^g({}^{g^{-1}n}a).$$

Therefore,

$$\tilde{\beta}(n) = ({}^g a)^{-1}\tilde{\alpha}(n)^n({}^g a).$$

Therefore, by Remark 2.16, ${}^g(\alpha,r) \sim {}^g(\beta,s)$. Thus, the action of G on $Der_c(N,(A,\mu))$ induces an action of G on $H^1(N,(A,\mu))$. Moreover if $H^1(N,(A,\mu))$ is a group, then $H^1(N,(A,\mu))$ is a G-module. It is sufficient to show for every $m \in N$, ${}^m(\alpha,r) \sim (\alpha,r)$. In fact, for every $n \in N$

$${}^m\alpha({}^{m^{-1}}n) = {}^m\alpha(m^{-1}nm) = {}^m(\alpha(m^{-1}){}^{m^{-1}}\alpha(n){}^{m^{-1}n}\alpha(m)) = \\ {}^m\alpha(m^{-1})\alpha(n)^n\alpha(m) = \alpha(m)^{-1}\alpha(n)^n\alpha(m).$$

Thus, $H^1(N,(A,\mu))$ is a G/N-set. Moreover if $H^1(N,(A,\mu))$ is a group, then $H^1(N,(A,\mu))$ is a G/N-module.

(ii) By a similar argument as in (i), for every $(\alpha,r) \in Der_c(G,(A,\mu))$

$${}^g\alpha({}^{g^{-1}}n) = \alpha(g)^{-1}\alpha(n)^n\alpha(g), \forall g \in G, n \in N,$$

whence, ${}^{gN}(\alpha \circ \imath, r) \sim (\alpha \circ \imath, r), \forall gN \in G/N$. □

Theorem 3.7. *Let (A,μ) be a partially crossed topological $G-R$-bimodule and N a normal subgroup of G. Then, there is an exact sequence*

$$1 \longrightarrow H^1(G/N,(A^N,\mu^N)) \xrightarrow{Inf^1} H^1(G,(A,\mu)) \xrightarrow{Res^1} H^1(N,(A,\mu))^{G/N}.$$

Proof. The map Inf^1 is one to one: If $(\alpha, r), (\beta, s) \in Der_c(G/N, A^N)$ and $Inf^1[(\alpha, r)] = Inf^1[(\beta, s)]$, then $(\alpha\pi, r) \sim (\beta\pi, s)$. Thus, there is an $a \in A$ such that $\beta\pi(g) = a^{-1}\alpha\pi(g)^g a, \forall g \in G$. Hence, $\beta(gN) = a^{-1}\alpha(gN)^g a, \forall gN \in G/N$. If $g \in N$, then $\alpha(gN) = \beta(gN) = 1$ and hence, $a \in A^N$. This implies that ${}^g a = {}^{(gN)}a, \forall g \in G$. Consequently, $(\alpha, r) \sim (\beta, s)$, i.e., Inf^1 is one to one.

Now we show that $KerRes^1 = ImInf^1$. Since $Res^1 Inf^1[(\alpha, r)] = [(\alpha(\pi\imath), r)] = [(\mathbf{1}, 1)]$, then $ImInf^1 \subset KerRes^1$.

Let $[(\alpha, r)] \in KerRes^1$. Then, there is an $a \in A$ such that $\alpha(n) = a^{-1n}a$, $\forall n \in N$. Consider $(\beta, \mu(a)r) \in Der_c(G, (A, \mu))$ with $\beta(g) = a\alpha(g)^g a^{-1}$, $\forall g \in G$. Since $\beta(n) = 1, \forall n \in N$ then, β induces the continuous crossed homomorphism $\gamma : G/N \to A$ via $\gamma(gN) = \beta(g)$. Also $Im\gamma \subset A^N$, since for all $n \in N$,

$$^n\gamma(gN) = {}^n\beta(g) = \beta(ng) = \beta(g)^g\beta(g^{-1}ng) = \beta(g) = \gamma(gN).$$

Clearly, $\mu(a)r \in H^0(N, R)$ and $(\gamma, \mu(a)r) \in Der_c(G/N, (A^N, \mu^N))$. Hence, $Inf^1[(\gamma, \mu(a)r)] = [(\gamma\pi, \mu(a)r)] = [(\beta, \mu(a)r)] = [(\alpha, r)]$. Consequently, $KerRes^1 \subset ImInf^1$. □

4. Coboundary Maps and Exact Sequence of Cohomologies.

In this section we will obtain a seven-term exact sequence of non-abelian cohomologies up to dimension 2.

Suppose that $1 \to (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \to 1$ is an exact sequence of partially crossed topological $G-R$-bimodules such that ι is an homeomorphic embedding. Thus, we can identify A with $\iota(A)$.

Now we define a coboundary map $\delta^0 : H^0(G, C) \to H^1(G, A)$.
Let $c \in H^0(G, C)$, $b \in B$ with $\pi(b) = c$. Then, we define $\delta^0(c)$ by $\delta^0(c)(g) = b^{-1g}b, \forall g \in G$. It is obvious that $\delta^0(c)$ is a continuous crossed homomorphism. Let $b' \in B$, $\pi(b') = c$. Then, $b' = ba$ for some $a \in A$. So,

$$(b')^{-1g}b' = a^{-1}b^{-1g}b^g a = a^{-1}\delta^0(c)(g)^g a.$$

Thus, the crossed homomorphism obtained from b' is cohomologous in A to the one obtained from b, i.e., δ^0 is well-defined.

Now, suppose that $1 \to (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \to 1$ is an exact sequence of partially crossed homomorphism such that ι is a homeomorphic embedding and in addition π has a continuous section $s : C \to B$.

We construct a coboundary map $\delta^1 : H^1(G, (C, \lambda)) \to H^2(G, A)$. Here, $H^2(G, A)$ is defined as in [3].
Let $\alpha \in H^1(G, (C, \lambda))$ and $s : C \to B$ be a continuous section for π.

Define δ^1 by $[(\alpha, r)] \mapsto [\tilde{\alpha}]$, where $\tilde{\alpha}(g,h) = s\alpha(g)\ {}^g(s\alpha(h))(s\alpha(gh))^{-1}$. Clearly, $\tilde{\alpha}$ is a continuous map.

We show that $\tilde{\alpha}$ is a factor set with values in A, and independent of the choice of the continuous section s. Also δ^1 is well-defined.
Since α is a crossed homomorphism, we get:

$$\pi(\tilde{\alpha}(g,h)) = \pi(s\alpha(g)\ {}^g s\alpha(h)(s\alpha(gh))^{-1} = \alpha(g)^g\alpha(h)(\alpha(gh))^{-1} = 1.$$

Thus, $\tilde{\alpha}$ has values in A.

Next, we show that $\tilde{\alpha}$ is a factor set, i.e.,

$$^g\tilde{\alpha}(h,k)\tilde{\alpha}(g,hk) = \tilde{\alpha}(gh,k)\tilde{\alpha}(g,h),\ \forall g,h,k \in G. \tag{4.1}$$

First we calculate the left hand side of (4.1). For simplicity, take $b_g = s\alpha(g)$, $\forall g \in G$. Since $A \subset Ker\mu$, then

$$^g\tilde{\alpha}(h,k)\tilde{\alpha}(g,hk) = {}^g(b_h{}^h b_k b_{hk}^{-1})(b_g{}^g b_{hk} b_{ghk}^{-1}) = b_g{}^g(b_h{}^h b_k b_{hk}^{-1})^g b_{hk} b_{ghk}^{-1}$$
$$= b_g{}^g(b_h{}^h b_k)^g b_{hk} b_{ghk}^{-1} = b_g{}^g b_h{}^{gh} b_k b_{ghk}^{-1},$$

On the other hand,

$$\tilde{\alpha}(gh,k)\tilde{\alpha}(g,h) = (b_{gh}{}^{gh} b_k b_{ghk}^{-1})(b_g{}^g b_h b_{gh}^{-1}) = b_g{}^g b_h{}^{gh} b_k b_{ghk}^{-1}.$$

Therefore, $\tilde{\alpha}$ is a factor set.

Next, we prove that $[\tilde{\alpha}]$ is independent of the choice of the continuous sections. Suppose that s and u are continuous sections for π. Set $b_g = s\alpha(g)$ and $b_g^{'} = u\alpha(g)$. Since $\pi(b_g^{'}) = \alpha(g) = \pi(b_g)$, then $b_g^{'} = b_g a_g$ for some $a_g \in A$. Obviously the function $\kappa : G \to A$, $g \mapsto a_g$, is continuous. Thus,

$$\bar{\alpha}(g,h) = b_g^{'}{}^g b_h^{'} b_{gh}^{'} = b_g\kappa(g)^g b_h{}^g\kappa(h)(\kappa(gh))^{-1} b_{gh}^{-1})$$
$$=(\kappa(g)^g\kappa(h)(\kappa(gh))^{-1})(b_g{}^g b_h b_{gh}^{-1}) = \delta^1_G(\kappa)(g,h)\tilde{\alpha}(g,h),$$

where $\delta^1_G(\kappa)(g,h) = {}^g\kappa(h)(\kappa(gh))^{-1}\kappa(g)$. Consequently, $\bar{\alpha}$ and $\tilde{\alpha}$ are cohomologous.

Suppose that (α, r) and (β, s) are cohomologous in $Der_c(G,(C,\lambda))$. Then, there is $c \in C$ such that $\beta(g) = c^{-1}\alpha(g)^g c$, $\forall g \in G$.
Let $s : C \to A$ be a continuous section for π. Since

$$\pi(s(c^{-1}\alpha(g)^g c)) = \pi(s(c)^{-1} s\alpha(g)^g s(c)),$$

then, there exists a unique $\gamma(g) \in ker\pi = A$ such that

$$\gamma(g)(s(c)^{-1} s\alpha(g)^g s(c)) = s(c^{-1}\alpha(g)^g c).$$

It is clear that the map $\gamma : G \to A$, $g \mapsto \gamma(g)$ is continuous. Therefore,

$$\tilde{\beta}(g,h) = s\beta(g).{}^g s\beta(h).(s\beta(gh))^{-1}$$

$$= s(c^{-1}\alpha(g)^g c).{}^g s(c^{-1}\alpha(h)^h c).(s(c^{-1}\alpha(gh)^{gh} c))^{-1}$$

$$= \gamma(g)[s(c)^{-1}s\alpha(g)^g s(c)].^g(\gamma(h)[s(c)^{-1}s\alpha(h)^h s(c)])$$

$$.(\gamma(gh)[s(c)^{-1}s\alpha(gh)^{gh}s(c)])^{-1} = \ ^g\gamma(h)\gamma(gh)^{-1}\gamma(g)[s(c)^{-1}s\alpha(g)^g s(c)]$$

$$.^g[s(c)^{-1}s\alpha(h)^h s(c)].[s(c)^{-1}s\alpha(gh)^{gh}s(c)]^{-1}$$

$$= \delta^1_G(\gamma)(g,h)[s(c)^{-1}s\alpha(g)^g s\alpha(h)(a\alpha(gh))^{-1}s(c)]$$

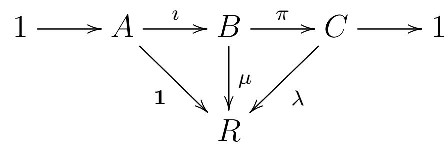
$$= \delta^1_G(\gamma)(g,h)[s(c)^{-1}\delta^1(\alpha)(g,h)s(c)] = \delta^1_G(\gamma)(g,h)\tilde{\alpha}(g,h).$$

The last equality is obtained from the fact that $\tilde{\alpha}(g,h) \in Ker\mu$ and $s(c) \in B$. Now, note that $\tilde{\alpha}$ is cohomologous to $\tilde{\beta}$, when (α, r) is cohomologous to (β, s). Thus, δ^1 is well-defined.

Recall that a short sequence

$$1 \to (A, \mathbf{1}) \xrightarrow{\imath} (B, \mu) \xrightarrow{\pi} (C, \lambda) \to 1$$

of partially crossed topological $G-R$-bimodules is exact, if the following diagram is commutative and the top row is exact.



The above exact sequence is called a proper extension with continuous sections, if $1 \to A \xrightarrow{\imath} B \xrightarrow{\pi} C \to 1$ is a proper extension and π has a continuous section.

Theorem 4.1. *Let* $1 \to (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \to 1$ *be a proper extension of partially crossed topological* $G-R$*-bimodules with continuous sections. Then, the following sequence is exact.*

$$0 \to H^0(G, A) \xrightarrow{\iota^0} H^0(G, B) \xrightarrow{\pi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A) \xrightarrow{\iota^1}$$
$$H^1(G, (B, \mu)) \xrightarrow{\pi^1} H^1(G, (C, \lambda)) \xrightarrow{\delta^1} H^2(G, A)$$

Proof. We may assume that ι is the inclusion map.

1. By [8, Theorem 4.1], the sequence

$$0 \to H^0(G, A) \xrightarrow{\iota^0} H^0(G, B) \xrightarrow{\pi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A)$$

is exact.

2. Exactness at $H^1(G, A)$: Let $c \in H^0(G, C)$. Then, there is $b \in B$ such that $\pi(b) = c$. So,

$$\iota^1\delta^0(c)(g) = \iota(\delta^0(c)(g)) = \iota(b^{-1g}b) = b^{-1g}b.$$

Consequently, $\iota^1\delta^0(c) \sim \mathbf{1}$. Conversely, let $[\alpha] \in Ker\iota^1$. Then, there is $b \in B$ such that $\alpha(g) = b^{-1g}b, \forall g \in G$. So, $\pi(b^{-1g}b) = 1, \forall g \in G$. Take $c = \pi(b)$. Hence, $c \in H^0(G, C)$. Thus, $\delta^0(c) \sim \alpha$.

3. Exactness at $H^1(G,(B,\mu))$: Since $\pi^1\iota^1([\alpha]) = \pi^1([(\alpha,1)] = [(\pi \circ \alpha, 1)] = [(\mathbf{1}, 1)] = 1$, then, $Im\iota^1 \subset Ker\pi^1$. Conversely, let $[(\beta, r)] \in ker\pi^1$. Then, there is $c \in C$ such that $\pi\beta(g) = c^{-1g}c$, for all $g \in G$ and $r = \lambda(c)^{-1}z$, for some $z \in H^0(G, R)$. Let $b \in B$ and $c = \pi(b)$. Therefore, $\pi(\beta(g)) = \pi(b^{-1g}b)$, $\forall g \in G$. On the other hand, the map $\tau_b : A \to A$, $a \mapsto b^{-1}ab$, is a topological isomorphism, because A is a normal subgroup of B. So, for every $g \in G$ there is a unique element $a_g \in G$ such that $\beta(g) = (b^{-1}a_gb)(b^{-1g}b)$. Thus, $\beta(g) = b^{-1}a_g$-${}^gb$, $\forall g \in G$. Hence, $a_g = b\beta(g)^gb^{-1}$, $\forall g \in G$. Obviously, the map $\alpha : G \to A$, $g \mapsto a_g$, is a continuous crossed homomorphism, and $\iota^1([\alpha]) = [(\alpha, 1)] = [(\alpha, z)] = [(\alpha, \lambda(c)r)] = [(\alpha, \mu(b)r] = [(\beta, r)]$.

4. Exactness at $H^1(G,(C,\lambda))$: Let $[(\beta, r)] \in H^1(G, B)$ and s be a continuous section for π. Then $\delta^1 \circ \pi^1([(\beta, r)]) = [\widetilde{\pi\beta}]$. There is a continuous map $z : G \to A$ such that $s\pi\beta(g) = \beta(g)z(g)$. Thus,

$$\begin{gathered}\widetilde{\pi\beta}(g,h) = s(\pi\beta(g))^g s(\pi\beta(h))(s(\pi\beta(gh)))^{-1} = \\ \beta(g)^g\beta(h)\beta(gh)^{-1}\delta^1_G(z)(g,h) = \delta^1_G(z)(g,h).\end{gathered}$$

So, $Im\pi^1 \subset Ker\delta^1$. Conversely, let $[(\gamma, r)] \in ker\delta^1$. Then, there is a continuous function $\alpha : G \to A$ such that $\tilde{\gamma} = \delta^1_G(\alpha)$, where $[\tilde{\gamma}] = \delta^1([(\gamma, r)])$. Thus,

$$\tilde{\gamma}(g,h) = s\gamma(g)^g s\gamma(h)(s\gamma(gh))^{-1} = {}^g\alpha(h)\alpha(gh)^{-1}\alpha(g), \forall g, h \in G.$$

Assume $\beta(g) = s\gamma(g)\alpha(g)^{-1}, \forall g \in G$. Since $A \subset Ker\mu$, then β is a continuous crossed homomorphism from G to B, and $\pi\beta = \gamma$. Also $(\beta, r) \in Der_c(G,(B,\mu))$ because $\mu\beta(g) = \mu(s\gamma(g)\alpha(g)^{-1}) = \mu(s\gamma(g)) = \lambda\gamma(g) = r^gr^{-1}$. Hence, $\pi^1([(\beta, r)]) = [(\gamma, r)]$. This completes the proof. □

5. Principal homogeneous spaces over (A,μ) - a new definition of $H^1(G,(A,\mu))$

Serre [9] showed that if A is a topological G-module in which A is discrete and G a profinite group then there is a bijection between the set, $P(A)$, of all classes of principal homogeneous spaces over A and $H^1(G, A)$. Similarly, Inassaridze [4] (algebraically) defined the G-torsor over a crossed G-module (A,μ) and showed that there is a natural isomorphism between the group, $E(G, A)$, of all classes of G-torsors over (A,μ) and the Guin's first non-abelian cohomology group $H^1(G,(A,\mu))$

(see [4, Theorem 4.2]). We will show that the first non-abelian cohomology of topological groups is closely related with principle homogeneous spaces.

Definition 5.1. A principal homogeneous space (or topological torsor) over a partially crossed topological $G-R$-bimodule (A, μ) is a pair (P, f) consisting of a G-space P, on which A acts on the right (in a manner compatible with G) so that for any $p \in P$ the natural map $e_p : A \to P$, $a \mapsto pa$ is a homeomorphism, and f is a G-map from P to R such that $f(pa) = \mu^{-1}(a)f(p)$ for any $p \in P$, $a \in A$.

Definition 5.2. It is said that principal homogeneous spaces (P, f) and (Q, g) over a partially crossed topological $G-R$-bimodule (A, μ) are isomorphic if there is a homemorphism $\nu : P \to Q$ compatible with the actions of G and A such that $f(p) = g \circ \nu(p) \mod H^0(G, R)$ for any $p \in P$.

Obviously, the isomorphism in Definition 5.2 is an equivalence relation in the set of all principle homogenous spaces over (A, μ). We denote by $\mathcal{P}(A, \mu)$ the set of all classes of principal homogeneous spaces over (A, μ). Suppose that A acts on the right on itself by translations. Obviously, (A, μ^{-1}) is a principal homogeneous space over (A, μ). We call it trivial topological torsor over (A, μ). Thus, $\mathcal{P}(A, \mu) \neq \emptyset$.

Theorem 5.3. *Let (A, μ) be a partially crossed topological $G-R$-bimodule. There is a bijection between $\mathcal{P}(A, \mu)$ and $H^1(G, (A, \mu))$.*

Proof. If $[(P, f)] \in \mathcal{P}(A, \mu)$, we choose a point $p \in P$. If $g \in G$, one has ${}^g p \in P$, therefore there exists a unique $\alpha_p^P(g) \in A$ such that ${}^g p = p\alpha_p^P(g)$. One can see that $g \mapsto \alpha_p^P(g)$ is a continuous crossed homomorphism and $\mu\alpha_p^P(g) = f(p)f(p\alpha_p^P(g))^{-1} = f(p)f({}^g p)^{-1} = f(p){}^g f(p)^{-1}$. Thus, $(\alpha_p^P, f(p)) \in Der_c(G, (A, \mu))$. On the one hand, substituting pa for p changes this crossed homomorphism into $g \mapsto a^{-1}\alpha_p^P(g){}^g a$, since ${}^g(pa) = {}^g p {}^g a = p\alpha_p^P(g){}^g a = (pa)a^{-1}\alpha_p^P(g){}^g a$. Also, $f(pa) = \mu(a)^{-1}f(p)$. Therefore, $(\alpha_q^P, f(q)) \sim (\alpha_p^P, f(p))$ for any $q \in P$. On the other hand, let (P', f') be isomorphic to (P, f). Thus, there is a homeomorphism $\nu : P \to P'$ with properties in Definition 5.2. Let $p' \in P'$ and $\nu(p) = p'$. Then ${}^g p' = {}^g\nu(p) = \nu({}^g p) = \nu(p\alpha_p^P(g)) = \nu(p)\alpha_p^P(g)$. So, $\alpha_{p'}^{P'} = \alpha_p^P$. Therefore by Remark 2.16, $(\alpha_{p'}^{P'}, f'(p')) \sim (\alpha_p^P, f(p))$. One may thus define $\lambda : \mathcal{P}(A, \mu) \to H^1(G, (A, \mu))$ via $[(P, f)] \mapsto [(\alpha_p^P, f(p))]$.

Vise versa, one defines $\gamma : H^1(G, (A, \mu)) \to \mathcal{P}(A, \mu)$ as follows:
If $(\alpha, r) \in Der_c(G, (A, \mu))$, denote by P_α the group A on which G acts

by the following *twisted formula*:

$$ga = \alpha(g)^g a.$$

Let now A act on the right on P_α by translations. Define $f_r : P_\alpha \to R$ by $f_r(a) = \mu^{-1}(a)r$. We have

$$f_r(ga) = \mu^{-1}(\alpha(g)^g a)r = {}^g\mu(a)^{-1}\mu^{-1}(\alpha(g))r = {}^g(\mu(a)^{-1}r) = {}^g f_r(a)$$

for any $g \in G$. Thus, f_r is a G-map. In addition, for any $a, b \in A$,

$$f_r(ab) = \mu(ab)^{-1}r = \mu(b)^{-1}(\mu(a)^{-1}r) = \mu(b)^{-1}f_r(a).$$

Therefore, (P_α, f_r) is a principal homogeneous space over (A, μ).

If $(\alpha, r) \sim (\beta, s)$, then there is $a \in A$ such that $\beta(g) = a^{-1}\alpha(g)^g a$ for any $g \in G$, and $s = \mu(a)^{-1}rt$ for some $t \in H^0(G, R)$. Define $\nu : P_\alpha \to P_\beta$ by $p \mapsto a^{-1}p$. For every $g \in G, p \in P_\alpha$, then

$$\nu(gp) = \nu(\alpha(g)^g p) = a^{-1}\alpha(g)^g p = \beta(g)^g a^{-1g} p = \beta(g)^g(a^{-1}p) = g\nu(p).$$

Thus, ν is a G-map. Obviously, ν is compatible with the action A on P_α and P_β.

For $p \in P_\alpha$, we get

$$f_s(\nu(p)) = f_s(a^{-1}p) = \mu(p^{-1}a)s = \mu^{-1}(p)\mu(a)\mu(a)^{-1}rt = \mu^{-1}(p)rt.$$

Therefore, (P_α, f_r) is isomorphic to (P_β, f_s). Consequently, one can define γ by $\gamma([(\alpha, r)]) = [(P_\alpha, f_r)]$.

We will show that $\gamma\lambda = Id_{P(A,\mu)}$. Let (Q, g) be a principle homogeneous space over (A, μ). Fix $q \in Q$. We define $\nu : Q \to P_{\alpha_q^Q}$ by $p \mapsto e_q^{-1}(p)$. Obviously ν is a homeomorphism. For any $h \in G$,

$$\nu({}^h p) = e_q^{-1}({}^h p) = e_q^{-1}({}^h(qe_q^{-1}(p)) = e_q^{-1}({}^h q {}^h e_q^{-1}(p)) = {}^h e_q^{-1}(p) = {}^h\nu(p).$$

i.e., ν is a G-map. Also for any $a \in A$,

$$\nu(pa) = e_q^{-1}(pa) = e_q^{-1}(qe_q^{-1}(p)a) = e_q^{-1}(p)a.$$

In addition for any $p \in P$,

$$f_{g(q)} \circ \nu(p) = \mu^{-1}(\nu(p))g(q) = g(q\nu(p)) = g(qe_q^{-1}(p)) = g(p).$$

This implies that $\gamma\lambda = Id_{P(A,\mu)}$.

Conversely, let $(\beta, s) \in Der_c(G, (A, \mu))$. For every $p \in P_\beta$, $f_s(p) = \mu^{-1}(p)s$ and also for every $g \in G$, $p\alpha_p^{P_\beta}(g) = gp = \alpha(g)^g p$. Thus, $(\beta, s) \sim (\alpha_p^{P_\beta}, f_s(p))$. This shows that $\lambda\gamma = Id_{H^1(G,(A,\mu))}$. ☐

Remark 5.4. In Definition 5.1, we may consider the G-map $f : P \to R$ as a continuous one.

Notice 5.5. If $Der_c(G,(A,\mu))$ induces naturally a group structure on $H^1(G,(A,\mu))$, then we will define a group product on $\mathcal{P}(A,\mu)$. Let $[(P_1,f_1)],[(P_2,f_2)] \in \mathcal{P}(A,\mu)$. Fix $p_1 \in P_1$ and $p_2 \in P_2$. Set $P = A$ and let A act on the right on P by translations. Consider a new action of G on P by the formula:

$$ga = {}^{f_1(p_1)}\alpha_{p_2}^{P_2}(g)\alpha_{p_1}^{P_1}(g)^g a$$

Define the map $f : P \to R$ by $f(a) = \mu^{-1}(a)f_1(p_1)f_2(p_2)$. We will show that (P,f) is a principle homogeneous space over (A,μ). Using λ, γ in the proof of Theorem 2.23, we consider the classes $[(\alpha_{p_1}^{P_1}, f_1(p_1)]$ and $[(\alpha_{p_2}^{P_2}, f_2(p_2)]$ corresponding to the classes $[(P_1,f_1)]$ and $[(P_2,f_2)]$, respectively. Since $H^1(G,(A,\mu))$ is group, then

$$[(\alpha_{p_1}^{P_1}, f_1(p_1)][(\alpha_{p_2}^{P_2}, f_2(p_2)] = [({}^{f_1(p_1)}\alpha_{p_2}^{P_2}\alpha_{p_1}^{P_1}, f_1(p_1)f_2(p_2))].$$

Obviously $\gamma([({}^{f_1(p_1)}\alpha_{p_2}^{P_2}\alpha_{p_1}^{P_1}, f_1(p_1)f_2(p_2))]) = [(P,f)]$. Thus, one can define a group product $\circ$ as follows:

$$[(P_1,f_1)] \circ [(P_2,f_2)] = [(P,f)].$$

Corollary 5.6. *Let (A,μ) be a partially crossed topological G-module. Then, $(\mathcal{P}(A,\mu),\circ)$ is a group.*

Proof. It is clear that by Theorem 2.23, $H^1(G,(A,\mu))$ is a group and Notice 5.5 implies that $\mathcal{P}(A,\mu)$ is a group by the action $\circ$. □

References

[1] D. Guin, *Cohomologie et homologie non abÉliennes des groupes*, C. R. Acad. Sci. Paris, (7) **301** (1985), 337-340.
[2] D. Guin, *Cohomologie et homologie non abÉliannes des groups*, J. Pure Appl. Algebra, **50** (1988), 109-137.
[3] S. T. Hu, *Cohomology theory in topological groups*, Michigan Math. J., (1) **1** (1952), 11-59.
[4] H. Inassaridze, *Higher Non-abelian cohomology of groups*, Glasgow Math J, (3) **44** (2002), 497-520.
[5] H. Inassaridze, *Non-abelian cohomology of groups*, Georgian Math. J. (4) **4** (1997), 313-332.
[6] H. Inassaridze, *Non-abelian cohomology with coefficients in crossed bimodules*, Georgian Math. J. (6) **4** (1997), 509-522.
[7] H. Sahleh and H. E. Koshkoshi, *On Topological Structure of the First Non-abelian Cohomology of Topological Groups*, British Journal of Mathematics and Computer Science, (12) **4** (2014), 1758-1770.
[8] H. Sahleh and H. E. Koshkoshi, *First Non-Abelian Cohomology of Topological Groups*, Gen. Math. Notes, (1) **12** (2012), 20-34.
[9] J.-P. Serre, *Galois cohomology*, Springer-Verlag, Berlin, 1997.

H. Sahleh
Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O.Box 1914, Rasht, Iran.
Email: sahleh@guilan.ac.ir

H. E. Koshkoshi
Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran.
Email: h.e.koshkoshi@guilan.ac.ir