A NOTE ON MAXIMAL NON-PRIME IDEALS

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ABSTRACT. The rings considered in this article are commutative with identity $1 \neq 0$. We say that a proper ideal $I$ of a ring $R$ is a maximal non-prime ideal if $I$ is not a prime ideal of $R$ but any proper ideal $A$ of $R$ with $I \subseteq A$ and $I \neq A$ is a prime ideal. That is, among all the proper ideals of $R$, $I$ is maximal with respect to the property of being not a prime ideal. The concept of maximal non-maximal ideal and maximal non-primary ideal of a ring can be similarly defined. The aim of this article is to characterize ideals $I$ of a ring $R$ such that $I$ is a maximal non-prime (respectively, a maximal non-maximal, a maximal non-primary) ideal of $R$.

1. INTRODUCTION

The rings considered in this article are nonzero commutative with identity. If $R$ is a subring of a ring $T$ with identity $1$, then we assume that $1 \in R$. If a set $A$ is a subset of a set $B$ and $A \neq B$, we denote it symbolically using the notation $A \subset B$. Let $P$ be a property of rings. Let $R$ be a subring of a ring $T$. Recall from [4] that $R$ is a maximal non-$P$, if $R$ does not have $P$, whereas each subring $S$ of $T$ with $R \subset S$ has property $P$. The concept of maximal non-Noetherian subring of a ring $T$ was investigated in [3]. There are other interesting research articles which appeared in the literature focussing on maximal non-$P$ subring of a ring $T$ (see for example, [2, 4]). Let $R$ be a non-zero commutative ring with identity. A proper ideal $I$ of a ring $R$ is said to be a maximal non-prime ideal of $R$ if the following conditions hold: (i) $I$ is not a

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prime ideal of $R$ and (ii) If $A$ is any proper ideal of $R$ such that $A$ contains $I$ properly, then $A$ is a prime ideal of $R$. Similarly, we can define the concept of a maximal non-maximal (respectively, a maximal non-primary) ideal of $R$. Motivated by the above mentioned works on maximal non-$P$ subrings, in this article, we focus our attempt on characterizing maximal non-prime (respectively, maximal non-maximal, maximal non-primary) ideals of a ring $R$. Let $I$ be a proper radical ideal of a ring $R$. It is proved in Proposition 3.2 that $I$ is a maximal non-primary ideal of $R$ if and only if $I$ is a maximal non-prime ideal of $R$ if and only if $I = M_1 \cap M_2$ for some distinct maximal ideals $M_1, M_2$ of $R$. Let $I$ be a proper ideal of $R$ such that $I \neq \sqrt{I}$. It is shown in Proposition 4.1 that $I$ is a maximal non-prime ideal of $R$ if and only if $I$ is a maximal non-maximal ideal of $R$ if and only if $\sqrt{I} = M$ is a maximal ideal of $R$ with $M^2 \subseteq I$, and $M = Rx + I$ for any $x \in M \setminus I$. Moreover, it is proved in Proposition 4.2 that $I$ is a maximal non-primary ideal of $R$ if and only if $\sqrt{I} = P$ is a prime ideal of $R$ such that $R/I$ is a quasilocal one-dimensional ring and $P/I$ is a minimal ideal of $R/I$.

By a quasilocal ring we mean a ring which admits only one maximal ideal. A Noetherian quasilocal ring is referred to as a local ring. By dimension of a ring $R$, we mean its Krull dimension and we use the abbreviation $\text{dim} R$ to denote the dimension of a ring $R$. We denote the nilradical of a ring $R$ by $\text{nil}(R)$. A ring $R$ is said to be reduced if $\text{nil}(R) = (0)$.

2. Some preliminary results

As mentioned in the introduction the rings considered in this article are commutative with identity $1 \neq 0$. We begin with the following lemma.

**Lemma 2.1.** Let $R$ be a ring. If $P_1, P_2$ are incomparable prime ideals of $R$ under inclusion, then $P_1 \cap P_2$ is not a primary ideal of $R$.

**Proof.** Let $I = P_1 \cap P_2$. Since $P_1$ and $P_2$ are incomparable under inclusion, there exist $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. Note that $ab \in I$. By the choice of $a, b$, it is clear that $a \notin I$ and no power of $b \in I$. This proves that $I = P_1 \cap P_2$ is not a primary ideal of $R$. \hfill $\Box$

**Lemma 2.2.** Let $R$ be a reduced ring which is not an integral domain. If every nonzero proper ideal of $R$ is primary, then $R$ has exactly two prime ideals and both of them are maximal ideals of $R$. 
Proof. Since $R$ is reduced but not an integral domain, it follows that $R$ has at least two minimal prime ideals. Let $P_1, P_2$ be distinct minimal prime ideals of $R$. Now we obtain from Lemma 2.1 and the hypothesis that $P_1 \cap P_2 = (0)$. We prove that $P_1, P_2$ are maximal ideals of $R$. Let $M$ be a maximal ideal of $R$ such that $P_1 \subseteq M$. We claim that $M = P_1$. Suppose that $P_1 \not= M$. Then $M \not\subseteq P_1 \cup P_2$. Let $a \in M \setminus (P_1 \cup P_2)$ and $b \in P_2 \setminus P_1$. As $ab \notin P_1$, it follows that $ab \not= 0$. Hence $R_{ab}$ is a primary ideal of $R$. Note that $R_{ab} \subseteq P_2$. Hence it follows from the choice of $a$ that no power of $a \in R_{ab}$. Therefore, $b \in R_{ab}$. This implies that $b = rab$ for some $r \in R$ and so $b(1-ra) = 0$. As $b \notin P_1$, it follows that $1-ra \in P_1 \subset M$. From $a \in M$, we obtain that $1 = 1-ra + ra \in M$. This is a contradiction. Therefore, $P_1 = M$ is a maximal ideal of $R$. Similarly, it follows that $P_2$ is a maximal ideal of $R$. From $P_1 \cap P_2 = (0)$, we get that $R$ has exactly two prime ideals which are $P_1$ and $P_2$ and moreover, both are maximal ideals of $R$.\hfill\□

Lemma 2.3. Let $R$ be a ring such that every nonzero proper ideal of $R$ is primary. Then $\text{dim} R \leq 1$. Moreover, if $R$ is not a reduced ring, then $R$ is necessarily quasilocal.

Proof. Suppose that $\text{dim} R > 1$. Then there exists a chain of prime ideals $P_1 \subset P_2 \subset P_3$ of $R$. Let $a \in P_2 \setminus P_1$ and $b \in P_3 \setminus P_2$. Since $ab \notin P_1$, it is clear that $ab \not= 0$ and hence $R_{ab} \not= (0)$. Observe that $R_{ab} \subseteq P_2$. By hypothesis, $R_{ab}$ is a primary ideal of $R$. From the choice of the element $b$, it is clear that no power of $b$ can belong to $R_{ab}$. Hence $a \in R_{ab}$. This implies that $a = rab$ for some $r \in R$ and so $a(1-rb) = 0$. Since $a \notin P_1$, it follows that $1-rb \in P_1 \subseteq P_3$. From $b \in P_3$, we obtain that $1 = 1-rb + rb \in P_3$. This is a contradiction. Therefore, $\text{dim} R \leq 1$.

We next prove the moreover assertion. Suppose that $R$ is not quasilocal. Then there exist at least two distinct maximal ideals $M_1, M_2$ of $R$. As we are assuming that $R$ is not a reduced ring, it follows that $M_1 \cap M_2 \not= (0)$. Hence by hypothesis, $M_1 \cap M_2$ is a primary ideal of $R$. This contradicts Lemma 2.1. Therefore, $R$ is necessarily quasilocal.\hfill\□

Lemma 2.4. Let $R$ be a ring which is not reduced. Suppose that $(0)$ is not a primary ideal of $R$. If every nonzero proper ideal of $R$ is primary, then $\text{nil}(R)$ is a minimal prime ideal of $R$. Indeed, $\text{nil}(R)$ is a minimal ideal of $R$.

Proof. We know from Lemma 2.3 that $R$ is necessarily quasilocal. Let $M$ be the unique maximal ideal of $R$. Since $R$ is not reduced, $\text{nil}(R) \not= (0)$. Hence $\text{nil}(R)$ is a primary ideal of $R$ and so it follows from [1,
Proposition 4.1] that \( \sqrt{\text{nil}(R)} = \text{nil}(R) \) is a prime ideal of \( R \). Since \( \text{nil}(R) \subseteq P \) for any prime ideal of \( R \), it follows that \( \text{nil}(R) \) is a minimal prime ideal of \( R \). As \( (0) \) is not a primary ideal of \( R \), it follows from [1, Proposition 4.2] that \( \sqrt{(0)} \) is not a maximal ideal of \( R \). Thus \( \text{nil}(R) \subseteq M. \) We prove that for any nonzero \( a \in \text{nil}(R), \text{nil}(R) = Ra \).

First we verify that for any \( b \in \text{nil}(R) \setminus (0) \) and for any \( m \in M \setminus \text{nil}(R), \) \( bm = 0. \) Suppose that \( bm \neq 0. \) By hypothesis, \( Rbm \) is a primary ideal of \( R. \) In fact \( Rbm \) is a \( \text{nil}(R) \)-primary ideal of \( R. \) Since no power of \( m \in \text{nil}(R), \) we obtain that \( b \in Rbm. \) This implies that \( b = rbm \) for some \( r \in R. \) Thus \( b(1 - rm) = 0. \) As \( 1 - rm \) is a unit in \( R, \) it follows that \( b = 0. \) This is a contradiction. Hence for any nonzero \( b \in \text{nil}(R) \) and \( m \in M \setminus \text{nil}(R), \) \( bm = 0. \) Let \( x \in \text{nil}(R). \) We assert that \( x \in Ra. \) This is clear if \( x = 0. \) If \( x \neq 0, \) then \( xm = 0 \in Ra. \) Now \( Ra \) is a \( \text{nil}(R) \)-primary ideal of \( R \) and no power of \( m \in \text{nil}(R). \) Hence it follows that \( x \in Ra. \) This proves that for any nonzero \( a \in \text{nil}(R), \) \( \text{nil}(R) = Ra. \) This shows that \( \text{nil}(R) \) is a minimal ideal of \( R. \)

**Lemma 2.5.** Let \( R \) be a quasilocal ring with \( M \) as its unique maximal ideal. Suppose that \( R \) is not reduced and \( \text{nil}(R) \) is a prime ideal of \( R \) with \( \text{nil}(R) \neq M. \) If \( \text{nil}(R) \) is a minimal ideal of \( R, \) then \( (0) \) is not a primary ideal of \( R. \)

**Proof.** Let \( a \in \text{nil}(R), a \neq 0. \) Let \( b \in M \setminus \text{nil}(R). \) Since \( \text{nil}(R) \) is a simple \( R \)-module, it follows that \( M(\text{nil}(R)) = (0) \) and so \( ab = 0. \) Now \( a \neq 0 \) and as \( b \notin \text{nil}(R), \) it follows that \( b^n \neq 0 \) for all \( n \geq 1. \) This proves that \( (0) \) is not a primary ideal of \( R. \)

**Lemma 2.6.** Let \( R \) be a ring which is not reduced. If every nonzero proper ideal of \( R \) is a prime ideal of \( R, \) then \( R \) is quasilocal with \( \text{nil}(R) \) as its unique maximal ideal and \( \text{nil}(R)^2 = (0). \) Moreover, for any \( x \in \text{nil}(R) \setminus \{0\}, \) \( \text{nil}(R) = Rx. \)

**Proof.** Since any prime ideal is primary, it follows from Lemma 2.3 that \( R \) is necessarily quasilocal. Let \( M \) be the unique maximal ideal of \( R. \) We prove that \( M = \text{nil}(R). \) Let \( m \in M. \) We assert that \( m^2 = 0. \) Suppose that \( m^2 \neq 0. \) Then \( Rm^2 \) is a prime ideal of \( R. \) Therefore, \( m \in Rm^2. \) This implies that \( m = rm^2 \) for some \( r \in R \) and so \( m(1 - rm) = 0. \) From \( 1 - rm \) is a unit in \( R, \) it follows that \( m = 0. \) This is a contradiction. Thus for any \( m \in M, \) \( m^2 = 0 \) and so \( M = \text{nil}(R). \) Hence \( M \) is the only prime ideal of \( R. \) Let \( a, b \in M. \) We show that \( ab = 0. \) This is clear if either \( a = 0 \) or \( b = 0. \) Suppose that \( a \neq 0 \) and \( b \neq 0. \) Then \( Ra, Rb \) are prime ideals of \( R. \) Therefore, \( Ra = Rb = M. \) This implies that \( a = ub \) for some unit \( u \in R. \) It follows from \( b^2 = 0 \) that \( ab = 0. \) This proves that \( M^2 = (\text{nil}(R))^2 = (0). \)
We next prove the moreover part. Let \( x \in \text{nil}(R) \setminus \{0\} \). Then \( Rx \) is a prime ideal of \( R \). From the fact that \( \text{nil}(R) \) is the only prime ideal of \( R \), it follows that \( \text{nil}(R) = Rx \). \( \square \)

3. Radical non-maximal prime ideals

The aim of this section is to determine proper radical ideals \( I \) of a ring \( R \) such that \( I \) is a maximal non-prime ideal. We start with the following lemma.

**Lemma 3.1.** Let \( D \) be an integral domain which is not a field. Then it admits nonzero proper ideals which are not prime ideals.

**Proof.** Let \( d \in D \) be a nonzero nonunit. Then for any \( n \geq 2 \), \( Dd^n \) is a proper nonzero ideal of \( D \) which is not a prime ideal of \( D \). \( \square \)

**Proposition 3.2.** Let \( R \) be a ring and \( I \) be a proper radical ideal of \( R \). Then the following statements are equivalent:

(i) \( I \) is a maximal non-primary ideal of \( R \).
(ii) \( I = M_1 \cap M_2 \) for some distinct maximal ideals \( M_1, M_2 \) of \( R \).
(iii) \( I \) is a maximal non-maximal ideal of \( R \).
(iv) \( I \) is a maximal non-prime ideal of \( R \).

**Proof.**
(i) \( \Rightarrow \) (ii) Note that \( R/I \) is a reduced ring and as \( I \) is not primary, it follows that \( I \) is not a prime ideal of \( R \) and so \( R/I \) is not an integral domain. Since \( I \) is a maximal non-primary ideal of \( R \), it follows that every nonzero proper ideal of \( R/I \) is primary. Hence we obtain from Lemma 2.2 that there exist distinct maximal ideals \( M_1, M_2 \) of \( R \) such that \( I = M_1 \cap M_2 \).

(ii) \( \Rightarrow \) (iii) We know from Lemma 2.1 that \( I = M_1 \cap M_2 \) is not a primary ideal and hence it is not a maximal ideal of \( R \). Let \( A \) be any proper ideal of \( R \) such that \( M_1 \cap M_2 \subset A \). Then either \( A \not\subset M_1 \) or \( A \not\subset M_2 \). Without loss of generality we may assume that \( A \not\subset M_1 \). Then \( A + M_1 = R \). Hence \( 1 = a + x \) for some \( a \in A \) and \( x \in M_1 \).

Now for any \( y \in M_2, y = ay + xy \in A + M_1M_2 = A \). This proves that \( M_2 \subset A \) and so \( A = M_2 \). Thus the only proper ideals \( A \) of \( R \) which contain \( I \) properly are \( M_1 \) and \( M_2 \) and both are maximal ideals of \( R \). Therefore, we obtain that \( I \) is a maximal non-maximal ideal of \( R \).

(iii) \( \Rightarrow \) (iv) Let \( A \) be a proper ideal of \( R \) with \( I \subset A \). Then by (iii) \( A \) is a maximal ideal of \( R \). Hence \( A \) is a prime ideal of \( R \). We claim that \( I \) is not a prime ideal of \( R \). Suppose that \( I \) is a prime ideal of \( R \). Since \( R/I \) is not a field, it follows from Lemma 3.1 that \( R/I \) admits nonzero proper ideals which are not maximal ideals. This contradicts (iii). Therefore, \( I \) is not a prime ideal of \( R \). This shows that \( I \) is a maximal non-prime ideal of \( R \).
(iv) ⇒ (i) Let $A$ be any proper ideal of $R$ with $I \subseteq A$. Then by (iv) $A$ is a prime ideal and hence is a primary ideal of $R$. Since $I$ is a radical ideal of $R$ and is not a prime ideal of $R$, we get that $I$ is not a primary ideal of $R$. This proves that $I$ is a maximal non-primary ideal of $R$. □

4. Non-radical maximal non-prime ideals

The aim of this section is to determine ideals $I$ of a ring $R$ such that $I \neq \sqrt{I}$ and $I$ is a maximal non-prime ideal of $R$.

**Proposition 4.1.** Let $I$ be a proper ideal of a ring $R$ such that $I \neq \sqrt{I}$. Then the following statements are equivalent:

(i) $I$ is a maximal non-prime ideal of $R$.

(ii) $\sqrt{I}$ is a maximal ideal of $R$, $(\sqrt{I})^2 \subseteq I$, and $\sqrt{I} = Rx + I$ for any $x \in \sqrt{I} \setminus I$.

(iii) $I$ is a maximal non-maximal ideal of $R$.

**Proof.** (i) ⇒ (ii) Note that $R/I$ is a non-reduced ring in which any non-zero proper ideal is a prime ideal. Hence we obtain from Lemma 2.6 that $R/I$ is a quasilocal ring with $\sqrt{I}/I$ as its unique maximal ideal, $(\sqrt{I}/I)^2 = I/I$, and moreover, $\sqrt{I}/I = R/I(x + I)$ for any $x \in \sqrt{I} \setminus I$. Therefore, $\sqrt{I}$ is a maximal ideal of $R$, $(\sqrt{I})^2 \subseteq I$, and $\sqrt{I} = Rx + I$ for any $x \in \sqrt{I} \setminus I$.

(ii) ⇒ (iii) Since $I \subseteq \sqrt{I}$, it follows that $I$ is not a maximal ideal of $R$. Let $A$ be any proper ideal of $R$ such that $I \subseteq A$. From $(\sqrt{I})^2 \subseteq I \subseteq A$, it follows that $\sqrt{I} \subseteq \sqrt{A}$. Since $\sqrt{I}$ is a maximal ideal of $R$, we obtain $\sqrt{I} = \sqrt{A}$. Let $a \in A \setminus I$. Then $a \in \sqrt{I}$. Hence $\sqrt{I} = Ra + I \subseteq A$ and so $A = \sqrt{I}$ is a maximal ideal of $R$. This proves that $I$ is a maximal non-maximal ideal of $R$.

(iii) ⇒ (i) As $I \subseteq \sqrt{I}$, it follows that $I$ is not a prime ideal of $R$. Let $A$ be any proper ideal of $R$ with $I \subseteq A$. Then $A$ is a maximal ideal and hence is a prime ideal of $R$. This shows that $I$ is a maximal non-prime ideal of $R$. □

We next proceed to characterize proper ideals $I$ of a ring $R$ such that $I \neq \sqrt{I}$ and $I$ is a maximal non-primary ideal of $R$.

**Proposition 4.2.** Let $I$ be a proper ideal of a ring $R$ such that $I \neq \sqrt{I}$. Then the following statements are equivalent:

(i) $I$ is a maximal non-primary ideal of $R$.

(ii) $\sqrt{I}$ is a prime ideal of $R$, $R/I$ is quasilocal, $\dim(R/I) = 1$, and $\sqrt{I}/I$ is a simple $R/I$-module.
Proof. (i) $\Rightarrow$ (ii) As $I \neq \sqrt{I}$ and $I$ is a maximal non-primary ideal of $R$, it follows that $I$ is not a primary ideal of $R$, whereas $\sqrt{I}$ is a primary ideal of $R$. Hence $\sqrt{\sqrt{I}} = \sqrt{I}$ is a prime ideal of $R$. Let us denote $\sqrt{I}$ by $P$. Note that $R/I$ is not a reduced ring, the zero-ideal of $R/I$ is not primary but each proper nonzero ideal of $R/I$ is primary. Hence we obtain from Lemma 2.3 that $R/I$ is quasilocal, $\dim(R/I) \leq 1$, and moreover, it follows from Lemma 2.4 that $P/I$ is a minimal ideal of $R/I$ (that is, $P/I$ is a simple $R/I$-module). Let $M/I$ denote the unique maximal ideal of $R/I$. Since $I$ is not a primary ideal of $R$, it follows from [1, Proposition 4.2] that $\sqrt{I}$ is not a maximal ideal of $R$. Therefore, $P/I \subset M/I$ and so $\dim(R/I) = 1$.

(ii) $\Rightarrow$ (i) Note that the ring $R/I$ satisfies the hypotheses of Lemma 2.5. Hence it follows from Lemma 2.5 that the zero-ideal of $R/I$ is not a primary ideal. Hence $I$ is not a primary ideal of $R$. Let $A$ be any proper ideal of $R$ such that $I \subset A$. We consider two cases:

Case(1) $A \subseteq \sqrt{I}$

In this case $A/I$ is a nonzero ideal of $R/I$ and $A/I \subseteq \sqrt{I}/I$. As $\sqrt{I}/I$ is a minimal ideal of $R/I$, we obtain that $A/I = \sqrt{I}/I$ and so $A = \sqrt{I}$ is a prime ideal of $R$. Hence $A$ is a primary ideal of $R$.

Case(2) $A \nsubseteq \sqrt{I}$

Let us denote the unique maximal ideal of $R/I$ by $M/I$. Note that $M$ is the only prime ideal of $R$ containing $A$. Hence it follows that $\sqrt{A} = M$. Since $M$ is a maximal ideal of $R$, we obtain from [1, Proposition 4.2] that $A$ is a primary ideal of $R$.

This proves that $I$ is a maximal non-primary ideal of $R$. $\square$

Recall from [1, p.52] that a proper ideal $I$ of a ring $R$ is said to be decomposable if $I$ admits a primary decomposition (that is, $I$ can be expressed as the intersection of a finite number of primary ideals of $R$). The following proposition characterizes decomposable ideals $I$ of a ring $R$ such that $I \neq \sqrt{I}$ and $I$ is a maximal non-primary ideal.

**Proposition 4.3.** Let $I$ be a proper ideal of a ring $R$ such that $I \neq \sqrt{I}$ and $I$ is decomposable. The following statements are equivalent:

(i) $I$ is a maximal non-primary ideal of $R$.

(ii) $\sqrt{I}$ is a prime ideal of $R$, $(R/I,M/I)$ is quasilocal, $\dim(R/I) = 1$, $I = \sqrt{I} \cap q$, where $q$ is a $M$-primary ideal of $R$, $q \neq M$, and $\sqrt{I}/I$ is a simple $R/I$-module.

Proof. (i) $\Rightarrow$ (ii) It follows from (i) $\Rightarrow$ (ii) of Proposition 4.2 that $\sqrt{I}$ is a prime ideal of $R$, $R/I$ is quasilocal, $\dim(R/I) = 1$, and $\sqrt{I}/I$ is a simple $R/I$-module. Let $M/I$ denote the unique maximal ideal of $R/I$. 


We are assuming that $I$ is decomposable. Let $I = q_1 \cap \cdots \cap q_n$ be an irredundant primary decomposition of $I$ in $R$ with $q_i$ is a $P_i$-primary ideal of $R$ for each $i \in \{1, \ldots, n\}$. Since $I$ is not a primary ideal of $R$, it follows that $n \geq 2$. Note that $\sqrt{I} = \bigcap_{i=1}^{n} P_i$. As $\sqrt{I}$ is a prime ideal of $R$, it follows that $\sqrt{I} = P_i$ for some $i \in \{1, 2, \ldots, n\}$. Without loss of generality we may assume that $\sqrt{I} = P_1$. Since $P_i \neq P_j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$, it follows that $P_1 \subseteq P_j$ for all $j \in \{2, \ldots, n\}$. As $P_1/I$ and $I/M$ are the only prime ideals of $R/I$, it follows that $n = 2$ and $P_2 = M$. Note that $I \subseteq P_1 \cap q_2$. We assert that $I = P_1 \cap q_2$. Since $q_1 \not\subseteq q_2$, it follows that $P_1 \not\subseteq q_2$. Let $x \in P_1 \setminus q_2$ and let $y \in q_2 \setminus P_1$. Observe that $xy \in P_1 \cap q_2$ but no power of $y$ belongs to $P_1 \cap q_2$ and $x \not\in P_1 \cap q_2$. Hence $P_1 \cap q_2$ is not a primary ideal of $R$. As we are assuming that $I$ is a maximal non-primary ideal of $R$, it follows that $I = P_1 \cap q_2$. Since $I \neq \sqrt{I}$, it follows that $q_2 \neq M$.

(ii) $\Rightarrow$ (i) This follows immediately from (ii) $\Rightarrow$ (i) of Proposition 4.2.

Example 4.4. Let $R = K[[X, Y]]$ be the power series ring in two variables $X, Y$ over a field $K$. It is well-known that $R$ is a local ring with $M = RX + RY$ as its unique maximal ideal. Let $I = RX^2 + RXY$. Observe that $I = RX \cap M^2$. Note that $\sqrt{I} = RX$ is a prime ideal of $R$, $M^2 \neq M$ is a $M$-primary ideal of $R$, $dim(R/I) = 1$, and $RX/I$ is a simple $R/I$-module. Hence it follows from (ii) $\Rightarrow$ (i) of Proposition 4.3 that $I$ is a maximal non-primary ideal of $R$.

5. Maximal non-irreducible ideals

Recall that an ideal $I$ of a ring $R$ is irreducible, if $I$ is not the intersection of any ideals $I_1, I_2$ of $R$ with $I \subset I_i$ for each $i \in \{1, 2\}$. The aim of this section is to determine proper ideals $I$ of a ring $R$ such that $I$ is a maximal non-irreducible ideal of $R$. We first characterize proper radical ideals $I$ of $R$ such that $I$ is a maximal non-irreducible ideal of $R$.

Proposition 5.1. Let $I$ be a proper radical ideal of a ring $R$. Then the following statements are equivalent:

(i) $I$ is a maximal non-irreducible ideal of $R$.

(ii) $I = M_1 \cap M_2$ for some distinct maximal ideals $M_1, M_2$ of $R$.

Proof. (i) $\Rightarrow$ (ii) Since $I$ is a proper radical ideal of $R$, it follows from [1, Proposition 1.14] that $I$ is the intersection of all the prime ideals $P$ of $R$ such that $P \supseteq I$. Let $C$ be the collection of all prime ideals $P$ of $R$ such that $P$ is minimal over $I$. Observe that we obtain from [5, Theorem 10] that $I$ is the intersection of all members of $C$. Since $I$ is not irreducible
and any prime ideal is irreducible, we get that $C$ contains at least two elements. Let $P_1, P_2 \in C$ be distinct. We assert that $C = \{P_1, P_2\}$. Suppose that there exists $P_3 \in C$ such that $P_3 \notin \{P_1, P_2\}$. Then it is clear that $I \subset P_2 \cap P_3$ and $P_2 \cap P_3$ is non-irreducible. This is in contradiction to the assumption that $I$ is a maximal non-irreducible ideal of $R$. Therefore, $C = \{P_1, P_2\}$ and so $I = P_1 \cap P_2$. We next show that $P_1$ and $P_2$ are maximal ideals of $R$. Towards showing it, we first prove that $P_1 + P_2 = R$. Suppose that $P_1 + P_2 \neq R$. Let $M$ be a maximal ideal of $R$ such that $P_1 + P_2 \subseteq M$. Since $P_1$ and $P_2$ are not comparable under the inclusion relation, there exist $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. Consider the ideals $J_1 = I + Ra + Rb^2$ and $J_2 = I + Ra^2 + Rb$ of $R$. It is clear that $I \subseteq J_1 \cap J_2$. As $a^2 \in (J_1 \cap J_2) \setminus I$, it follows that $I \subset J_1 \cap J_2$. Since $I$ is a maximal non-irreducible ideal of $R$, we obtain that $J_1 \cap J_2$ is irreducible. Therefore, either $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$. If $J_1 \subseteq J_2$, then $a = x + ra^2 + sb$ for some $x \in I = P_1 \cap P_2$ and $r, s \in R$. This implies that $a(1 - ra) = x + sb \in P_2$. As $a \notin P_2$, we obtain that $1 - ra \in P_2$. Therefore, $1 = ra + 1 - ra \in P_1 + P_2 \subseteq M$. This is a contradiction. Observe that we get a similar contradiction if $J_2 \subseteq J_1$. Hence $P_1 + P_2 = R$. Let $M_1$ be a maximal ideal of $R$ such that $P_1 \subseteq M_1$. Since $P_1 + P_2 = R$, it follows that the ideal $M_1 \cap P_2$ is not irreducible. As $I \subseteq M_1 \cap P_2$, we obtain that $I = P_1 \cap P_2 = M_1 \cap P_2$. Since $P_1 \not\supseteq P_2$, it follows that $P_1 \supseteq M_1$ and so $P_1 = M_1$ is a maximal ideal of $R$. Similarly it can be shown that $P_2$ is a maximal ideal of $R$. Thus $I = M_1 \cap M_2$ for some distinct maximal ideals $M_1, M_2$ of $R$.

(ii) ⇒ (i) If $I = M_1 \cap M_2$ for some distinct maximal ideals $M_1, M_2$ of $R$, then it is clear that $I$ is not irreducible. It is verified in the proof of (ii) ⇒ (iii) of Proposition 3.2 that $M_1$ and $M_2$ are the only proper ideals $J$ of $R$ such that $I \subset J$. Since $M_1$ and $M_2$ are both irreducible, we obtain that $I$ is a maximal non-irreducible ideal of $R$.

Let $I$ be a proper ideal of a ring $R$ such that $I \neq \sqrt{I}$. We next attempt to characterize such ideals $I$ in order that $I$ is a maximal non-irreducible ideal of $R$. We do not know the precise characterization of such ideals. However, we have the following partial results.

**Lemma 5.2.** Let $I$ be a proper ideal of a ring $R$ such that $I \neq \sqrt{I}$. If $I$ is a maximal non-irreducible ideal of $R$, then $\sqrt{I}$ is a prime ideal of $R$ and moreover, $R/I$ is quasilocal.

**Proof.** Let $C$ be the collection of all prime ideals $P$ of $R$ such that $P$ is minimal over $I$. We assert that $C$ is singleton. Let $P, Q \in C$. Since $I \neq \sqrt{I}$, it is clear that $I \subset P \cap Q$. As $I$ is a maximal non-irreducible ideal of $R$, it follows that $P \cap Q$ is irreducible. Hence either $P \subseteq Q$ or
Lemma 5.3. Let \((T, N)\) be a quasilocal ring such that \((0) \neq \sqrt{(0)}\) and \((0)\) is a maximal non-irreducible ideal of \(T\). Then \(\dim_{T/N}(N/N^2) \leq 2\).

**Proof.** Suppose that \(\dim_{T/N}(N/N^2) \geq 3\). Let \(\{a, b, c\} \subseteq N\) be such that \(\{a+N^2, b+N^2, c+N^2\}\) is linearly independent over \(T/N\). Consider the ideals \(J_1 = Ta + Tc\) and \(J_2 = Tb + Tc\). By the choice of \(a, b, c\), it is clear that \(J_1 \nsubseteq J_2\), \(J_2 \nsubseteq J_1\) and so \(J_1 \cap J_2\) is not an irreducible ideal of \(T\). Moreover, as \(c \in J_1 \cap J_2\), it follows that \(J_1 \cap J_2 \neq (0)\). This contradicts the hypothesis that \((0)\) is a maximal non-irreducible ideal of \(T\). Therefore, \(\dim_{T/N}(N/N^2) \leq 2\). \(\square\)

Lemma 5.4. Let \((T, N)\) be a quasilocal ring such that \((0) \neq \sqrt{(0)}\) and \(\dim_{T/N}(N/N^2) = 2\). Then the following statements are equivalent:

(i) \((0)\) is a maximal non-irreducible ideal of \(T\).

(ii) \(N^2 = (0)\).

**Proof.** By hypothesis, \(\dim_{T/N}(N/N^2) = 2\). Let \(\{a, b\} \subseteq N\) be such that \(\{a + N^2, b + N^2\}\) is a basis of \(N/N^2\) as a vector space over \(T/N\).

(i) \(\Rightarrow\) (ii) Consider the ideals \(J_1 = N^2 + Ta\) and \(J_2 = N^2 + Tb\). By the choice of the elements \(a, b\), it is clear that \(J_1 \nsubseteq J_2\) and \(J_2 \nsubseteq J_1\). Hence the ideal \(J_1 \cap J_2\) is not irreducible. Since \((0)\) is a maximal non-irreducible ideal of \(T\), it follows that \(J_1 \cap J_2 = (0)\). As \(N^2 \subseteq J_1 \cap J_2\), we obtain that \(N^2 = (0)\).

(ii) \(\Rightarrow\) (i) It follows from \(N^2 = (0)\) and from the choice of the elements \(a, b\) that \(Ta \nsubseteq Tb, Tb \nsubseteq Ta,\) and \(Ta \cap Tb = (0)\). This implies that \((0)\) is not an irreducible ideal of \(T\). Let \(J\) be any nonzero proper ideal of \(T\). Then either \(\dim_{T/N}(J) = 1\) or \(2\). If \(\dim_{T/N}(J) = 2\), then \(J = N\) is irreducible. Suppose that \(\dim_{T/N}(J) = 1\). Let \(A, B\) be proper ideals of \(T\) such that \(J = A \cap B\). If \(J \neq A\) and \(J \neq B\), then we get that \(A = B = N\) and so \(J = N\). This is a contradiction. Hence either \(J = A\) or \(J = B\). This shows that \(J\) is irreducible. Hence \((0)\) is a maximal non-irreducible ideal of \(T\). \(\square\)
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