Journal of Algebra and Related Topics
Vol. 11, No 1, (2023), pp 93-109

ON GRADED 2-ABSORBING PRIMARY HYPERIDEALS OF A GRADED MULTIPLICATIVE HYPERRING

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Abstract. Let G be a group with identity e and R be a multiplicative hyperring. We introduce and study the notions of graded 2-absorbing and graded 2-absorbing primary hyperideals of a graded multiplicative hyperring R which are generalizations of prime hyperideals. We present basic properties and characterizations of these graded hyperideals and homogeneous components. Among various results, we prove that the intersection of two graded prime hyperideals is a graded 2-absorbing hyperideal.

1. Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hyperstructures have many applications to several sectors of both pure and applied mathematics, for instance in geometry, lattices, cryptography, automata, graphs and hypergraphs, fuzzy set, probability and rough set theory and so on (see [9]). The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [20], at the 8th Congress of Scandinavian Mathematicians. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication

MSC(2010): Primary: 20N20; Secondary: 16W50, 11Y50
Keywords: graded multiplicative hyperring, graded 2-absorbing hyperideal, graded 2-absorbing primary hyperideal.
Received: 25 October 2022, Accepted: 9 January 2023.
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is an operation [17]. Prime, primary, and maximal subhypermodules of a hypermodule in the sense of krasner hyperring R were discussed by M. M. Zahedi and R. Ameri in [30]. R. Ameri *et al*. in [2] introduced Krasner (m, n)-hyperrings and in [3] studied prime and primary subhypermodules of (m, n) hypermodules. Also, K. Hila *et al*. in [16] introduced and studied (k, n)-absorbing hyperideals in Krasner (m, n)-hyperrings. The notion of multiplicative hyperrings is an important class of algebraic hyperstructures which is a generalization of rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation [26]. Procesi and Rota introduced and studied in brief the prime hyperideals of multiplicative hyperrings [22, 23, 24] and this idea is further generalized in a paper by Dasgupta [11]. R. Ameri *et al*. in [1] described multiplicative hyperring of fractions and coprime hyperideals. Later on, many researches have observed generalizations of prime hyperideals in multiplicative hyperrings [4, 5, 27, 29]. The principal notions of algebraic hyperstructure theory can be found in [8, 9, 10, 25]. Furthermore, the study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction [28]. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [12, 13, 18, 21]). Theory of greded hyperrings can be considered as an extension theory of hyperrings. The notion of 2-absorbing ideals over commutative rings which is a generalization of prime ideals has been introduced and investigated by A. Badawi in [6]. In this paper, we define the notions of G-graded multiplicative hyperrings and graded hyperideals of graded multiplicative hyperrings, also we intend to study extensively graded prime(primary) hyperideals of a graded multiplicative hyperring $(R, +, \circ)$ with absorbing zero and prove some results regarding them. In the last section, we define and study the notions of graded 2-absorbing and graded 2-absorbing primary hyperideals of a graded multiplicative hyperring R which are generalizations of graded prime hyperideals. We give some results and several properties of them. For example, we show that every graded 2-absorbing hyperideal is a graded 2-absorbing primary hyperideal, but the converse is not true in general. Also, we prove that every graded primary hyperideal of a graded multiplicative hyperring R is a graded 2-absorbing primary hyperideal of R.

2. Preliminaries

Definition 2.1. [26] Let R be a non-empty set. By $P^*(R)$, we mean the set of all non-empty subset of R. Let $\circ$ be a hyperoperation from $R \times R$ to $P^*(R)$. Rota called $(R, +, \circ)$ a *multiplicative hyperring*, if it has the following properties:

(i) $(R, +)$ is an abelian group;
(ii) $(R, \circ)$ is a hypersemigroup;
(iii) For all $a, b, c \in R$, $a \circ (b+c) \subseteq a \circ b + a \circ c$ and $(b+c) \circ a \subseteq b \circ a + c \circ a$;
(iv) $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

If in (iii) we have equalities instead of inclusions, then we say that the multiplicative hyperring is strongly distributive.

Here, we mean a *hypersemigroup* by a non-empty set R with an associative hyperoperation $\circ$, i.e.,

$$a \circ (b \circ c) = \bigcup_{t \in (b \circ c)} a \circ t = \bigcup_{s \in (a \circ b)} s \circ c = (a \circ b) \circ c$$

for all $a, b, c \in R$.

Further, if R is a multiplicative hyperring with $a \circ b = b \circ a$ for all $a, b \in R$, then R is called a commutative multiplicative hyperring.

Example 2.2. [23] Let K be a field and V be a vector space over K. If for all $a, b \in V$ we denote by (a, b) the subspace generated by the subset $\{a, b\}$ of V, then we can consider the following hyperoperation on V: for all $a, b \in V$, $a \circ b = (a, b)$. It follows that $(V, +, \circ)$ is a multiplicative hyperring, which is not strongly distributive.

Definition 2.3. [23] (a) Let $(R, +, \circ)$ be a multiplicative hyperring and S be a non-empty subset of R. Then S is said to be a *subhyperring* of R if $(S, +, \circ)$ is itself a multiplicative hyperring.

(b) A subhyperring I of a multiplicative hyperring R is a *hyperideal* of $(R, +, \circ)$ if $I - I \subseteq I$ and for all $x \in I$, $r \in R$; $x \circ r \cup r \circ x \subseteq I$.

Definition 2.4. [11] Let C be the class of all finite products of elements of a multiplicative hyperring R i.e., $C = \{r_1 \circ r_2 \circ \cdots \circ r_n : r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a C-ideal of R if for any $A \in C$, $A \cap I \neq \emptyset$, then $A \subseteq I$.

Definition 2.5. [10] (a) A proper hyperideal M of a multiplicative hyperring R is maximal in R, if for any hyperideal I of R, $M \subset I \subseteq R$, then $I = R$.

(*b*) A proper hyperideal P of a multiplicative hyperring R is said to be a prime hyperideal of R, if for any $a, b \in R$, $a \circ b \subseteq P$, then $a \in P$ or $b \in P$.

(*c*) A proper hyperideal Q of a multiplicative hyperring R is said to be a primary hyperideal of R, if for any $a, b \in R$, $a \circ b \subseteq Q$, then $a \in Q$ or $b^n \subseteq Q$ for some $n \in \mathbb{N}$.

Definition 2.6. [11] Let I be a hyperideal of a multiplicative hyperring $(R, +, \circ)$. The intersection of all prime hyperideals of R containing I is called the radical of I, denoted by $Rad(I)$. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $Rad(I) = R$. Also, the hyperideal $\{r \in R : r^n \subseteq I \text{ for some } n \in \mathbb{N}\}$ will be designated by $D(I)$ and note that the inclusion $D(I) \subseteq Rad(I)$ always holds. In addition, if I is a C-ideal of R, other inclusion holds by [**?**, 18].

Definition 2.7. [10] Let $(R, +, \circ)$ and $(S, +, \circ)$ be two multiplicative hyperrings the function $f : R \to S$ is called a homomorphism, if
(*i*) for all $a, b \in R$, $f(a + b) = f(a) + f(b)$,
(*ii*) for all $a, b \in R$, $f(a \circ b) \subseteq f(a) \circ f(b)$.

In particular, f is called good homomorphism in case $f(a \circ b) = f(a) \circ f(b)$. The kernel of a homomorphism is defined as $Ker(f) = f^{-1}(\langle 0 \rangle) = \{r \in R : f(r) \in \langle 0 \rangle\}$ and note that $f(r)$ may not be a zero element.

Throughout this paper, we assume that all hyperrings are commutative multiplicative hyperrings with absorbing zero, i.e., $0 \in R$ such that $x = 0 + x$ and $0 \in x \cdot 0 = 0 \cdot x$ for all $x \in R$.

3. Graded hyperrings and properties graded hyperideals

In this section, first we study the concept of graded multiplicative hyperrings. Then, several properties of graded prime and graded primary hyperideals in a graded multiplicative hyperring are given.

Definition 3.1. [15] Let G be a group with identity element e. A multiplicative hyperring (R, G) is called a G-graded multiplicative hyperring, if there exists a family $\{R_g\}_{g \in G}$ of additive subgroups of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$ where $R_g R_h = \bigcup \{r_g \circ r_h : r_g \in R_g,\ r_h \in R_h\}$. For simplicity, we will denote the graded multiplicative hyperring (R, G) by R.

An element of a graded hyperring R is called homogeneous if it belongs to $\bigcup_{g \in G} R_g$ and this set of homogeneous elements is denoted by

$h(R)$. If $x \in R_g$ for some $g \in G$, then we say that x is of degree g, and it is denoted by x_g. If $x \in R$, then there exist unique elements $x_g \in h(R)$ such that $x = \sum_{g \in G} x_g$.

In fact, every hyperring is trivially a G-graded hyperring by letting $R_e = R$ and $R_g = 0$ for all $g \neq e$.

Lemma 3.2. *If $R = \bigoplus_{g \in G} R_g$ is a graded multiplicative hyperring, then R_e is a subhyperring of R where e is the identity element of group G.*

Proof. As $R_e R_e \subseteq R_e$, so for any $x_e, y_e \in R_e$ we have $x_e \circ y_e \subseteq R_e R_e \subseteq R_e$. Therefore R_e is closed under multiplicative and thus is a subhyperring of R. □

Example 3.3. Let $(R, +, \cdot)$ be a ring and $x_1, \dots, x_d$ indeterminate over R. For $m = (m_1, \dots, m_d) \in \mathbb{N}^d$, let $X^m = x_1^{m_1} \dots x_d^{m_d}$. Consider the polynomial ring $S = R[x_1, \dots, x_d]$. Assume that $A \in P^*(S)$ such that $|A| \geq 2$. Then there exists a multiplicative hyperring with absorbing zero $(S_A, +, \circ)$, where $S_A = S$ and for any $a, b \in S$, $a \circ b = \{a\alpha \cdot b : \alpha \in A\}$.
(a) Let $S_A = \mathbb{Z}[x_1, \dots, x_d]$ where $A = \{2, 3, -1\}$ and $G = (\mathbb{Z}, +)$ be the integers group. Then $S_A = \bigoplus S_n$ is a G- graded multiplicative hyperring such that $S_n = \{\sum_{m \in \mathbb{N}^d} r_m X^m \mid r_m \in \mathbb{Z}, m_1 + \dots + m_d = n\}$. Notice that $S_0 = R = \mathbb{Z}$ and deg $x_i = 1$ for all i.
(b) Let $S_A = \mathbb{Z}[x_1, \dots, x_d]$ where $A = \{2, x_1, 4\}$ and $G = (\mathbb{Z}, +)$ be the integers group. We know that $S_n = \{\sum_{m \in \mathbb{N}^d} r_m X^m \mid r_m \in \mathbb{Z}, m_1 + \dots + m_d = n\}$ are all subgroups of $(S_A, +)$ and $S_A = \bigoplus S_n$, but we can easily to see that $S_1 S_1 \nsubseteq S_2$, then $(S_A, +, \circ)$ is not a G- graded multiplicative hyperring.

Example 3.4. In Definition 3.14, let $G = (\mathbb{Z}_2, +)$ be the cyclic group of order 2 and $R = \{a, b, c, d\}$. Consider the multiplicative hyperring $(R, +, \circ)$, where operation $+$ and hyperoperation $\circ$ are defined on R as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\circ$	a	b	c	d
a	{a}	{a}	{a}	{a}
b	{a}	{c, d}	{b, d}	{a, d}
c	{a}	{b, d}	{c, d}	{ a, d}
d	{a}	{a, d}	{a, d}	{a}

It is easy to see that $R_0 = \{a, c\}$, $R_1 = \{a, b\}$ and $R_2 = \{a, d\}$ are all non-trivial subgroups of $(R, +)$. We can show that R is not a $\mathbb{Z}_2$-graded multiplicative hyperring.

Definition 3.5. Let $R = \bigoplus_{g\in G} R_g$ be a graded multiplicative hyperring. A subhyperring S of R is called a graded subhyperring of R, if $S = \bigoplus_{g\in G}(S \cap R_g)$. Equivalently, S is graded if for every element $f \in S$, all the homogeneous components of f (as an element of R) are in S.

Example 3.6. Let $R_A = \mathbb{Z}[x, y]$ with $A = \{-3, 4\}$ and $G = (\mathbb{Z}, +)$ be the integers group. Then the polynomial multiplicative hyperring $R_A = \mathbb{Z}[x, y]$ is the $\mathbb{Z}$-graded multiplicative hyperring. Consider the subhyperring $S = \mathbb{Z}[x^3, x^2 + y^3]$ of $R_A = \mathbb{Z}[x, y]$. Then it is easy to verify that $S = \mathbb{Z}[x^3, x^2 + y^3]$ is a graded subhyperring of $R_A = \mathbb{Z}[x, y]$, where deg x=3 and deg y=2.

Definition 3.7. Let I be a hyperideal of a graded multiplicative hyperring R. Then I is a graded hyperideal, if $I = \bigoplus_{g\in G}(I \cap R_g)$. For any $a \in I$ and for some $r_g \in h(R)$ that $a = \sum_{g\in G} r_g$, then $r_g \in I \cap R_g$ for all $g \in G$.

Example 3.8. Let $R = M_2(\mathbb{Z}_5)$ the ring of all 2×2 matrices with entries from the field $(\mathbb{Z}_5, +, \cdot)$. For all $x, y \in R$ we define the hyperoperation $x \circ y = \{2xy, 3xy\}$. Then $(R, +, \circ)$ is a multiplicative hyperring, which is not strongly distributive. Let $G = \mathbb{Z}_4$ the group of integers modulo 4. Then, multiplicative hyperring $(R, +, \circ)$ is G-graded by

$$R_0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, R_2 = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix},\ R_1 = R_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } a, b, c, d \in \mathbb{Z}_5.$$

Consider the hyperideal $I = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle$ of multiplicative hyperring $(R, +, \circ)$. Note that, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in I$ such that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If I is a graded hyperideal of multiplicative hyperring $(R, +, \circ)$, then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in I$ which is a contradiction. So I is not a graded hyperideal of multiplicative hyperring $(R, +, \circ)$.

Lemma 3.9. [15] *Let I and J be graded hyperideals of a graded multiplicative hyperring R. Then*

(i) *$I \cap J$ is a graded hyperideal of R.*
(ii) *$IJ = \cup\{\sum_{i=1}^{n} a_i \circ b_i : a_i \in I, b_i \in J \text{and } n \in \mathbb{N}\}$ is a graded hyperideal of R.*
(iii) *$I \cup J$ is a graded hyperideal of R if and only if $I \subseteq J$ or $J \subseteq I$.*
(iv) *$I + J$ is a graded hyperideal of R.*

Definition 3.10. Let I be a graded hyperideal of a graded multiplicative hyperring $(R, +, \circ)$. The intersection of all graded prime hyperideals of R containing I is called the graded radical of I, denoted by $Grad(I)$. If the graded multiplicative hyperring R does not have any graded prime hyperideal containing I, we define $Grad(I) = R$.

Let I be a graded hyperideal of a graded multiplicative hyperring R. We define $D(I) = \{r \in R : \text{for any } g \in G,\ r_g^{n_g} \subseteq I \text{ for some } n_g \in \mathbb{N}\}$. It is clear that $D(I)$ is a graded hyperideal of R.

Definition 3.11. Let R be a graded multiplicative hyperring and C be the class of all finite products of homogeneous elements of R i. e. $C = \{r_1 \circ r_2 \circ \cdots \circ r_n : r_i \in h(R), n \in \mathbb{N}\} \subseteq P^*(h(R))$. A graded hyperideal I of R is said to be a C^{gr}-ideal of R if for any $A \in C$, $A \cap I \neq \emptyset$, then $A \subseteq I$.

Theorem 3.12. [15] *Let $I = \bigoplus_{g \in G} I_g = \bigoplus_{g \in G}(I \cap R_g)$ be a graded hyperideal of a commutative graded multiplicative hyperring $R = \bigoplus_{g \in G} R_g$. Then $D(I) \subseteq Grad(I)$. The equality holds when I is a C^{gr}-ideal of R.*

Definition 3.13. Let $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$ be two graded multiplicative hyperring. The function $f : R \to S$ is called a graded homomorphism, if

(i) for any $a, b \in R$, $f(a + b) = f(a) + f(b)$,
(ii) for any $a, b \in R$, $f(a \circ b) \subseteq f(a) \circ f(b)$,
(iii) $f(R_g) \subseteq S_g$ for any $g \in G$.

In particular, f is called graded good homomorphism in case $f(a \circ b) = f(a) \circ f(b)$. The kernel of a graded homomorphism is defined as $Ker(f) = f^{-1}(\langle 0 \rangle) = \{r \in R : f(r) \in \langle 0 \rangle\}$ and note that $f(r)$ may not be a zero element.

If Q is a graded hyperideal of S and $f : R \to S$ is a graded good homomorphism, then $f^{-1}(Q)$ is a graded hyperideal of R. If I is a graded hyperideal of R and $f : R \to S$ is an onto graded good homomorphism, then $f(I)$ is a graded hyperideal of S.

Definition 3.14. (a) A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded prime hyperideal of R if, for any $a_g, b_h \in h(R)$, $a_g \circ b_h \subseteq I$, then $a_g \in I$ or $b_h \in I$.
(b) A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded primary hyperideal of R if, for any $a_g, b_h \in h(R)$, $a_g \circ b_h \subseteq I$, then $a_g \in I$ or $b_h^n \in I$ for some $n \in \mathbb{N}$.

Lemma 3.15. *Let I be a graded prime hyperideal of a graded multiplicative hyperring R and J be a subset of $h(R)$. For any $a_g \in h(R)$, $a_g J \subseteq I$ and $a_g \notin I$ imply that $J \subseteq I$.*

Proof. Let $a_g J \subseteq I$ and $a_g \notin I$ where $a_g \in h(R)$. Hence we have $a_g J = \bigcup_{b_h \in J}(a_g \circ b_h) \subseteq I$. Let $b_h \in J$. Then $a_g \circ b_h \subseteq a_g J \subseteq I$. Since I is a graded prime hyperideal of R and $a_g \notin I$, we have $b_h \in I$. Thus $J \subseteq I$. □

Lemma 3.16. *Let I be a graded primary hyperideal of a graded multiplicative hyperring R and J be a subset of $h(R)$. For any $a_g \in h(R)$, $a_g J \subseteq I$ and $a_g \notin I$ imply that $J \subseteq Grad(I)$ (or $a_g J \subseteq I$ and $J \nsubseteq I$ imply that $a_g \in Grad(I)$).*

Proof. Let $a_g J \subseteq I$ and $a_g \notin I$ where $a_g \in h(R)$. Hence we have $a_g J = \bigcup_{b_h \in J}(a_g \circ b_h) \subseteq I$. Let $b_h \in J$. Then $a_g \circ b_h \subseteq a_g J \subseteq I$. Since I is a graded primary hyperideal of R and $a_g \notin I$, we have $b_h \in Grad(I)$. Thus $J \subseteq Grad(I)$. The proof of the other argument is similar. □

Proposition 3.17. *Let I be a graded prime hyperideal of a graded multiplicative hyperring R and A, B be subsets of $h(R)$. If $AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$.*

Proof. Suppose that $AB \subseteq I$ and $A \nsubseteq I$. Hence there exists $a_g \in A$ such that $a_g \notin I$. Let $b_h \in B$. Thus $a_g \circ b_h \subseteq AB \subseteq I$, then $b_h \in I$ because I is a graded prime hyperideal of R and $a_g \notin I$. Hence $B \subseteq I$, as needed. □

Definition 3.18. Let I be a graded hyperideal of a graded multiplicative hyperring R and P be a graded prime hyperideal such that $I \subseteq P$. If there is no graded prime hyperideal P' such that $I \subseteq P' \subseteq P$, then P is called minimal graded prime hyperideal of I. The set of all minimal graded prime hyperideals of I is denoted by $Min_{gr}(I)$.

Proposition 3.19. *If P is a graded prime hyperideal of a graded multiplicative hyperring R, then $Min_{gr}(P) = P$.*

Proof. The proof is clear. □

4. Graded 2-absorbing primary hyperideals

In this section, we introduce and study graded 2-absorbing primary hyperideals of a graded multiplicative hyperring and investigate the properties of this notion in commutative graded multiplicative hyperrings.

Definition 4.1. A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded 2-absorbing hyperideal of R, if for any $a_g, b_h, c_k \in h(R)$, $a_g \circ b_h \circ c_k \subseteq I$, then $a_g \circ b_h \subseteq I$ or $b_h \circ c_k \subseteq I$ or $a_g \circ c_k \subseteq I$.

Example 4.2. In the graded multiplicative polynomial hyperring $R_A = \mathbb{Z}[x,y]$ with $A = \{2,3\}$, the graded hyperideal $J = \langle 6, 2x, 2y, xy\rangle$ is a graded 2-absorbing hyperideal of R_A which is not a 2-absorbing hyperideal. To see this, let $f_1 = 3$, $f_2 = x+2$ and $f_3 = y+2$. Then $f_1 \circ f_2 \circ f_3 = (\bigcup_{a\in A}(f_1 \cdot a \cdot f_2)) \circ f_3 = \bigcup_{b\in A}(\bigcup_{a\in A}(f_1 \cdot a \cdot f_2)) \cdot b \cdot f_3 = \{12xy + 24x + 24y + 48, 18xy + 36x + 36y + 72, 18xy + 36x + 24y + 48, 27xy + 54x + 36y + 72\} \subseteq J$, but $f_1 \circ f_2 = \bigcup_{a\in A}(f_1 \cdot a \cdot f_2) = \{6x+12, 9x+18\} \nsubseteq J$, $f_1 \circ f_3 = \bigcup_{a\in A}(f_1 \cdot a \cdot f_3) = \{6y+12, 9y+18\} \nsubseteq J$ and $f_2 \circ f_3 = \bigcup_{a\in A}(f_2 \cdot a \cdot f_3) = \{2xy+4x+4y+8, 3xy+6x+6y+12\} \nsubseteq J$. Thus J is not a 2-absorbing hyperideal of R_A.

Example 4.3. Consider the $\mathbb{Z}$-graded multiplicative polynomial hyperring $R_A = \mathbb{R}[x,y,z]$ with $A = \{-4,1,5\}$. Then $J = \langle xyz, x^2y^2\rangle$ is a graded hyperideal of R_A generated by homogeneous elements xyz, x^2y^2. Since

$$\begin{aligned} x \circ y \circ z &= \bigcup_{b\in A}(\bigcup_{a\in A}(x \cdot a \cdot y)) \cdot b \cdot z \\ &= \{xyz, -4xyz, 5xyz, 16xyz, -20xyz, 25xyz\} \subseteq J \end{aligned}$$

but $x \circ y = \{xy, -4xy, 5xy\} \nsubseteq J$, $x \circ z = \{xz, -4xz, 5xz\} \nsubseteq J$ and $y \circ z = \{yz, -4yz, 5yz\} \nsubseteq J$ we conclude that J is not a graded 2-absorbing hyperideal of R_A.

Definition 4.4. A proper graded hyperideal I of a graded multiplicative hyperring R is called a graded 2-absorbing primary hyperideal of R, if for any $a_g, b_h, c_k \in h(R)$, $a_g \circ b_h \circ c_k \subseteq I$, then $a_g \circ b_h \subseteq I$ or $b_h \circ c_k \subseteq Grad(I)$ or $a_g \circ c_k \subseteq Grad(I)$.

It is clear that every graded 2-absorbing hyperideals is a graded 2-absorbing primary hyperideal. The converse is not true, as is shown in the following example.

Example 4.5. Let $G = (\mathbb{Z}_2, +)$ be the cyclic group of order 2, $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Then $(R, +, \circ) = \mathbb{Z}_A[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ with $A = \{-1, 2\}$ is a graded multiplicative hyperring, where $\mathbb{Z}_A = \mathbb{Z}$ and for any $x, y \in \mathbb{Z}_A$, $x \circ y = \{x \cdot a \cdot y : a \in A\}$. Let $I = \langle 12\rangle \oplus \langle 0\rangle$. Then I is a graded hyperideal of R and a graded 2-absorbing primary hyperideal of R. Although I is not a graded 2-absorbing hyperideal of R. Since, for all $\alpha, \beta \in A$ we have $(2,0) \circ (2,0) \circ (3i,0) = ((2,0) \cdot \alpha \cdot (2,0)) \cdot \beta \cdot (3i,0) = \{12i, -24i, 48i\} \subseteq I$ but $(2,0) \circ (2,0) = \{-2, 8\} \nsubseteq I$ and $(2,0) \circ (3i,0) = \{-6i, 12i\} \nsubseteq I$.

Proposition 4.6. *Every graded primary hyperideal of a graded multiplicative hyperring R is a graded 2-absorbing primary hyperideal of R.*

Proof. Let I be a graded primary hyperideal of R. Suppose that $a_g \circ b_h \circ c_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$ where $a_g, b_h, c_k \in h(R)$. Since I is a graded primary hyperideal of R, then by Lemma 3.16, $c_k \in Grad(I)$. Since $Grad(I)$ is a graded hyperideal of R, so $a_g \circ c_k \subseteq Grad(I)$ and $b_h \circ c_k \subseteq Grad(I)$. Thus I is a graded 2-absorbing primary hyperideal of R. □

Example 4.7. In the graded multiplicative hyperring $R = \mathbb{Z}_A[i]$ with $A = \{2, 3\}$, the graded hyperideal $J = \langle 6 \rangle \oplus \langle 0 \rangle$ of R is a graded 2-absorbing primary hyperideal, but it is not a graded primary hyperideal. Since, for all $\alpha \in A$ we have $(2, 0) \circ (3i, 0) = (2, 0) \cdot \alpha \cdot (3i, 0) = \{12i, 18i\} \subseteq J$ but $(2, 0) \notin J$ and $(3i, 0) \notin Grad(J)$. This example shows that a graded 2-absorbing primary hyperideal of a graded multiplicative hyperring R is not necessarily a graded primary hyperideal of R.

Theorem 4.8. *Let I be a graded hyperideal of a graded multiplicative hyperring R. If $Grad(I)$ is a graded prime hyperideal of R, then I is a graded 2-absorbing primary hyperideal of R.*

Proof. Suppose that $a_g \circ b_h \circ c_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$ where $a_g, b_h, c_k \in h(R)$. Since R is a commutative graded hyperring, we have $(a_g \circ c_k)(b_h \circ c_k) = a_g \circ b_h \circ c_k^2 \subseteq I \subseteq Grad(I)$, since $Grad(I)$ is a graded prime hyperideal of R, so $a_g \circ c_k \subseteq Grad(I)$ or $b_h \circ c_k \subseteq Grad(I)$ by Proposition 3.17. Hence I is a graded 2-absorbing primary hyperideal of R. □

Theorem 4.9. *Let P be a graded hyperideal of a graded multiplicative hyperring R and $I_1, I_2, \dots I_n$ be 2-absorbing primary hyperideals of R such that $Grad(I_i) = P$ for all $i = 1, 2, \dots, n$. Then $\bigcap_{i=1}^n I_i$ is a graded 2-absorbing primary hyperideal and $Grad(\bigcap_{i=1}^n I_i) = P$.*

Proof. Let $I = \bigcap_{i=1}^n I_i$. Clearly,

$$Grad(I) = Grad(\bigcap_{i=1}^{n} I_i) = \bigcap_{i=1}^{n} Grad(I_i) = P.$$

Suppose that $a_g \circ b_h \circ c_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$ where $a_g, b_h, c_k \in h(R)$. Hence $a_g \circ b_h \nsubseteq I_i$ for some i. Since I_i is a graded 2-absorbing primary hyperideal of R and $a_g \circ b_h \circ c_k \subseteq I \subseteq I_i$, then $a_g \circ c_k \subseteq Grad(I_i) = P$ or $b_h \circ c_k \subseteq Grad(I_i) = P$. Thus we conclude $a_g \circ c_k \subseteq Grad(I)$ or $b_h \circ c_k \subseteq Grad(I)$. Thus I is a graded 2-absorbing primary hyperideal of R. □

Proposition 4.10. *If P_1 and P_2 are graded prime hyperideals of a graded multiplicative hyperrring R, then $P_1 \cap P_2$ is a graded 2-absorbing hyperideal of R.*

Proof. Let $a_g, b_h, c_k \in h(R)$ such that $a_g \circ b_h \circ c_k \subseteq P_1 \cap P_2$, $a_g \circ b_h \nsubseteq P_1 \cap P_2$ and $b_h \circ c_k \nsubseteq P_1 \cap P_2$. Then $a_g, b_h, c_k \notin P_1 \cap P_2$. Assume that $a_g \in P_1 \cap P_2$, then $a_g \in P_1$ and $a_g \in P_2$. Since P_1 and P_2 are graded hyperideals of R, we have $a_g \circ b_h \subseteq P_1$ and $a_g \circ b_h \subseteq P_2$. Then $a_g \circ b_h \subseteq P_1 \cap P_2$ which is a contradiction. Thus $a_g \notin P_1 \cap P_2$. Similarly, $b_h, c_k \notin P_1 \cap P_2$. We consider three cases.

Case1: Suppose that $a_g \notin P_1$ and $a_g \notin P_2$. Since $c_k \notin P_1 \cap P_2$, we have three cases again. Assume that $c_k \notin P_1$ and $c_k \notin P_2$. Since P_1 is a graded prime hyperideal of R and $a_g \circ c_k \nsubseteq P_1$, $a_g \circ b_h \circ c_k \subseteq P_1$, then $b_h \in P_1$ by Lemma 3.15. Hence $a_g \circ b_h \subseteq P_1$. Similarly, Since P_2 is a graded prime hyperideal of R and $a_g \circ c_k \nsubseteq P_2$, $a_g \circ b_h \circ c_k \subseteq P_2$, then $b_h \in P_2$ by Lemma 3.15. Thus $a_g \circ b_h \subseteq P_2$. So $a_g \circ b_h \subseteq P_1 \cap P_2$ which is a contradiction. Then $c_k \in P_1$ or $c_k \in P_2$. Now, assume that $c_k \notin P_1$ and $c_k \in P_2$. Since P_1 is a graded prime hyperideal of R and $a_g \circ c_k \nsubseteq P_1$, $a_g \circ b_h \circ c_k \subseteq P_1$, then $b_h \in P_1$ by Lemma 3.15. Thus $b_h \circ c_k \subseteq P_1$. Since $c_k \in P_2$, then $b_h \circ c_k \subseteq P_2$ and so $b_h \circ c_k \subseteq P_1 \cap P_2$ which is a contradiction. Finally, assume that $c_k \notin P_2$ and $c_k \in P_1$. Since P_2 is a graded prime hyperideal of R and $a_g \circ c_k \nsubseteq P_2$, $a_g \circ b_h \circ c_k \subseteq P_2$, then $b_h \in P_2$ and so $b_h \circ c_k \subseteq P_2$. Since $c_k \in P_1$, we conclude $b_h \circ c_k \subseteq P_1$. Therefore $b_h \circ c_k \subseteq P_1 \cap P_2$ which is a contradiction. Thus if $a_g \notin P_1 \cap P_2$, implies that $a_g \in P_1$ or $a_g \in P_2$.

Case 2: Suppose that $a_g \in P_1$ and $a_g \notin P_2$. We show that $c_k \in P_2$. Assume that $c_k \notin P_2$. Since P_2 is a graded prime hyperideal of R, we have $a_g \circ c_k \nsubseteq P_2$. Since $a_g \circ b_h \circ c_k \subseteq P_2$, $a_g \circ c_k \nsubseteq P_2$ and P_2 is a graded prime hyperideal of R, then $b_h \in P_2$ by Lemma 3.15. Hence $a_g \circ b_h \subseteq P_1 \cap P_2$ which is a contradiction. Thus $c_k \in P_2$. Since $c_k \notin P_1 \cap P_2$, we get $c_k \notin P_1$. Therefore $a_g \circ c_k \subseteq P_1 \cap P_2$.

Case 3: Suppose that $a_g \in P_2$ and $a_g \notin P_1$. We show that $c_k \in P_1$. Assume that $c_k \notin P_1$. Since P_1 is a graded prime hyperideal of R, we have $a_g \circ c_k \nsubseteq P_1$. Since $a_g \circ b_h \circ c_k \subseteq P_1$, $a_g \circ c_k \nsubseteq P_1$ and P_1 is a graded prime hyperideal of R, then $b_h \in P_1$ by Lemma 3.15. Hence $a_g \circ b_h \subseteq P_1 \cap P_2$ which is a contradiction. Thus $c_k \in P_1$. Since $c_k \notin P_1 \cap P_2$, we get $c_k \notin P_2$. Therefore $a_g \circ c_k \subseteq P_1 \cap P_2$. Consequently, $P_1 \cap P_2$ is a graded 2-absorbing hyperideal of R. □

Theorem 4.11. *Let I_1 be a P_1-graded primary C^{gr}-ideal and I_2 be a P_2-graded primary C^{gr}-ideal of a graded multiplicative hyperring R.Then the following statements hold:*

(i) *$I_1 \cap I_2$ is a graded 2-absorbing primary hyperideal of R.*
(ii) *$I_1 I_2$ is a graded 2-absorbing primary hyperideal of R.*

Proof. (i) Let $I_1 \cap I_2 = K$. Then $Grad(K) = P_1 \cap P_2$. Now we show that K is a graded 2-absorbing primary hyperideal of R. Suppose that

$a_g \circ b_h \circ c_k \subseteq K$, $a_g \circ c_k \nsubseteq Grad(K)$ and $b_h \circ c_k \nsubseteq Grad(K)$ where $a_g, b_h, c_k \in h(R)$. Since $Grad(K)$ is a graded hyperideal of R, we have $a_g \notin Grad(K)$, $b_h \notin Grad(K)$ and $c_k \notin Grad(K)$. Since P_1 and P_2 are graded prime hyperideals of R, by Proposition 4.10, we conclude that $Grad(K) = P_1 \cap P_2$ is a graded 2-absorbing hyperideal of R. Thus $a_g \circ b_h \subseteq Grad(K) \subseteq P_1$. Since P_1 is a graded prime hyperideal of R, we have $a_g \in P_1$ or $b_h \in P_1$. We may assume that $a_g \in P_1$. Hence $a_g \notin P_2$ since $a_g \notin Grad(K) = P_1 \cap P_2$. One can easily show that $b_h \notin P_1$. We claim that $a_g \in I_1$ and $b_h \in I_2$. Suppose that $a_g \notin I_1$. Since I_1 is a P_1-graded primary hyperideal of R, $a_g \circ b_h \circ c_k \subseteq I_1$ and $a_g \notin I_1$, then $b_h \circ c_k \subseteq Grad(I_1) = P_1$ by Lemma 3.16. Since $b_h \in P_2$, hence $b_h \circ c_k \subseteq P_2$, and so $b_h \circ c_k \subseteq P_1 \cap P_2 = Grad(K)$ which is a contradiction. Hence $a_g \in I_1$. Now, let $b_h \notin I_2$. Since I_2 is a P_2-graded primary hyperideal of R, $a_g \circ b_h \circ c_k \subseteq I_2$ and $b_h \notin I_2$, then $a_g \circ c_k \subseteq Grad(I_2) = P_2$ by Lemma 3.16. Since $a_g \in P_1$, hence $a_g \circ c_k \subseteq P_1$, and so $a_g \circ c_k \subseteq P_1 \cap P_2 = Grad(K)$ which is a contradiction. Thus $b_h \in I_2$. Therefore $a_g \circ b_h \subseteq I_1 \cap I_2 = K$.

(ii) We have $Grad(I_1 I_2) = Grad(I_1) \bigcap Grad(I_2) = P_1 \cap P_2$ ([15]). Let $a_g \circ b_h \circ c_k \subseteq I_1 I_2$ and $a_g \circ b_h, b_h \circ c_k \nsubseteq Grad(I_1 I_2) = P_1 \cap P_2$ where $a_g, b_h, c_k \in h(R)$. We show that $a_g \circ c_k \subseteq I_1 I_2$. Then $a_g, b_h, c_k \notin Grad(I_1 I_2) = P_1 \cap P_2$. Moreover, we have $a_g \circ c_k \subseteq Grad(I_1 I_2) = P_1 \cap P_2$ since $P_1 \cap P_2$ is a graded 2-absorbing hyperideal of R. Since $a_g \circ c_k \subseteq Grad(I_1 I_2) = P_1 \cap P_2 \subseteq P_1$ and P_1 is a graded prime hyperideal, we get $a_g \in P_1$ or $c_k \in P_1$. We may assume that $a_g \in P_1$. Since $a_g \notin P_1 \cap P_2$, we have $a_g \notin P_2$. Also, $c_k \in P_2$ and $c_k \notin P_1$ since P_2 is a graded prime hyperideal and $a_g \circ c_k \subseteq Grad(I_1 I_2) = P_1 \cap P_2 \subseteq P_2$. We claim that $a_g \in I_1$ and $c_k \in I_2$. Let $a_g \notin I_1$. Since $a_g \circ b_h \circ c_k \subseteq I_1$, $a_g \notin I_1$ and I_1 is a graded primary hyperideal, we get $b_h \circ c_k \subseteq Grad(I_1) = P_1$. Since $c_k \in P_2$, we have $b_h \circ c_k \subseteq P_2$, and so $b_h \circ c_k \subseteq P_1 \cap P_2 \subseteq Grad(I_1 I_2)$ which is a contradiction. Thus $a_g \in I_1$. Similarly, we conclude that $c_k \in I_k$. Consequently, $a_g \circ c_k \subseteq I_1 I_2$. □

Lemma 4.12. *Let $f : R \to S$ be an onto graded good homomorphism of graded multiplicative hyperrings. If I is a graded hyperideal of R, then $f(Grad(I)) \subseteq Grad(f(I))$.*

Proof. Let $y \in f(Grad(I))$. Hence $y = f(x)$ for some $x \in Grad(I)$. So we can write $x = \sum_{g \in G} x_g$ where $x_g \in Grad(I) \cap h(R)$. Thus $y = f(\sum_{g \in G} x_g) = \sum_{g \in G} f(x_g)$ because f is a graded good homomorphism. Since $x \in Grad(I)$, then for any $g \in G$, there exists $n_g > 0$ such that $x_g^{n_g} \subseteq I$. Therefore $f(x_g^{n_g}) = (f(x_g))^{n_g} \subseteq f(I)$ since f is a good homomorphism. Hence $y = \sum_{g \in G} f(x_g) \in Grad(f(I))$. □

Lemma 4.13. *Let $f : R \to S$ be a graded good homomorphism of graded multiplicative hyperrings. If J is a graded hyperideal of S, then $f^{-1}(Grad(J)) = Grad(f^{-1}(J))$.*

Proof. Let $x \in f^{-1}(Grad(J))$. Thus we can write $x = \sum_{g\in G} x_g$ where $x_g \in f^{-1}(Grad(I)) \cap h(R)$. Then $f(x) = \sum_{g\in G} f(x_g) \in Grad(J)$. Hence for any $g \in G$, there exists $n_g > 0$ such that $f(x_g)^{n_g} = f(x_g^{n_g}) \subseteq J$. Therefore for any $g \in G$, there exists $n_g > 0$ such that $x_g^{n_g} \subseteq f^{-1}(J)$. Thus $x \in Grad(f^{-1}(J))$. The converse can be shown similarly. □

Theorem 4.14. *Let R and S be graded multiplicative hyperrings and let $f : R \to S$ be an onto graded good homomorphism. If I is a graded 2-absorbing primary hyperideal of R such that $ker(f) \subseteq I$, then $f(I)$ is a graded 2-absorbing primary hyperideal of S.*

Proof. Let $s_g \circ s_h \circ s_k \subseteq f(I)$ where $s_g, s_h, s_k \in h(S)$. Thus since f is onto, $f(r_g) = s_g$, $f(r_h) = s_h$ and $f(r_k) = s_k$ for some $r_g, r_h, r_k \in h(R)$. Since f is a graded good homomorphism we have $s_g \circ s_h \circ s_k = f(r_g) \circ f(r_h) \circ f(r_k) = f(r_g \circ r_h \circ r_k) \subseteq f(I)$. We show that $r_g \circ r_h \circ r_k \subseteq I$. Suppose that $x \in r_g \circ r_h \circ r_k$, then $f(x) \in f(r_g \circ r_h \circ r_k) \subseteq f(I)$, and so $f(x) = f(a)$ for some $a \in I$. Thus $f(x) - f(a) = f(x - a) = 0 \in \langle 0\rangle$, so $x - a \in Ker(f) \subseteq I$. Hence $x \in I$ since $a \in I$, then $r_g \circ r_h \circ r_k \subseteq I$. Since I is a graded 2-absorbing primary hyperideal of R, we get $r_g \circ r_h \subseteq I$ or $r_g \circ r_k \subseteq Grad(I)$ or $r_h \circ r_k \subseteq Grad(I)$. By lemma 4.12, $s_g \circ s_h \subseteq f(I)$ or $s_g \circ s_k \subseteq f(Grad(I)) \subseteq Grad(f(I))$ or $s_h \circ s_k \subseteq f(Grad(I)) \subseteq Grad(f(I))$. Therefore $f(I)$ is a graded 2-absorbing primary hyperideal of S. □

Theorem 4.15. *Let $f : R \to S$ be an graded good homomorphism of graded multiplicative hyperrings. If J is a graded 2-absorbing primary hyperideal of S, then $f^{-1}(J)$ is a graded 2-absorbing primary hyperideal of R.*

Proof. Let $a_g \circ b_h \circ c_k \subseteq f^{-1}(J)$ where $a_g, b_h, c_k \in h(R)$. Since $f(a_g \circ b_h \circ c_k) = f(a_g) \circ f(b_h) \circ f(c_k) \subseteq J$ and J is a graded 2-absorbing primary hyperideal of S, we have $f(a_g) \circ f(b_h) \subseteq J$ or $f(a_g) \circ f(c_k) \subseteq Grad(J)$ or $f(b_h) \circ f(c_k) \subseteq Grad(J)$. Thus $a_g \circ b_h \subseteq f^{-1}(J)$ or $a_g \circ c_k \subseteq f^{-1}(Grad(J))$ or $b_h \circ c_k \subseteq f^{-1}(Grad(J))$. By equality $Grad(f^{-1}(J)) = f^{-1}(Grad(J))$, we have $a_g \circ b_h \subseteq f^{-1}(J)$ or $a_g \circ c_k \subseteq Grad(f^{-1}(J))$ or $b_h \circ c_k \subseteq Grad(f^{-1}(J))$, so $f^{-1}(J)$ is a graded 2-absorbing primary hyperideal of R. □

Suppose that I is a graded hyperideal of a graded multiplicative hyperring $R = \bigoplus_{g\in G} R_g$. Then quotient group $R/I = \{a + I : a \in R\}$ becomes a multiplicative hyperring with the multiplication $(a + I) \circ$

$(b+I)=\{r+I : r\in a\circ b\}$. One can easily prove that R/I is a graded hyperring with $R/I=\bigoplus_{g\in G}(R/I)_g$ where for all $g\in G$, $(R/I)_g=(R_g+I)/I$. Also, all graded hyperideals of R/I are of the form J/I, where J is a graded hyperideal of R containing I since the natural graded homomorphism $\phi : R\to R/I$ is a graded good epimorphism ([15]).

Theorem 4.16. *Let I, J be graded hyperideals of a graded multiplicative hyperring R such that $J\subseteq I$. If I is a graded 2-absorbing primary hyperideal of R, then I/J is a graded 2-absorbing primary hyperideal of R/J.*

Proof. A mapping $f : R\to R/J$ with $f(x)=x+J$ for all $x\in R$ is an onto graded good homomorphism. Then the proof hold by Theorem 4.14. □

Lemma 4.17. *Let I be a graded 2-absorbing primary hyperideal of a strongly distributive graded multiplicative hyperring R. Let $k\in G$ and J_k be a subgroup of R_k. If $a_g\circ b_hJ_k\subseteq I$ and $a_g\circ b_h\nsubseteq I$ for $a_g, b_h\in h(R)$, then $a_gJ_k\subseteq Grad(I)$ or $b_hJ_k\subseteq Grad(I)$.*

Proof. Suppose that $a_gJ_k\nsubseteq Grad(I)$ and $b_hJ_k\nsubseteq Grad(I)$. We have $a_gJ_k=\bigcup_{j_k\in J_k}a_g\circ j_k\nsubseteq Grad(I)$ and $b_hJ_k=\bigcup_{j_k\in J_k}b_h\circ j_k\nsubseteq Grad(I)$. Hence there exist $c_k, d_k\in J_k$ such that $a_g\circ c_k\nsubseteq Grad(I)$ and $b_h\circ d_k\nsubseteq Grad(I)$. Since $a_g\circ b_h\circ c_k\subseteq I$, $a_g\circ b_h\nsubseteq I$, $a_g\circ c_k\nsubseteq Grad(I)$ and I is a graded 2-absorbing primary hyperideal of R, then $b_h\circ c_k\subseteq Grad(I)$. Similarly, Since $a_g\circ b_h\circ d_k\subseteq I$, $a_g\circ b_h\nsubseteq I$, $b_h\circ d_k\nsubseteq Grad(I)$ and I is a graded 2-absorbing primary hyperideal of R, then $a_g\circ d_k\subseteq Grad(I)$. Now since $a_g\circ b_h\circ(c_k+d_k)\subseteq I$, $a_g\circ b_h\nsubseteq I$ and I is a graded 2-absorbing primary hyperideal of R, then $a_g\circ(c_k+d_k)\subseteq Grad(I)$ or $b_h\circ(c_k+d_k)\subseteq Grad(I)$. Suppose that $a_g\circ(c_k+d_k)=a_g\circ c_k+a_g\circ d_k\subseteq Grad(I)$. Since $a_g\circ d_k\subseteq Grad(I)$, we conclude that $a_g\circ c_k\subseteq Grad(I)$ which is a contradiction. Similarly, let $b_h\circ(c_k+d_k)=b_h\circ c_k+b_h\circ d_k\subseteq Grad(I)$. Since $b_h\circ c_k\subseteq Grad(I)$, we conclude that $b_h\circ d_k\subseteq Grad(I)$ which is a contradiction. Thus $a_gJ_k\subseteq Grad(I)$ or $b_hJ_k\subseteq Grad(I)$. □

Theorem 4.18. *Let I be a graded hyperideal of a strongly distributive graded multiplicative hyrperring R. Then I is a graded 2-absorbing primary hyperideal of R if and only if for any subgroups J_g, K_h, L_k of R_g, R_h, R_k respectively, $J_gK_hL_k\subseteq I$, then $J_gK_h\subseteq I$ or $J_gL_k\subseteq Grad(I)$ or $K_hL_k\subseteq Grad(I)$.*

Proof. Let I be a graded 2-absorbing primary hyperideal of R and $J_gK_hL_k\subseteq I$ and $J_gK_h\nsubseteq I$. We show that $J_gL_k\subseteq Grad(I)$ or $K_hL_k\subseteq Grad(I)$. Suppose that $J_gL_k\nsubseteq Grad(I)$ and $K_hL_k\nsubseteq Grad(I)$. Hence

$j_gL_k \nsubseteq Grad(I)$ and $k_hL_k \nsubseteq Grad(I)$ for some $j_g \in J_g$ and $k_h \in K_h$. By Lemma 4.17, we get $j_g \circ k_h \subseteq I$. Since $J_gK_h \nsubseteq I$, so there exist $a_g \in J_g$ and $b_h \in K_h$ such that $a_g \circ b_h \nsubseteq I$. Since $(a_g \circ b_h)L_k \subseteq J_gK_hL_k \subseteq I$ and $a_g \circ b_h \nsubseteq I$, by Lemma 4.17, $a_gL_k \subseteq Grad(I)$ or $b_hL_k \subseteq Grad(I)$.

Case 1: Suppose that $a_gL_k \subseteq Grad(I)$ and $b_hL_k \nsubseteq Grad(I)$. Since $(j_g \circ b_h)L_k \subseteq J_gK_hL_k \subseteq I$, $b_hL_k \nsubseteq Grad(I)$ and $j_gL_k \nsubseteq Grad(I)$, we have $j_g \circ b_h \subseteq I$ by Lemma 4.17. Since $((a_g + j_g) \circ b_h)L_k \subseteq J_gK_hL_k \subseteq I$ and $b_hL_k \nsubseteq Grad(I)$, we have $(a_g+j_g)L_k \subseteq Grad(I)$ or $(a_g+j_g)\circ b_h \subseteq I$ by Lemma 4.17. Assume that $(a_g + j_g)L_k \subseteq Grad(I)$. Then for every $l_k \in L_k$, we have $(a_g + j_g) \circ l_k = a_g \circ l_k + j_g \circ l_k \subseteq Grad(I)$. Since $a_gL_k \subseteq Grad(I)$ and $Grad(I)$ is a graded hyperideal of R, we get $j_gL_k \subseteq Grad(I)$ which is a contradiction. Now, let $(a_g + j_g) \circ b_h = a_g \circ b_h + j_g \circ b_h \subseteq I$. Since $j_g \circ b_h \subseteq I$ and I is a graded hyperideal of R, then $a_g \circ b_h \subseteq I$, a contradiction.

Case 2: Suppose that $a_gL_k \nsubseteq Grad(I)$ and $b_hL_k \subseteq Grad(I)$. Then $a_g \circ k_h \subseteq I$ by Lemma 4.17. Since $a_g \circ (b_h + k_h)L_k \subseteq J_gK_hL_k \subseteq I$ but $a_gL_k \nsubseteq Grad(I)$, we have $a_g\circ(b_h+k_h) \subseteq I$ or $(b_h+k_h)L_k \subseteq Grad(I)$ by Lemma 4.17. Suppose that $(b_h + k_h)L_k \subseteq Grad(I)$, so $(b_h + k_h) \circ l_k = b_h \circ l_k + k_h \circ l_k \subseteq Grad(I)$ for every $l_k \in L_k$. Since $b_hL_k \subseteq Grad(I)$ and $Grad(I)$ is a graded hyperideal of R, we get $k_hL_k \subseteq Grad(I)$ which is a contradiction. Now, let $a_g \circ (b_h + k_h) = a_g \circ b_h + a_g \circ j_g \subseteq I$. Since $a_g \circ k_h \subseteq I$ and I is a graded hyperideal of R, then $a_g \circ b_h \subseteq I$ which is a contradiction.

Case 3: Suppose that $a_gL_k \subseteq Grad(I)$ and $b_hL_k \subseteq Grad(I)$. Since $b_hL_k \subseteq Grad(I)$ and $k_hL_k \nsubseteq Grad(I)$, we have $(b_h + k_h)L_k \nsubseteq Grad(I)$. By Lemma 4.17, we conclude that $j_g \circ (b_h + k_h) = j_g \circ b_h + j_g \circ k_h \subseteq I$, and since $j_g \circ k_h \subseteq I$, so $j_g \circ b_h \subseteq I$. Since $a_gL_k \subseteq Grad(I)$ and $j_gL_k \nsubseteq Grad(I)$, we get $(a_g + j_g)L_k \nsubseteq Grad(I)$. Hence $(a_g + j_g) \circ k_h = a_g \circ k_h + j_g \circ k_h \subseteq I$ by Lemma 4.17. Since $j_g \circ k_h \subseteq I$ and $a_g \circ k_h + j_g \circ k_h \subseteq I$, we have $a_g \circ k_h \subseteq I$. Thus $(a_g + j_g) \circ (b_h + k_h) = a_g\circ b_h+a_g\circ k_h+b_h\circ j_g+j_g\circ k_h \subseteq I$ by Lemma 4.17. Hence $a_g\circ b_h \subseteq I$ since $a_g\circ b_h+a_g\circ k_h+b_h\circ j_g+j_g\circ k_h \subseteq I$ and $a_g\circ k_h+b_h\circ j_g+j_g\circ k_h \subseteq I$ which is a contradiction. Consequently, we conclude that $J_gL_k \subseteq Grad(I)$ or $K_hL_k \subseteq Grad(I)$.

Conversely, suppose that $a_g \circ b_h \circ c_k \subseteq I$ where $a_g, b_h, c_k \in h(R)$. Then $\langle a_g \circ b_h \circ c_k\rangle \subseteq \langle a_g\rangle \circ \langle a_g\rangle \circ \langle a_g\rangle \subseteq I$ where $\langle a_g\rangle = \{na_g : n \in \mathbb{Z}\}$, $\langle b_h\rangle = \{nb_h : n \in \mathbb{Z}\}$ and $\langle c_k\rangle = \{nc_k : n \in \mathbb{Z}\}$ are subgroups of R_g, R_h and R_k respectively. Therefore $\langle a_g\rangle \circ \langle b_h\rangle \subseteq I$ or $\langle a_g\rangle \circ \langle c_k\rangle \subseteq Grad(I)$ or $\langle b_h\rangle \circ \langle c_k\rangle \subseteq Grad(I)$ by Lemma 4.17. Thus $a_g \circ b_h \subseteq I$ or $a_g \circ c_k \subseteq Grad(I)$ or $b_h \circ c_k \subseteq Grad(I)$, as needed. □

5. Conclusions

In this article, we introduced and studied the notions of graded 2-absorbing and graded 2-absorbing primary hyperideals of a graded multiplicative hyperring R which are generalizations of graded prime hyperideals. We showed that the concepts of 2-absorbing primary hyperideals and graded 2-absorbing primary hyperideals are totally different. Several properties, examples and characterizations of graded 2-absorbing primary hyperideals have been investigated. Moreover, we investigated the properties and the behavior of this structure under homogeneous components, graded hyperring homomorphisms. Among various results we proved that the intersection of two graded prime hyperideals is a graded 2-absorbing hyperideal and also showed that every graded primary hyperideal of a graded multiplicative hyperring R is a graded 2-absorbing primary hyperideal of R.

Acknowledgments

The author would like to thank the anonymous Editor/referee(s) for their careful reading and comments, which helped to significantly improve the paper.

References

1. R. Ameri, A. Kordi and S. Hoskova-Mayerova, *Multiplicative hyperring of fractions and coprime hyperideals*, An. St. Univ. Ovidius Constanta, (1) **25** (2017), 5–23.
2. R. Ameri and M. Norouzi, *Prime and primary hyperideals in Krasner* (m, n)*-hyperrings*, European Journal of Combinatorics, **34** (2013), 379–390.
3. R. Ameri, M. Norouzi and V. Leoreanu-Fotea, *On prime and primary subhypermodules of* (m, n)*-hypermodules*, European Journal of Combinatorics, **44** (2015), 175–190.
4. M. Anbarloei, *On 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring*, Cogent Mathematics, **4** (2017), 1–8.
5. L. K. Ardekani and B. Davvaz, *Differential multiplicative hyperrings*, J. Algebr. Syst., (1) **2** (2014), 21–35.
6. A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., (3) **75** (2007), 417–429.
7. M. Cohen and S. Montgomery, *Group-graded rings, smash products, and group actions*, Trans. Amer. Math. Soc., (1) **282** (1984), 237–258.
8. P. Corsini, *Prolegomena of Hypergroup Theory*, Second ed., Aviani Editore, 1993.
9. P. Corsini and V. Leoreanu-Fotea, *Applications of Hyperstructures Theory*, Adv. Math., Kluwer Academic Publishers, 2013.
10. B. Davvaz and V. Leoreanu-Fotea, *Hyperring theory and applications*, Internationl Academic Press, USA, 2007.

11. U. Dasgupta, *On prime and primary hyperideals of a multiplicative hyperring*, Annals of the Alexandru Ioan Cuza University-Mathematices, (1) **58** (2012), 19–36.
12. F. Farzalipour and P. Ghiasvand, *On the Union of Graded Prime Submodules*, Thai. J. Math., (1) **9** (2011), 49–55.
13. F. Farzalipour and P. Ghiasvand, *On graded semiprime and graded weakly semiprime ideals*, Intern. Electronic J. Algebra, **13** (2013), 15–22.
14. P. Ghiasvand and F. Farzalipour, *A generalization of graded prime submodules over non-Commutative graded rings*, J. Algebra Relat. Topics, (1) **8** (2020), 39–50.
15. P. Ghiasvand, F. Farzalipour and S. Mirvakili, *On expansions of graded 2-absorbing hyperideals in graded multiplicative hyperrings*, Filomat, **35(9)** (2021), 3033–3045.
16. K. Hila, K. Naka and B. Davvaz, *On (k, n)-absorbing hyperideals in Krasner (m, n)-hyperrings*, Quart. J. Math., **69** (2018), 1035–1046.
17. M. Krasner, *A class of hyperrings and hyperfield*, Intern. J. Math. Math. Sci., (2) **6** (1983), 307–312.
18. A. Macdonal, *Commutative Algebra*, Addison Wesley Publishing Company, 1969.
19. N. H. MacCoy, *A note on finite union of ideals and subgroups*, Proc. Amer. Math. Soc., (4) **8** (1969), 633–637.
20. F. Marty, *Sur une generalization de la notion de groupe*, in: 8iem Congres Math. Scandinaves, Stockholm, (1934), 45-49.
21. N. Nastasescu and F. Van Oystaeyen, *Graded Rings Theory*, Mathematical Library 28, North Holland, Amsterdam, 1937.
22. R. Procesi and R. Rota, *Complementary multiplicative hyperrings*, Discrete Math., **208/209** (1999), 485–497.
23. R. Procesi and R. Rota, *On some classes of hyperstructures*, Combinatories Discrete Math., **1** (1987), 71–80.
24. R. Procesi-Ciampi and R. Rota, *The hyperring spectrum*, Riv. Mat. Pura Appl., **1** (1987), 71–80.
25. R. Rota, *Strongly distributive multiplicative hyperrings*, J. Geometry, **39** (1990), 130–138.
26. R. Rota, *Sugli iperanelli moltiplicativi*, Rend. Di Mat, (2) **Series 7** (1982), 711–724.
27. E. Sengelen servim, B. A. Ersoy and B. Davvaz, *Primary hyperideals of multiplicative hyperrings*, Intern. Balkan J. Math., **1(1)** (2018), 43–49.
28. J. P. Serre, *Local Algebra*, Springer Verlag, 2000.
29. G. Ulucak, *On expansions of prime and* 2*-absorbing hyperideals in multiplicative hyperrings*, Turkish J. Math., **43** (1984), 1504–1517.
30. M. M. Zahedi and R. Ameri, *On the prime, primary and maximal subhypermodules*, Italian J. Pure and Applied Math., **5** (1999), 61–80.

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