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SOME NUMERICAL RESULTS ON TWO CLASSES OF FINITE GROUPS

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Abstract. In this paper, we consider the finitely presented groups G_m and $K(s,l)$ as follows;

$$G_m = \langle a,b | a^m = b^m = 1,\ [a,b]^a = [a,b],\ [a,b]^b = [a,b]\rangle$$

$$K(s,l) = \langle a,b | ab^s = b^l a,\ ba^s = a^l b\rangle;$$

and find the n^{th}-commutativity degree for each of them. Also we study the concept of n-abelianity on these groups, where m,n,s and l are positive integers, $m,n \geq 2$ and $g.c.d(s,l) = 1$.

1. Introduction

Let G be a finite group. The n^{th}-*commutativity degree* of G, written $P_n(G)$, is defined as the ratio

$$\frac{|\{(x,y) \in G \times G | x^n y = y x^n\}|}{|G|^2}.$$

The n^{th}-commutativity degree, first defined by Mohd. Ali and Sarmin [11]. In 1945, F. Levi introduced n-abelian groups [10]. For $n > 1$, a group G is called n-abelian if $(xy)^n = x^n y^n$ for all x and y in G. Abelian groups and groups of exponent dividing n are clearly n-abelian. Some other studies of this concept can be seen in [1, 2, 5]. In [9], the n^{th}-commutativity degree and n-abelianity of 2-generated p-groups of nilpotency class two have been studied. Through this paper, the standard notation $[a,b] = a^{-1}b^{-1}ab$ is used for the commutator of the

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elements a and b.

Here, we consider the following finitely presented groups:

$$G_m = \langle a, b | a^m = b^m = 1,\ [a,b]^a = [a,b],\ [a,b]^b = [a,b] \rangle,\ m \geq 2;$$

$$K(s,l) = \langle a, b | ab^s = b^l a,\ ba^s = a^l b \rangle,\ \textit{where } g.c.d(s,l) = 1,$$

which are nilpotent groups of nilpotency class two.
In Section 2, we state some lemmas and theorems are needed in the proofs of main results. Section 3 is devoted to compute the n^{th}-commutativity degree of G_m and K_m, where $m = l - s + 1$ and $K_m = K(1,m)$. The n-abelianity of these groups have been studied as well.

2. Preliminary

In this section, we state some lemmas and theorems which will be used in other sections. First, we state a lemma without proof that establishes some properties of groups of nilpotency class two.

Lemma 2.1. *If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$:*
(i) $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$;
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$;
(iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.

The following lemma can be seen in [6].

Lemma 2.2. *Let $G_m = \langle a, b | a^m = b^m = 1,\ [a,b]^a = [a,b],\ [a,b]^b = [a,b] \rangle$ where $m \geq 2$, then we have*
(i) every element of G_m can be uniquely presented by $a^i b^j [a,b]^t$, where $1 \leq i, j, t \leq m$.
(ii) $|G_m| = m^3$.

Now, we state some lemmas which can be found in [3, 4].

Lemma 2.3. *The groups $K(s,l) = \langle a, b | ab^s = b^l a,\ ba^s = a^l b \rangle$, where $g.c.d(s,l) = 1$, have the following properties:*
(i) $|K(s,l)| = |l - s|^3$, *if $g.c.d(s,l) = 1$ and is infinite otherwise;*
(ii) if $g.c.d(s,l) = 1$, then $|a| = |b| = (l-s)^2$ and $a^{l-s} = b^{s-l}$.

Lemma 2.4. *(i) For every $l \geq 3$, $K(s,l) \cong K(1, 2-l)$.*
(ii) For every $l \geq 2$ and $g.c.d(s,l) = 1$, $K(s, s+l) \cong K(1, l+1)$.

Lemma 2.5. *Every element of K_m can be uniquely presented by $x = a^{\beta}b^{\gamma}a^{(m-1)\delta}$, where $1 \leq \beta, \gamma, \delta \leq m-1$.*

Lemma 2.6. *In K_m, $[a,b] = b^{m-1} \in Z(K_m)$.*

Also we recall the following lemma of [8].

Lemma 2.7. *For the integers α, β, $(0 \leq \alpha \leq \beta)$ and variables x, z and u, the number of solutions of the equation $p^{\alpha}x \equiv zu \ (mod\ p^{\beta})$ is*

$$p^{2\beta-1}((\alpha+1)p-\alpha).$$

Proposition 2.8. *For the integer β and variables x, y, z and u, the number of solutions of the equation $xy \equiv zu \ (mod\ p^{\beta})$ is*

$$p^{2\beta-1}(p^{\beta+1}+p^{\beta}-1).$$

Proof. By Lemma 2.7, the number of solutions of $xy \equiv ij \ (mod\ p^{\beta})$ is

$$\sum_{\alpha=0}^{\beta} \phi(p^{\beta-\alpha})p^{2\beta-1}((\alpha+1)p-\alpha).$$

To complete the proof, we proceed as follows:

$$\begin{aligned}
&\sum_{\alpha=0}^{\beta} \phi(p^{\beta-\alpha})p^{2\beta-1}((\alpha+1)p-\alpha) \\
&= p^{2\beta-1}((\beta+1)p-\beta) + p^{3\beta-2}(p-1)\sum_{\alpha=0}^{\beta-1}\frac{(\alpha+1)p-\alpha}{p^{\alpha}} \\
&= p^{2\beta-1}((\beta+1)p-\beta) + p^{3\beta-2}(p-1)(p+2\sum_{\alpha=0}^{\beta-1}\frac{1}{p^{t}} - \frac{\beta+1}{p^{\beta-1}}) \\
&= p^{3\beta} - p^{3\beta-1} + 2p^{2\beta-1}(p-1)\frac{p^{\beta}-1}{p-1} + p^{2\beta-1} \\
&= p^{2\beta-1}(p^{\beta+1}+p^{\beta}-1).
\end{aligned}$$

□

Corollary 2.9. *For the integer $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$ and variables x, y, z and u, the number of solutions of the equation $xy \equiv zu \ (mod\ n)$ is $\prod_{i=1}^{k} {p_i}^{2\alpha_i-1}({p_i}^{\alpha_i+1}+{p_i}^{\alpha_i}-1)$.*

Corollary 2.10. *Let m, n be integers and x, y, z and u be variables where $1 \leq x, z \leq n$ and $1 \leq y, u \leq m$. Then the number of solutions of the equation $xy \equiv zu \pmod d$ is*

$$(\frac{m}{d})^2(\frac{n}{d})^2\prod_{i=1}^{k}{p_i}^{2\alpha_i-1}\left({p_i}^{\alpha_i+1}+{p_i}^{\alpha_i}-1\right)$$

where $d = g.c.d(m, n) = p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$.

3. The n^{th}-commutativity degree of G_m and K_m

In this section, by considering the groups G_m and K_m we get explicit formulas for their n^{th}-commutativity degrees. First, we prove the following proposition.

Proposition 3.1. *For the integers $m, n \geq 2$;*

(1) *If $G = G_m$ and $x \in G$, then we have*
$$x^n = a^{ni}b^{nj}[a, b]^{nt-\frac{n(n-1)}{2}ij};$$
(2) *If $G = K_m$ and $x \in G$, then we have*
$$x^n = a^{n\beta}b^{n\gamma}a^{n(m-1)\delta+\frac{n(n-1)}{2}(m-1)\beta\gamma}.$$

Proof. We use an induction method on n. By Lemma 2.2, the assertion holds for $n = 1$. Now, let
$$x^n = a^{ni}b^{nj}[a, b]^{nt-\frac{n(n-1)}{2}ij}.$$
Then
$$x^{n+1} = a^{ni}b^{nj}[a, b]^{nt-\frac{n(n-1)}{2}ij}a^ib^j[a, b]^t.$$
Since $G' \subseteq Z(G)$, by Lemma 2.1 we have
$$x^{n+1} = a^{(n+1)i}b^{(n+1)j}[a, b]^{(n+1)t-\frac{n(n+1)}{2}ij}.$$
The second part may be proved in a similar way. □

Theorem 3.2. *For $m, n \geq 2$, let $G = G_m$, $l = g.c.d(n, m)$ and $\frac{m}{l} = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_s}^{\alpha_s}$. Then we have:*

$$P_n(G) = \prod_{i=1}^{s}\frac{{p_i}^{\alpha_i+1}+{p_i}^{\alpha_i}-1}{{p_i}^{2\alpha_i+1}}.$$

Proof. Let $A_n = \{(x, y) \in G \times G|\ x^n y = yx^n\}$. By Lemma 2.2, we can write $x = a^{i_1} b^{j_1} [a, b]^{t_1}$ and $y = a^{i_2} b^{j_2} [a, b]^{t_2}$ where $1 \leq i_1, i_2, j_1, j_2, t_1, t_2 \leq m$. Then by Proposition 3.1 we have

$$x^n = a^{ni_1} b^{nj_1} [a, b]^{nt_1 - \frac{n(n-1)}{2} i_1 j_1}$$

So

$$x^n y = a^{ni_1} b^{nj_1} [a, b]^{nt_1 - \frac{n(n-1)}{2} i_1 j_1} a^{i_2} b^{j_2} [a, b]^{t_2}.$$

Since G is a group of nilpotency class two, $G' \subseteq Z(G)$. Hence by Lemma 2.1

$$\begin{aligned} x^n y &= a^{ni_1} b^{nj_1} a^{i_2} b^{j_2} [a, b]^{nt_1 + t_2 - \frac{n(n-1)}{2} i_1 j_1} \\ &= a^{ni_1 + i_2} [b, a]^{ni_2 j_1} b^{nj_1 + j_2} [a, b]^{nt_1 + t_2 - \frac{n(n-1)}{2} i_1 j_1} \\ &= a^{ni_1 + i_2} b^{nj_1 + j_2} [a, b]^{nt_1 + t_2 - \frac{n(n-1)}{2} i_1 j_1 - ni_2 j_1}. \end{aligned}$$

On the other hand, we obtain

$$yx^n = a^{ni_1 + i_2} b^{nj_1 + j_2} [a, b]^{nt_1 + t_2 - \frac{n(n-1)}{2} i_1 j_1 - ni_1 j_2}.$$

So if $x^n y = yx^n$, then

$$a^{ni_1 + i_2} b^{nj_1 + j_2} [a, b]^{nt_1 + t_2 - \frac{n(n-1)}{2} i_1 j_1 - ni_2 j_1} = a^{ni_1 + i_2} b^{nj_1 + j_2} [a, b]^{nt_1 + t_2 - \frac{n(n-1)}{2} i_1 j_1 - ni_1 j_2}.$$

Thus by uniqueness of presenting of $x^n y$ and yx^n, Lemma 2.2, we must have

$$[a, b]^{ni_1 j_2 - ni_2 j_1} = 1.$$

Because of $|G'| = m$, $ni_1 j_2 \equiv ni_2 j_1\ (mod\ m)$. Furthermore, if $l = g.c.d(n, m)$ then we have

$$i_1 j_2 \equiv i_2 j_1\ (mod\ \frac{m}{l}).\ (*)$$

Now, let $\frac{m}{l} = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_s}^{\alpha_s}$, where p_i are prime numbers and α_i are positive integers. So by Lemma 2.10, the number of solutions of congruence $(*)$ is

$$(\frac{m}{m/l})^2 (\frac{m}{m/l})^2 \prod_{i=1}^{s} {p_i}^{2\alpha_i - 1} \left({p_i}^{\alpha_i + 1} + {p_i}^{\alpha_i} - 1\right).$$

Now since the parameters k_1 and k_2 are free and $1 \leq k_1, k_2 \leq m$, we obtain

$$|A_n| = m^2 l^4 \prod_{i=1}^{s} {p_i}^{2\alpha_i - 1} \left({p_i}^{\alpha_i + 1} + {p_i}^{\alpha_i} - 1\right).$$

Finally

$$\begin{aligned}
P_n(G) &= \frac{|A_n|}{|G|^2} \\
&= \frac{m^2 l^4 \prod_{i=1}^{s} {p_i}^{2\alpha_i - 1} \left({p_i}^{\alpha_i+1} + {p_i}^{\alpha_i} - 1\right)}{m^6} \\
&= \frac{1}{(m/l)^4} \prod_{i=1}^{s} {p_i}^{2\alpha_i - 1} \left({p_i}^{\alpha_i+1} + {p_i}^{\alpha_i} - 1\right) \\
&= \frac{\prod_{i=1}^{s} {p_i}^{2\alpha_i - 1} \left({p_i}^{\alpha_i+1} + {p_i}^{\alpha_i} - 1\right)}{\prod_{i=1}^{s} p_i^{4\alpha_i}} \\
&= \prod_{i=1}^{s} \frac{{p_i}^{\alpha_i+1} + {p_i}^{\alpha_i} - 1}{{p_i}^{2\alpha_i+1}}.
\end{aligned}$$

□

The following table is a verified result of GAP [7], when $m = 6$.

n	l	m/l	The number of solutions	$P_n(G)$
1	1	6	330	$\frac{55}{216}$
2	2	3	528	$\frac{11}{27}$
3	3	2	810	$\frac{5}{8}$
4	2	3	528	$\frac{11}{27}$
5	1	6	330	$\frac{55}{216}$
6	6	1	1296	1
12	6	1	1296	1

Theorem 3.3. *For $m, n \geq 2$, let $G = K_m$. Then we have:*

$$P_n(G) = \prod_{i=1}^{s} \frac{{p_i}^{\alpha_i+1} + {p_i}^{\alpha_i} - 1}{{p_i}^{2\alpha_i+1}},$$

where $l = g.c.d(m-1, n)$ and $\frac{m-1}{l} = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_s}^{\alpha_s}$.

Proof. Let $G = K_m$ and $B_n = \{(x, y) \in G \times G | x^n y = y x^n\}$. By Lemma 2.3, we can write $x = a^{\beta_1} b^{\gamma_1} a^{(m-1)\delta_1}$ and $y = a^{\beta_2} b^{\gamma_2} a^{(m-1)\delta_2}$ where $1 \leq \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \leq m-1$. Then by Proposition 3.1, we have

$$x^n = a^{n\beta_1} b^{n\gamma_1} a^{n(m-1)\delta_1 - \frac{n(n-1)}{2}(m-1)\beta_1\gamma_1}.$$

Now by using this fact that $[a,b]=b^{m-1}=a^{1-m}$, we obtain

$$\begin{aligned} x^n y &= a^{n\beta_1} b^{n\gamma_1} a^{n(m-1)\delta_1+\frac{n(n-1)}{2}(m-1)\beta_1\gamma_1} a^{\beta_2} b^{\gamma_2} a^{(m-1)\delta_2} \\ &= a^{n\beta_1} b^{n\gamma_1} a^{\beta_2} b^{\gamma_2} a^{n(m-1)\delta_1+\frac{n(n-1)}{2}(m-1)\beta_1\gamma_1+(m-1)\delta_2} \\ &= a^{n\beta_1+\beta_2} b^{n\gamma_1+\gamma_2} a^{(m-1)(n\delta_1+\delta_2+\frac{n(n-1)}{2}\beta_1\gamma_1+n\beta_2\gamma_1)}. \end{aligned}$$

and also

$$yx^n = a^{n\beta_1+\beta_2} b^{n\gamma_1+\gamma_2} a^{n(m-1)\delta_1+(m-1)\delta_2+\frac{n(n-1)}{2}(m-1)\beta_1\gamma_1+n(m-1)\beta_1\gamma_2}.$$

So by uniqueness of presenting of $x^n y$ and yx^n, Lemma 2.5, if $x^n y = yx^n$ then

$$a^{n(m-1)(\beta_1\gamma_2-\beta_2\gamma_1)} = 1$$

and since $|a| = (m-1)^2$;

$$n\beta_1\gamma_2 \equiv n\beta_2\gamma_1 \ (mod\ m-1).$$

Suppose that $l = g.c.d(n, m-1)$. Then

$$\beta_1\gamma_2 \equiv \beta_2\gamma_1 \ (mod\ \frac{m-1}{l}). \ (**)$$

Now, let $\frac{m-1}{l} = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_s}^{\alpha_s}$, where p_i are prime numbers and α_i are positive integers. So, by Lemma 2.10, we get

$$|B_n| = (m-1)^2 l^4 \prod_{i=1}^{s} {p_i}^{2\alpha_i-1}\left({p_i}^{\alpha_i+1}+{p_i}^{\alpha_i}-1\right).$$

Thus

$$\begin{aligned} P_n(G) &= \frac{|B_n|}{|G|^2} \\ &= \frac{1}{((m-1)/l)^4} \prod_{i=1}^{s} {p_i}^{2\alpha_i-1}\left({p_i}^{\alpha_i+1}+{p_i}^{\alpha_i}-1\right) \\ &= \prod_{i=1}^{s} \frac{{p_i}^{\alpha_i+1}+{p_i}^{\alpha_i}-1}{{p_i}^{2\alpha_i+1}}. \end{aligned}$$

□

4. n-ABELIANITY OF G_m AND K_m

Here, we consider 2-generated groups G_m and K_m and investigate when these groups are n-abelian.

Theorem 4.1. *For $m, n \geq 2$, G_m is n-abelian if and only if $m \mid \frac{n(n-1)}{2}$.*

Proof. Let $x = a^{i_1}b^{j_1}[a,b]^{t_1}$ and $y = a^{i_2}b^{j_2}[a,b]^{t_2}$ be two elements of G_m where $1 \leq i_1, i_2, j_1, j_2, t_1, t_2 \leq m$. By using Proposition 3.1 and Lemma 2.1, we obtain

$$\begin{aligned} x^n y^n &= a^{ni_1}b^{nj_1}[a,b]^{nt_1 - \frac{n(n-1)}{2}i_1j_1} a^{ni_2}b^{nj_2}[a,b]^{nt_2 - \frac{n(n-1)}{2}i_2j_2} \\ &= a^{ni_1}b^{nj_1}a^{ni_2}b^{nj_2}[a,b]^{n(t_1+t_2) - \frac{n(n-1)}{2}(i_1j_1+i_2j_2)} \\ &= a^{n(i_1+i_2)}b^{n(j_1+j_2)}[a,b]^{n(t_1+t_2) - \frac{n(n-1)}{2}(i_1j_1+i_2j_2) - n^2 i_2 j_1}. \end{aligned}$$

On the other hand, for computing $(xy)^n$ we use an induction method on n. Indeed

$$xy = a^{i_1+i_2}b^{j_1+j_2}[a,b]^{t_1+t_2-i_2j_1}.$$

and if

$$(xy)^n = a^{n(i_1+i_2)}b^{n(j_1+j_2)}[a,b]^{n(t_1+t_2) - \frac{n(n-1)}{2}(i_1+i_2)(j_1+j_2) - ni_2j_1},$$

then

$$\begin{aligned} (xy)^{n+1} &= a^{n(i_1+i_2)}b^{n(j_1+j_2)}a^{i_1+i_2}b^{j_1+j_2}[a,b]^{(n+1)(t_1+t_2) - \frac{n(n-1)}{2}(i_1+i_2)(j_1+j_2) - (n+1)i_2j_1} \\ &= a^{(n+1)(i_1+i_2)}b^{(n+1)(j_1+j_2)}[a,b]^{(n+1)(t_1+t_2) - \frac{n(n+1)}{2}(i_1+i_2)(j_1+j_2) - (n+1)i_2j_1}. \end{aligned}$$

By Lemma 2.2, each element of G_m has a unique expression in the form $a^ib^j[a,b]^t$, $1 \leq i,j,t \leq m$. So, G_m is n-abelian ($x^ny^n = (xy)^n$, for all $x, y \in G$) if and only if

$$\frac{n(n-1)}{2}(i_1j_1+i_2j_2)+n^2i_2j_1 \equiv \frac{n(n-1)}{2}(i_1+i_2)(j_1+j_2)+ni_2j_1 \ (mod\, m)$$

which is equivalent to

$$\frac{n^2-n}{2}(i_2j_1 - i_1j_2) \equiv 0 \ (mod\ m).$$

Now since this holds for all $x, y \in G_m$, the group G_m is n-abelian if and only if $m|\frac{n(n-1)}{2}$. □

Theorem 4.2. *For $m, n > 2$, K_m is n-abelian if and only if*

$$m-1 \mid \frac{n(n-1)}{2}.$$

Proof. Let $x = a^{\beta_1}b^{\gamma_1}a^{(m-1)\delta_1}$ and $y = a^{\beta_2}b^{\gamma_2}a^{(m-1)\delta_2}$ be two elements of K_m, where $1 \leq \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \leq m-1$. Then by Proposition 3.1 and Lemma 2.1, we get

$$x^n y^n = a^{n(\beta_1+\beta_2)}b^{n(\gamma_1+\gamma_2)}a^{(m-1)(n(\delta_1+\delta_2)+\frac{n(n-1)}{2}(\beta_1\gamma_1+\beta_2\gamma_2)+n^2\beta_2\gamma_1)}.$$

On the other hand by using induction on n, we obtain

$$(xy)^n = a^{n(\beta_1+\beta_2)}b^{n(\gamma_1+\gamma_2)}a^{(m-1)(n(\delta_1+\delta_2)+\frac{n(n-1)}{2}(\beta_1+\beta_2)(\gamma_1+\gamma_2)+n\beta_2\gamma_1)}.$$

Therefore by the uniqueness of presenting of $(xy)^n$ and $x^n y^n$, K_m is n-abelian if and only if

$$\frac{n(n-1)}{2}(\beta_1\gamma_1+\beta_2\gamma_2)+n^2\beta_2\gamma_1 \equiv \frac{n(n-1)}{2}(\beta_1+\beta_2)(\gamma_1+\gamma_2)+n\beta_2\gamma_1 \ (mod\, m-1).$$

Hence

$$\frac{n^2-n}{2}(\beta_2\gamma_1 - \beta_1\gamma_2) \equiv 0 \ (mod\ m-1).$$

This results that K_m is n-abelian if and only if $m-1|\frac{n^2-n}{2}$. □

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