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SINGLE VALUED NEUTROSOPHIC IDEALS OF PSEUDO MV-ALGEBRAS

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Abstract. After introducing the concept of single valued neutrosophic ideal in a pseudo MV-algebra, its properties are examined. The various conditions under which a single valued neutrosophic set can be a single valued neutrosophic ideal are examined. Characterizations of a single valued neutrosophic ideal of a pseudo MV-algebra are considered.

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy set to deal with problems with imprecise information (see [28]). Zadeh used one single value to represent the grade of membership of the fuzzy set defined in a universe. There is a difficulty that not all problems with imprecise information are expressed in the class of membership value by a single point. To overcome such difficulties, the notion of interval valued fuzzy sets is adopted by Turksen (see [26]). As an extended notion of fuzzy sets, Atanassov defined intuitionistic fuzzy sets which are characterized by grade of membership and non-membership functions (see [1]). In intuitionistic fuzzy sets, the membership (resp. non-membership) function represents truth (resp. false) part. Smarandache used indeterminacy membership function as an independent component to introduce neutrosophic sets by using three components: truth, indeterminacy and falsehood (see [22, 23]). Wang et al. introduced the

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notion of a single valued neutrosophic set which is an instance of neutrosophic sets which can be used in real scientific and engineering applications, etc. (see [27]). It is already well known that neutrosophic sets are being applied in almost every field of study. In particular, single valued neutrosophic sets are applied to BCK/BCI-algebras (see [3, 4, 12, 13, 14, 15, 16, 18, 24, 25]). In 1958, C.C. Chang introduced MV-algebras which are an extension of a two-valued reasoning (see [5]). As a non-commutative generalization of MV-algebras, Georgescu and Iorgulescu introduced pseudo MV-algebras (see [11]). Since then, many researchers have been studying various things about pseudo MV-algebras (see [6, 7, 8, 9, 10],).

The purpose of this paper is to apply the single valued neutrosophic set to the pseudo MV-algebra. We introduce the concepts of a single valued neutrosophic ideal in a pseudo MV-algebra, and investigate several properties. We present conditions under which a single valued neutrosophic set can be a single valued neutrosophic ideal. We discuss characterizations of a single valued neutrosophic ideal.

2. Preliminaries

This section lists well-known basic knowledge about pseudo MV-algebras and single valued neutrosophic sets required in this paper. For further information, please refer to references [11] and [27].

Let $\mathcal{M} := (M, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. We set a new binary operation $\odot$ on M via $x \odot y = (y^- \oplus x^-)^\sim$ for all $x, y \in M$. We will write $x \oplus y \odot z$ instead of $x \oplus (y \odot z)$, that is, the operation "$\odot$" is prior to the operation "$\oplus$".

A *pseudo MV-algebra* is an algebra $\mathcal{M} := (M, \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that

(M1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(M2) $x \oplus 0 = 0 \oplus x = x$,
(M3) $x \oplus 1 = 1 \oplus x = 1$,
(M4) $1^\sim = 0,\ 1^- = 0$,
(M5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
(M6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$,
(M7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,
(M8) $(x^-)^\sim = x$

for all $x, y, z \in M$. We define a binary relation "$\leq$" on $\mathcal{M}$ as follows:

$$(\forall x, y \in M) \left(x \leq y \ \Leftrightarrow \ x^- \oplus y = 1\right). \tag{2.1}$$

Then "$\leq$" is a partial order and $(\mathcal{M}, \leq)$ is a lattice in with the join $x \vee y$ and the meet $x \wedge y$ of any elements x and y are given as follows:

$$x \vee y = x \oplus x^{\sim} \odot y = x \odot y^{-} \oplus y,$$
$$x \wedge y = x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y.$$

Proposition 2.1. *Every pseudo MV-algebra* $\mathcal{M} := (M, \oplus, ^{-}, ^{\sim}, 0, 1)$ *satisfies:*

$$0^{\sim} = 0^{-} = 1, \tag{2.2}$$
$$(\forall x \in M) \left((x^{\sim})^{-} = x\right), \tag{2.3}$$
$$(\forall x \in M) \left(x \odot 1 = 1 \odot x = x\right), \tag{2.4}$$
$$(\forall x \in M) \left(x \odot 0 = 0 \odot x = 0\right), \tag{2.5}$$
$$(\forall x \in M) \left(x \oplus x^{\sim} = 1,\ x^{-} \oplus x = 1\right), \tag{2.6}$$
$$(\forall x \in M) \left(x \odot x^{-} = 0,\ x^{\sim} \odot x = 0\right), \tag{2.7}$$
$$(\forall x, y, z \in M) \left(x \leq y \ \Rightarrow\ x \odot z \leq y \odot z,\ z \odot x \leq z \odot y\right), \tag{2.8}$$

Definition 2.2. A subset F of a pseudo MV-algebra $\mathcal{M} := (M, \oplus, ^{-}, ^{\sim}, 0, 1)$ is called an *ideal* of $\mathcal{M}$ if it satisfies the following conditions:

(I1) $0 \in F$,
(I2) $(\forall x, y \in M)$ $(x, y \in F \ \Rightarrow x \oplus y \in F)$,
(I3) $(\forall x, y \in M)$ $(x \in F,\ y \leq x \ \Rightarrow\ y \in F)$.

Let M be a non-empty set. A *single valued neutrosophic set* (SVNS) in M is a structure of the form:

$$\mathcal{N}_{\sim} := \{\langle x; \widetilde{\mathcal{N}}_T(x), \widetilde{\mathcal{N}}_I(x), \widetilde{\mathcal{N}}_F(x)\rangle \mid x \in M\},$$

where $\widetilde{\mathcal{N}}_T : M \to [0,1]$ is a truth membership function, $\widetilde{\mathcal{N}}_I : M \to [0,1]$ is an indeterminate membership function, and $\widetilde{\mathcal{N}}_F : M \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $\mathcal{N}_{\sim} := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ for the SVNS

$$\mathcal{N}_{\sim} := \{\langle x; \widetilde{\mathcal{N}}_T(x), \widetilde{\mathcal{N}}_I(x), \widetilde{\mathcal{N}}_F(x)\rangle \mid x \in M\}.$$

Given an SVNS $\mathcal{N}_{\sim} := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in M, we consider the following sets.

$$\mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) := \{x \in M \mid \widetilde{\mathcal{N}}_T(x) \geq \alpha\},$$
$$\mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) := \{x \in M \mid \widetilde{\mathcal{N}}_I(x) \geq \beta\},$$
$$\mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma) := \{x \in M \mid \widetilde{\mathcal{N}}_F(x) \leq \gamma\},$$

which are called *SVNS level subsets* of M where $\alpha, \beta, \gamma \in [0,1]$.

We consider the following sets.

$$\Gamma^{TF}_{\widetilde{\mathcal{N}}} := \{(x,y) \in M \times M \mid \widetilde{\mathcal{N}}_T(x) \geq \widetilde{\mathcal{N}}_T(y), \widetilde{\mathcal{N}}_F(x) \leq \widetilde{\mathcal{N}}_F(y)\}, \tag{2.9}$$

$$\Gamma^{I}_{\widetilde{\mathcal{N}}} := \{(x,y) \in M \times M \mid \widetilde{\mathcal{N}}_I(x) \geq \widetilde{\mathcal{N}}_I(y)\}, \tag{2.10}$$

$$\Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} := \left\{ \tfrac{z}{\{x,y\}} \in \tfrac{M}{M\times M} \mid \begin{array}{l} \widetilde{\mathcal{N}}_T(z) \geq \min\{\widetilde{\mathcal{N}}_T(x), \widetilde{\mathcal{N}}_T(y)\} \\ \widetilde{\mathcal{N}}_F(z) \leq \max\{\widetilde{\mathcal{N}}_F(x), \widetilde{\mathcal{N}}_F(y)\} \end{array} \right\}, \tag{2.11}$$

$$\Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)} := \left\{ \tfrac{z}{\{x,y\}} \in \tfrac{M}{M\times M} \mid \widetilde{\mathcal{N}}_I(z) \geq \min\{\widetilde{\mathcal{N}}_I(x), \widetilde{\mathcal{N}}_I(y)\} \right\}. \tag{2.12}$$

It is clear that

$$(x,y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \Leftrightarrow \tfrac{x}{\{y,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}, \tag{2.13}$$

$$(x,y) \in \Gamma^{I}_{\widetilde{\mathcal{N}}} \Leftrightarrow \tfrac{x}{\{y,y\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)}, \tag{2.14}$$

$$(x,y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \cap \Gamma^{I}_{\widetilde{\mathcal{N}}} \Leftrightarrow \tfrac{x}{\{y,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)}, \tag{2.15}$$

$$(x,y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}},\ (y,z) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \Rightarrow (x,z) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}, \tag{2.16}$$

$$(x,y) \in \Gamma^{I}_{\widetilde{\mathcal{N}}},\ (y,z) \in \Gamma^{I}_{\widetilde{\mathcal{N}}} \Rightarrow (x,z) \in \Gamma^{I}_{\widetilde{\mathcal{N}}}, \tag{2.17}$$

$$(x,y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \cap \Gamma^{I}_{\widetilde{\mathcal{N}}},\ (y,z) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \cap \Gamma^{I}_{\widetilde{\mathcal{N}}} \Rightarrow (x,z) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \cap \Gamma^{I}_{\widetilde{\mathcal{N}}}, \tag{2.18}$$

for all $x, y, z \in M$.

Proposition 2.3 ([2]). *Let $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ be an SVNS in M. For any $a, x, y, z \in M$, we have*

$$\tfrac{a}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)},\ (y,z) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \Rightarrow \tfrac{a}{\{x,z\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}, \tag{2.19}$$

$$(a,x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}},\ \tfrac{x}{\{y,z\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \Rightarrow \tfrac{a}{\{y,z\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}. \tag{2.20}$$

Corollary 2.4 ([2]). *Let $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ be an SVNS in M. For any $a, x, y, z \in M$, we have*

$$\tfrac{a}{\{x,y\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)},\ (y,z) \in \Gamma^{I}_{\widetilde{\mathcal{N}}} \Rightarrow \tfrac{a}{\{x,z\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)}, \tag{2.21}$$

$$(a,x) \in \Gamma^{I}_{\widetilde{\mathcal{N}}},\ \tfrac{x}{\{y,z\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)} \Rightarrow \tfrac{a}{\{y,z\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)}. \tag{2.22}$$

3. Single valued neutrosophic ideals

In what follows, let $\mathcal{M} := (M, \oplus, ^-, ^\sim, 0, 1)$ denote a pseudo MV-algebra unless otherwise specified.

Definition 3.1. An SVNS $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in M is called a *single valued neutrosophic ideal* (briefly, SVN-ideal) of $\mathcal{M}$ if the sets $\mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha)$, $\mathcal{M}(\widetilde{\mathcal{N}}_I; \beta)$ and $\mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma)$ are ideals of $\mathcal{M}$ for all $\alpha, \beta, \gamma \in [0,1]$ with $\mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) \neq \emptyset$, $\mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) \neq \emptyset$ and $\mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma) \neq \emptyset$.

Example 3.2. For any subset G of M, let $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ be an SVNS in $\mathcal{M}$ defined as follows:

$$\widetilde{\mathcal{N}}_T : M \to [0,1],\ x \mapsto \begin{cases} 0.73 & \text{if } x \in G, \\ 0.45 & \text{if } x \in M \setminus G, \end{cases}$$

$$\widetilde{\mathcal{N}}_I : M \to [0,1],\ x \mapsto \begin{cases} 0.38 & \text{if } x \in G, \\ 0.19 & \text{if } x \in M \setminus G, \end{cases}$$

$$\widetilde{\mathcal{N}}_F : M \to [0,1],\ x \mapsto \begin{cases} 0.27 & \text{if } x \in G, \\ 0.72 & \text{if } x \in M \setminus G, \end{cases}$$

It is routine to veryfy that if G is an ideal of $\mathcal{M}$, then $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is an SVN-ideal of $\mathcal{M}$.

Example 3.3. Let $M := \{(1, y) \in \mathbb{R} \times \mathbb{R} \mid y \geq 0\} \cup \{(2, y) \in \mathbb{R} \times \mathbb{R} \mid y \leq 0\}$. For any $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, we define binary operation "$\oplus$" and unary operations "$^-$" and "$^\sim$" as follows:

$$\oplus : M \times M \to M,\ ((a,b),(c,d)) \mapsto \begin{cases} (1, b+d) & \text{if } a = c = 1, \\ (2, ad+b) & \text{if } ac = 2,\ ad+b \leq 0, \\ (2,0) & \text{otherwise}, \end{cases}$$

$$^- : M \to M,\ (a,b) \mapsto \left(\tfrac{2}{a}, -\tfrac{2b}{a}\right),$$

and

$$^\sim : M \to M,\ (a,b) \mapsto \left(\tfrac{2}{a}, -\tfrac{b}{a}\right).$$

Then $\mathcal{M} := (M, \oplus, ^-, ^\sim, 0^*, 1^*)$ is a pseudo MV-algebra where $0^* := (1, 0)$ and $1^* = (2, 0)$ (see [6], [10]). For two subsets $F := \{(1, y) \mid y > 0\}$ and $G := \{(2, y) \mid y < 0\}$ of M, we define an SVNS $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in $\mathcal{M} := (M, \oplus, ^-, ^\sim, 0^*, 1^*)$ as follows:

$$\widetilde{\mathcal{N}}_T : M \to [0,1],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x = 0^*, \\ \alpha_2 & \text{if } x \in F, \\ \alpha_3 & \text{if } x \in G \cup \{1^*\}, \end{cases}$$

$$\widetilde{\mathcal{N}}_I : M \to [0,1],\ x \mapsto \begin{cases} \beta_1 & \text{if } x = 0^*, \\ \beta_2 & \text{if } x \in F, \\ \beta_3 & \text{if } x \in G \cup \{1^*\}, \end{cases}$$

and

$$\widetilde{\mathcal{N}}_F : M \to [0,1],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x = 0^*, \\ \gamma_2 & \text{if } x \in F, \\ \gamma_3 & \text{if } x \in G \cup \{1^*\}, \end{cases}$$

where $\alpha_1 > \alpha_2 > \alpha_3$, $\beta_1 > \beta_2 > \beta_3$, $\gamma_1 < \gamma_2 < \gamma_3$ in $[0, 1]$. It is routine to verify that $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is an SVN-ideal of $\mathcal{M}$.

Proposition 3.4. *Every SVN-ideal* $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ *of* $\mathcal{M}$ *satisfies:*

$$(\forall x, y \in M)\left(\tfrac{x\oplus y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}\right), \tag{3.1}$$

$$(\forall x, y \in M)\left(x \leq y \;\Rightarrow\; (x,y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}\right). \tag{3.2}$$

Proof. Let $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ be an SVN-ideal of $\mathcal{M}$. Then the sets $\mathcal{M}(\widetilde{\mathcal{N}}_T;\alpha)$ and $\mathcal{M}(\widetilde{\mathcal{N}}_F;\gamma)$ are ideals of $\mathcal{M}$ for all $\alpha,\gamma \in [0,1]$ with $\mathcal{M}(\widetilde{\mathcal{N}}_T;\alpha) \neq \emptyset$ and $\mathcal{M}(\widetilde{\mathcal{N}}_F;\gamma) \neq \emptyset$. Suppose that $\frac{a\oplus b}{\{a,b\}} \notin \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$ for some $a, b \in M$. Then

$$\widetilde{\mathcal{N}}_T(a\oplus b) < \min\{\widetilde{\mathcal{N}}_T(a), \widetilde{\mathcal{N}}_T(b)\} \text{ or } \widetilde{\mathcal{N}}_F(a\oplus b) > \max\{\widetilde{\mathcal{N}}_F(a), \widetilde{\mathcal{N}}_F(b)\}.$$

If $\widetilde{\mathcal{N}}_T(a\oplus b) < \min\{\widetilde{\mathcal{N}}_T(a), \widetilde{\mathcal{N}}_T(b)\}$, then

$$a \in \mathcal{M}(\widetilde{\mathcal{N}}_T;\alpha_T) \text{ and } b \in \mathcal{M}(\widetilde{\mathcal{N}}_T;\alpha_T),$$

for $\alpha_T := \min\{\widetilde{\mathcal{N}}_T(a), \widetilde{\mathcal{N}}_T(b)\}$. But $a \oplus b \notin \mathcal{M}(\widetilde{\mathcal{N}}_T;\alpha_T)$, which is a contradiction. If $\widetilde{\mathcal{N}}_F(a\oplus b) > \max\{\widetilde{\mathcal{N}}_F(a), \widetilde{\mathcal{N}}_F(b)\}$, then $a \in \mathcal{M}(\widetilde{\mathcal{N}}_F;\gamma_F)$ and $b \in \mathcal{M}(\widetilde{\mathcal{N}}_F;\gamma_F)$ for $\gamma_F := \max\{\widetilde{\mathcal{N}}_F(a), \widetilde{\mathcal{N}}_F(b)\}$. But $a \oplus b \notin \mathcal{M}(\widetilde{\mathcal{N}}_F;\gamma_F)$, which is a contradiction. Hence $\frac{x\oplus y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$ for all $x, y \in M$. Let $x, y \in M$ be such that $x \leq y$. If $(x,y) \notin \Gamma^{TF}_{\widetilde{\mathcal{N}}}$, then $\widetilde{\mathcal{N}}_T(x) < \widetilde{\mathcal{N}}_T(y)$ or $\widetilde{\mathcal{N}}_F(x) > \widetilde{\mathcal{N}}_F(y)$, which imply that

$$\widetilde{\mathcal{N}}_T(x) < \alpha_T := \tfrac{1}{2}\left(\widetilde{\mathcal{N}}_T(x) + \widetilde{\mathcal{N}}_T(y)\right) < \widetilde{\mathcal{N}}_T(y)$$

and

$$\widetilde{\mathcal{N}}_F(x) > \gamma_F := \tfrac{1}{2}\left(\widetilde{\mathcal{N}}_F(x) + \widetilde{\mathcal{N}}_F(y)\right) > \widetilde{\mathcal{N}}_F(y).$$

It follows that $y \in \mathcal{M}(\widetilde{\mathcal{N}}_T;\alpha_T)$ and $x \notin \mathcal{M}(\widetilde{\mathcal{N}}_T;\alpha_T)$, or $y \in \mathcal{M}(\widetilde{\mathcal{N}}_F;\gamma_F)$ and $x \notin \mathcal{M}(\widetilde{\mathcal{N}}_F;\gamma_F)$. This is a contradiction, and therefore $(x,y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$. □

Corollary 3.5. *Every SVN-ideal* $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ *of* $\mathcal{M}$ *satisfies:*

$$(\forall x, y \in M)\left(\tfrac{x\oplus y}{\{x,y\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)}\right), \tag{3.3}$$

$$(\forall x, y \in M)\left(x \leq y \;\Rightarrow\; (x,y) \in \Gamma^{I}_{\widetilde{\mathcal{N}}}\right). \tag{3.4}$$

We present conditions under which an SVNS can be an SVN-ideal.

Theorem 3.6. *If an SVNS* $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ *in* $\mathcal{M}$ *satisfies* (3.1), (3.2), (3.3) *and* (3.4), *then* $\mathcal{N}_\sim$ *is an SVN-ideal of* $\mathcal{M}$.

Proof. Assume that $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in $\mathcal{M}$ satisfies (3.1), (3.2), (3.3) and (3.4). Let $\alpha, \beta, \gamma \in [0,1]$ be such that $\mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) \neq \emptyset$, $\mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) \neq \emptyset$ and $\mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma) \neq \emptyset$. It is clear that

$$0 \in \mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) \cap \mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) \cap \mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma).$$

If $x, y \in \mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) \cap \mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) \cap \mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma)$, then

$$x \oplus y \in \mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) \cap \mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) \cap \mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma),$$

by (3.1) and (3.3). Let $x, y \in M$ be such that $y \leq x$ and $x \in \mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) \cap \mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) \cap \mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma)$. Using (3.2) and (3.4), we have $(y, x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \cap \Gamma^{I}_{\widetilde{\mathcal{N}}}$ and thus

$$y \in \mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha) \cap \mathcal{M}(\widetilde{\mathcal{N}}_I; \beta) \cap \mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma).$$

Hence, $\mathcal{M}(\widetilde{\mathcal{N}}_T; \alpha)$, $\mathcal{M}(\widetilde{\mathcal{N}}_I; \beta)$ and $\mathcal{M}(\widetilde{\mathcal{N}}_F; \gamma)$ are ideals of $\mathcal{M}$, and therefore ,$\mathcal{N}_\sim$ is an SVN-ideal of $\mathcal{M}$. □

Proposition 3.7. *If an SVNS $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in $\mathcal{M}$ is an SVN-ideal of $\mathcal{M}$, then*

$$(\forall x \in M) \left((0, x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \right). \tag{3.5}$$

Proof. This is obtained directly from (3.2). □

Corollary 3.8. *Every SVN-ideal $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ of $\mathcal{M}$ satisfies:*

$$(\forall x \in M) \left((0, x) \in \Gamma^{I}_{\widetilde{\mathcal{N}}} \right). \tag{3.6}$$

Proposition 3.9. *Every SVN-ideal $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ of $\mathcal{M}$ satisfies:*

$$(\forall x, y \in M) \left((x \wedge y, x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \right), \tag{3.7}$$

$$(\forall x, y \in M) \left(\tfrac{x \odot y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \right), \tag{3.8}$$

$$(\forall x, y \in M) \left(\tfrac{x \wedge y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \right), \tag{3.9}$$

$$(\forall x, y \in M) \left(\tfrac{x \vee y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \right), \tag{3.10}$$

$$(\forall x, y \in M) \left(\tfrac{y}{\{x, x^{\sim} \odot y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \right). \tag{3.11}$$

Proof. Since $x \wedge y \leq x$ for all $x, y \in M$, we get $(x \wedge y, x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$ by (3.2). Since $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$ for all $x, y \in M$, we have $(x \odot y, x \oplus y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$, $(x \wedge y, x \oplus y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$ and $(x \vee y, x \oplus y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$ by (3.2). It follows from (2.20) and (3.1) that

$$\tfrac{x \odot y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}, \ \tfrac{x \wedge y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \text{ and } \tfrac{x \vee y}{\{x,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}.$$

Since $y \leq x \vee y = x \oplus x^{\sim} \odot y$ for all $x, y \in M$, we obtain $(y, x \oplus x^{\sim} \odot y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$ by (3.2). Using (3.1) induces $\frac{x \oplus x^{\sim} \odot y}{\{x, x^{\sim} \odot y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$, which implies from (2.20) that $\frac{y}{\{x, x^{\sim} \odot y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$. ∎

Corollary 3.10. *Every SVN-ideal $\mathcal{N}_{\sim} := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ of $\mathcal{M}$ satisfies:*

$$(\forall x, y \in M) \left((x \wedge y, x) \in \Gamma^{I}_{\widetilde{\mathcal{N}}} \right), \tag{3.12}$$

$$(\forall x, y \in M) \left(\tfrac{x \odot y}{\{x,y\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)} \right), \tag{3.13}$$

$$(\forall x, y \in M) \left(\tfrac{x \wedge y}{\{x,y\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)} \right), \tag{3.14}$$

$$(\forall x, y \in M) \left(\tfrac{x \vee y}{\{x,y\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)} \right), \tag{3.15}$$

$$(\forall x, y \in M) \left(\tfrac{y}{\{x, x^{\sim} \odot y\}} \in \Gamma^{I}_{\widetilde{\mathcal{N}}(\min,\max)} \right). \tag{3.16}$$

Proposition 3.11. *Let $\mathcal{N}_{\sim} := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ be an SVNS in $\mathcal{M}$. If $\mathcal{N}_{\sim}$ satisfies (3.5) and (3.11), then $\mathcal{N}_{\sim}$ satisfies (3.2) and*

$$(\forall x, y \in M) \left(\tfrac{y}{\{x, y \odot x^{-}\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \right). \tag{3.17}$$

Proof. Assume that $\mathcal{N}_{\sim}$ satisfies (3.5) and (3.11). Let $x, y \in M$ be such that $y \leq x$. Using (2.7) and (2.8), we have $x^{\sim} \odot y \leq x^{\sim} \odot x = 0$ and so $x^{\sim} \odot y = 0$. Thus $\frac{y}{\{x,0\}} = \frac{y}{\{x, x^{\sim} \odot y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$ by (3.11). It follows from (2.19) and (3.5) that $\frac{y}{\{x,x\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$, that is, $(y, x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$. Note that

$$(y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x) \leq (y \odot x^{-})^{\sim} \odot (y \odot x^{-}) \oplus x = 0 \oplus x = x,$$

which implies from (3.2) that

$$\left((y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x), x \right) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}.$$

Now, since $x^{\sim} \odot y \leq x \oplus x^{\sim} \odot y = y \odot x^{-} \oplus x$, we get $(x^{\sim} \odot y, y \odot x^{-} \oplus x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$ by (3.2). Combining this and (3.11), we have $\frac{y}{\{x, y \odot x^{-} \oplus x\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$. If we take $x := y \odot x^{-}$ and $y := y \odot x^{-} \oplus x$ in (3.11), then

$$\frac{y \odot x^{-} \oplus x}{\{y \odot x^{-},\ (y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x)\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)},$$

and so $\frac{y \odot x^{-} \oplus x}{\{y \odot x^{-},\ x\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$ by (2.19). On the other hand, since $y \leq y \vee x = y \odot x^{-} \oplus x$, we get $(y, y \odot x^{-} \oplus x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}}$ by (3.2). Hence using (2.20) induces $\frac{y}{\{x, y \odot x^{-}\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)}$. ∎

Corollary 3.12. *Let $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ be an SVNS in $\mathcal{M}$. If $\mathcal{N}_\sim$ satisfies (3.6) and (3.16), then $\mathcal{N}_\sim$ satisfies (3.4) and*

$$(\forall x, y \in M)\left(\tfrac{y}{\{x, y\odot x^-\}} \in \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}\right). \tag{3.18}$$

Theorem 3.13. *If an SVNS $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in $\mathcal{M}$ satisfies (3.5), (3.6), (3.17) and (3.18), then $\mathcal{N}_\sim$ is an SVN-ideal of $\mathcal{M}$.*

Proof. Let $x, y \in M$ be such that $x \leq y$. Then $x \odot y^- \leq y \odot y^- = 0$ by (2.7) and (2.8), and so $x \odot y^- = 0$. Hence $\frac{x}{\{y,0\}} = \frac{x}{\{y, x\odot y^-\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}$ by (3.17) and (3.18). Since $(0, y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \cap \Gamma^I_{\widetilde{\mathcal{N}}}$ by (3.5) and (3.6), it follows from (2.19) and (2.21) that $\frac{x}{\{y,y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}$, that is, $(x, y) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}$ which shows that (3.2) and (3.4) are valid. Note that $(x \oplus y) \odot y^- = (x \oplus (y^-)^\sim) \odot y^- = x \wedge y^- \leq x$ for all $x, y \in M$. Thus $((x \oplus y) \odot y^-, x) \in \Gamma^{TF}_{\widetilde{\mathcal{N}}} \cap \Gamma^I_{\widetilde{\mathcal{N}}}$ by (3.2) and (3.4). If we take $x := y$ and $y := x \oplus y$ in (3.17) and (3.18), then $\frac{x\oplus y}{\{y, (x\oplus y)\odot y^-\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}$. Hence $\frac{x\oplus y}{\{y,x\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}$ by (2.19) and (2.21). Using Theorem 3.6, we conclude that $\mathcal{N}_\sim$ is an SVN-ideal of $\mathcal{M}$. □

Theorem 3.14. *If an SVNS $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in $\mathcal{M}$ satisfies (3.5), (3.6) and*

$$(\forall x, y, z \in M)\left(\tfrac{x\odot y}{\{x\odot y\odot z,\, z^\sim\odot y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}\right), \tag{3.19}$$

then $\mathcal{N}_\sim$ is an SVN-ideal of $\mathcal{M}$.

Proof. If we change x, y and z in (3.19) to y, 1 and x^- respectively, then

$$\tfrac{y}{\{y\odot x^-, x\}} = \tfrac{y\odot 1}{\{y\odot 1\odot x^-, (x^-)^\sim\odot 1\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}.$$

Hence $\mathcal{N}_\sim$ is an SVN-ideal of $\mathcal{M}$ by Theorem 3.13. □

Theorem 3.15. *If an SVNS $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ in $\mathcal{M}$ satisfies (3.5), (3.6) and*

$$(\forall x, y, z \in M)\left(\tfrac{x\odot y}{\{x\odot y\odot z^-,\, z\odot y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}\right), \tag{3.20}$$

then $\mathcal{N}_\sim$ is an SVN-ideal of $\mathcal{M}$.

Proof. If we change z in (3.20) to $z^\sim$ and use (2.3), then

$$\tfrac{x\odot y}{\{x\odot y\odot z,\, z^\sim\odot y\}} = \tfrac{x\odot y}{\{x\odot y\odot (z^\sim)^-,\, z^\sim\odot y\}} \in \Gamma^{TF}_{\widetilde{\mathcal{N}}(\min,\max)} \cap \Gamma^I_{\widetilde{\mathcal{N}}(\min,\max)}.$$

Therefore $\mathcal{N}_\sim$ is an SVN-ideal of $\mathcal{M}$ by Theorem 3.14. □

The above results are combined to obtain the following characterization of an SVN-ideal:

Theorem 3.16. *Given an SVNS* $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ *in* $\mathcal{M}$*, the following are equivalent.*

(i) $\mathcal{N}_\sim$ *is an SVN-ideal of* $\mathcal{M}$*.*
(ii) $\mathcal{N}_\sim$ *satisfies* (3.1), (3.2), (3.3) *and* (3.4)*.*
(iii) $\mathcal{N}_\sim$ *satisfies* (3.5), (3.6), (3.11) *and* (3.16)*.*
(iv) $\mathcal{N}_\sim$ *satisfies* (3.5), (3.6), (3.17) *and* (3.18)*.*

Given a subset A of M, we consider two SVNSs $\mathcal{N}_\sim^A := (\widetilde{\mathcal{N}}_T^A, \widetilde{\mathcal{N}}_I^A, \widetilde{\mathcal{N}}_F^A)$ and $\chi_\sim := (\tilde{\chi}_T, \tilde{\chi}_I, \tilde{\chi}_F)$ in $\mathcal{M}$ in which

$$\widetilde{\mathcal{N}}_T^A : M \to [0,1],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in A, \\ \alpha_2 & \text{if } x \in M \setminus A, \end{cases}$$

$$\widetilde{\mathcal{N}}_I^A : M \to [0,1],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in A, \\ \beta_2 & \text{if } x \in M \setminus A, \end{cases}$$

$$\widetilde{\mathcal{N}}_F^A : M \to [0,1],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in A, \\ \gamma_2 & \text{if } x \in M \setminus A, \end{cases}$$

where $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$ and $\gamma_1 < \gamma_2$ in $[0,1]$, and

$$\tilde{\chi}_T : M \to [0,1],\ x \mapsto \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in M \setminus A, \end{cases}$$

$$\tilde{\chi}_I : M \to [0,1],\ x \mapsto \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in M \setminus A, \end{cases}$$

$$\tilde{\chi}_F : M \to [0,1],\ x \mapsto \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in M \setminus A. \end{cases}$$

We know that the SVNS $\mathcal{N}_\sim^A := (\widetilde{\mathcal{N}}_T^A, \widetilde{\mathcal{N}}_I^A, \widetilde{\mathcal{N}}_F^A)$ is a generalization of the SVNS $\chi_\sim := (\tilde{\chi}_T, \tilde{\chi}_I, \tilde{\chi}_F)$, which is called the *characteristic SVNS*.

Proposition 3.17. *Every subset* A *is an ideal of* $\mathcal{M}$ *if and only if the SVNS* $\mathcal{N}_\sim^A := (\widetilde{\mathcal{N}}_T^A, \widetilde{\mathcal{N}}_I^A, \widetilde{\mathcal{N}}_F^A)$ *is an SVN-ideal of* $\mathcal{M}$*.*

Proof. Straightforward. □

Corollary 3.18. *Every subset* F *is an ideal of* $\mathcal{M}$ *if and only if the characteristic SVNS* $\chi_\sim := (\tilde{\chi}_T, \tilde{\chi}_I, \tilde{\chi}_F)$ *is an SVN-ideal of* $\mathcal{M}$*.*

Theorem 3.19. *If* $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ *is an SVN-ideal of* $\mathcal{M}$*, then the sets*

$$\mathcal{M}_{\widetilde{\mathcal{N}}_T} := \{x \in M \mid \widetilde{\mathcal{N}}_T(x) = \widetilde{\mathcal{N}}_T(0)\} \ \text{ and } \mathcal{M}_{\widetilde{\mathcal{N}}_F} := \{x \in M \mid \widetilde{\mathcal{N}}_F(x) = \widetilde{\mathcal{N}}_F(0)\},$$

are ideals of $\mathcal{M}$*.*

Proof. Straightforward. □

Corollary 3.20. *If $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is an SVN-ideal of $\mathcal{M}$, then the set*

$$\mathcal{M}_{\widetilde{\mathcal{N}}_I} := \{x \in M \mid \widetilde{\mathcal{N}}_I(x) = \widetilde{\mathcal{N}}_I(0)\},$$

is an ideal of $\mathcal{M}$.

Corollary 3.21. *If $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is an SVN-ideal of $\mathcal{M}$, then the set*

$$\mathcal{M}_{\mathcal{N}_\sim} := \mathcal{M}_{\widetilde{\mathcal{N}}_T} \cap \mathcal{M}_{\widetilde{\mathcal{N}}_I} \cap \mathcal{M}_{\widetilde{\mathcal{N}}_F},$$

is an ideal of $\mathcal{M}$.

The converse of Corollary 3.21 may not be true in general as seen in the following example.

Example 3.22. Consider the pseudo MV-algebra $\mathcal{M}$ in Example 3.3. Let $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ be an SVNS in $\mathcal{M}$ which is given as follows:

$$\widetilde{\mathcal{N}}_T : M \to [0,1],\ x \mapsto \begin{cases} \frac{4}{5} & \text{if } x = 0^*, \\ \frac{1}{2} & \text{if } x \neq 0^*, \end{cases}$$

$$\widetilde{\mathcal{N}}_I : M \to [0,1],\ x \mapsto \begin{cases} \frac{1}{3} & \text{if } x = 0^*, \\ \frac{3}{4} & \text{if } x \neq 0^*, \end{cases}$$

$$\widetilde{\mathcal{N}}_F : M \to [0,1],\ x \mapsto \begin{cases} \frac{3}{5} & \text{if } x = 0^*, \\ \frac{1}{5} & \text{if } x \neq 0^*, \end{cases}$$

Then $\mathcal{M}_{\mathcal{N}_\sim} = \{0^*\}$ which is an ideal of $\mathcal{M}$. But $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is not an SVN-ideal of $\mathcal{M}$ since $\big(0^*, (1, \frac{3}{5})\big) \notin \Gamma^I_{\widetilde{\mathcal{N}}}$ and/or $\big(0^*, (2, -\frac{3}{4})\big) \notin \Gamma^{TF}_{\widetilde{\mathcal{N}}}$.

Theorem 3.23. *Given a subset A of M, the SVNS $\mathcal{N}^A_\sim := (\widetilde{\mathcal{N}}^A_T, \widetilde{\mathcal{N}}^A_I, \widetilde{\mathcal{N}}^A_F)$ is an SVN-ideal of $\mathcal{M}$ if and only if $\mathcal{M}_{\mathcal{N}^A_\sim}$ is an ideal of $\mathcal{M}$.*

Proof. It is clear since $\mathcal{M}_{\mathcal{N}^A_\sim} = A$. □

Proposition 3.24. *If $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is an SVN-ideal of $\mathcal{M}$, then the sets $\widetilde{\mathcal{N}}^+_T := \{x \in M \mid \widetilde{\mathcal{N}}_T(x) > 0\}$ and $\widetilde{\mathcal{N}}^-_F := \{x \in M \mid \widetilde{\mathcal{N}}_T(x) < 1\}$ are ideals of $\mathcal{M}$ when they are nonempty.*

Proof. Assume that $\widetilde{\mathcal{N}}^+_T \neq \emptyset \neq \widetilde{\mathcal{N}}^-_F$. Obviously, $0 \in \widetilde{\mathcal{N}}^+_T \cap \widetilde{\mathcal{N}}^-_F$. Assume that $x, y \in \widetilde{\mathcal{N}}^+_T \cap \widetilde{\mathcal{N}}^-_F$ for all $x, y \in M$. Then $\widetilde{\mathcal{N}}_T(x) > 0$, $\widetilde{\mathcal{N}}_T(y) > 0$, $\widetilde{\mathcal{N}}_F(x) < 1$ and $\widetilde{\mathcal{N}}_F(y) < 1$. It follows from (3.1) that

$$\widetilde{\mathcal{N}}_T(x \oplus y) \geq \min\{\widetilde{\mathcal{N}}_T(x), \widetilde{\mathcal{N}}_T(y)\} > 0 \text{ and } \widetilde{\mathcal{N}}_F(x \oplus y) \leq \max\{\widetilde{\mathcal{N}}_F(x), \widetilde{\mathcal{N}}_F(y)\} < 1.$$

Hence $x \oplus y \in \widetilde{\mathcal{N}}_T^+ \cap \widetilde{\mathcal{N}}_F^-$. Let $x, y \in M$ be such that $x \leq y$ and $y \in \widetilde{\mathcal{N}}_T^+ \cap \widetilde{\mathcal{N}}_F^-$. Then $(x, y) \in \Gamma_{\widetilde{\mathcal{N}}}^{TF}$ by (3.2), and so $\widetilde{\mathcal{N}}_T(x) \geq \widetilde{\mathcal{N}}_T(y) > 0$ and $\widetilde{\mathcal{N}}_F(x) \leq \widetilde{\mathcal{N}}_F(y) < 1$. Thus $x \in \widetilde{\mathcal{N}}_T^+ \cap \widetilde{\mathcal{N}}_F^-$. Therefore $\widetilde{\mathcal{N}}_T^+$ and $\widetilde{\mathcal{N}}_F^-$ are ideals of $\mathcal{M}$. □

Corollary 3.25. *If $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is an SVN-ideal of $\mathcal{M}$, then the set*

$$\widetilde{\mathcal{N}}_I^+ := \{x \in M \mid \widetilde{\mathcal{N}}_I(x) > 0\},$$

is an ideal of $\mathcal{M}$ when it is nonempty.

Corollary 3.26. *If $\mathcal{N}_\sim := (\widetilde{\mathcal{N}}_T, \widetilde{\mathcal{N}}_I, \widetilde{\mathcal{N}}_F)$ is an SVN-ideal of $\mathcal{M}$, then the set $\widetilde{\mathcal{N}}_T^+ \cap \widetilde{\mathcal{N}}_I^+ \cap \widetilde{\mathcal{N}}_F^-$ is an ideal of $\mathcal{M}$ when it is nonempty.*

4. Conclusions and Future Studies

MV-algebras were defined by Chang (1958) as an algebraic counterpart of many-valued reasoning. Pseudo MV-algebras generalize MV-algebras, and pseudo MV-algebras are an algebraic counterpart of non-commutative reasoning. Using an indeterminacy membership function as an independent component, Smarandache introduced the notion of neutrosophic sets which are a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. It is necessary to specify the neutrosophic set from a scientific or engineering perspective. Otherwise, it will be difficult to apply it to actual applications. So, Wang et al. introduced single valued neutrosophic sets to enhance its ease in the real application. The aim of this paper was to study the ideal theory of pseudo MV-algebras using single valued neutrosophic sets. We have introduced the concepts of a single valued neutrosophic ideal in a pseudo MV-algebra, and have investigated several properties. We have presented conditions under which a single valued neutrosophic set can be a single valued neutrosophic ideal, and have discussed characterizations of a single valued neutrosophic ideal. Using the ideas and results of this paper, we will study the neutroscopic set theory in the related algebraic structures: pseudo effect algebras, pseudo BCK-algebras, pseudo BE-algebras, pseudo hoops, pseudo equality algebras, etc., in the future.

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