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A CLASS OF J-QUASIPOLAR RINGS

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Abstract. In this paper, we introduce a class of J-quasipolar rings. Let R be a ring with identity. An element a of a ring R is called *weakly J-quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p$ or $a - p$ are contained in $J(R)$ and the ring R is called *weakly J-quasipolar* if every element of R is weakly J-quasipolar. We give many characterizations and investigate general properties of weakly J-quasipolar rings. If R is a weakly J-quasipolar ring, then we show that (1) $R/J(R)$ is weakly J-quasipolar, (2) $R/J(R)$ is commutative, (3) $R/J(R)$ is reduced. We use weakly J-quasipolar rings to obtain more results for J-quasipolar rings. We prove that the class of weakly J-quasipolar rings lies between the class of J-quasipolar rings and the class of quasipolar rings. Among others it is shown that a ring R is abelian weakly J-quasipolar if and only if R is uniquely clean.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Given a ring R, the symbol $U(R)$ and $J(R)$ stand for the group of units and the Jacobson radical of R, respectively.

Let R be a ring and $a \in R$. We adopt the notations $comm(a) = \{b \in R \mid ab = ba\}$ while the *second commutant* $comm^2(a) = \{b \in R \mid bc = cb$ for all $c \in comm(a)\}$ and $R^{qnil} = \{a \in R \mid 1 + ax$ is invertible for each $x \in comm(a)\}$. An element a of a ring R is called *quasipolar* (see [8]) if there exists $p^2 = p \in R$ such that $p \in comm^2(a)$, $a + p \in U(R)$ and

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$ap \in R^{qnil}$. Any idempotent p satisfying the above conditions is called a *spectral idempotent* of a, and this term is borrowed from spectral theory in Banach algebra and it is unique for a. Quasipolar rings have been studied by many ring theorists (see [5],[7], [8] and [12]). Recently, J-quasipolar rings are introduced in [6]. For an element a of a ring R, if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in J(R)$, then a is called *J-quasipolar* and a ring R is called J-quasipolar, if every element of R is J-quasipolar. It is proved that every J-quasipolar ring is quasipolar.

Motivated by these classes of polarity versions of rings, we introduce weakly J-quasipolar rings, generalizing J-quasipolar rings. Throughout this paper, some basic properties of weakly J-quasipolar ring are studied, also examples and counter examples are given. We show that the class of weakly J-quasipolar rings lies properly between the class of J-quasipolar rings and the class of quasipolar rings. It is proved that R is J-quasipolar if and only if R is weakly J-quasipolar and $2 \in J(R)$. Then some of the main results of J-quasipolar rings are special cases of our results for this general setting. Given a ring R, if $M_n(R)$ and $T_n(R)$ denote the ring of all $n \times n$ matrices and triangular matrices over R, then we investigate necessary and sufficient conditions as to weakly J-quasipolarity of $T_2(R)$ over a commutative local ring R. Further, it is proven that $M_n(R)$ is not weakly J-quasipolar for $n \geq 2$. Finally, we determine under what conditions a 2×2 matrix over a commutative local ring is weakly J-quasipolar.

In what follows, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers, the ring of integers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. The notations $detA$ and trA denote the determinant and the trace of a square matrix A over a commutative ring and I_n denotes the $n \times n$ identity matrix.

2. Weakly J-Quasipolar Rings

In this section, we introduce a class of quasipolar rings which is a generalization of J-quasipolar rings. By using weakly J-quasipolar rings, we obtain more results for J-quasipolar rings. It is clear that every J-quasipolar ring is weakly J-quasipolar and we supply an example to show that the converse does not hold in general (see Example 2.9). Moreover, it is shown that the class of weakly J-quasipolar rings lies strictly between the class of J-quasipolar rings and the class of quasipolar rings (see Example 2.9, Corollary 2.11 and Example 2.12). We investigate general properties of weakly J-quasipolar rings.

Definition 2.1. Let R be a ring and $a \in R$. The element a is called *weakly J-quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a+p \in J(R)$ or $a - p \in J(R)$. The idempotent which satisfies the above condition is called a *weakly J-spectral idempotent* and R is called *weakly J-quasipolar* if every element of R is weakly J-quasipolar.

Lemma 2.2 shows that weakly J-quasipolar elements and rings are abundant.

Lemma 2.2. *Let R be a ring. Then we have the followings.*

(1) *Every idempotent in R is weakly J-quasipolar.*
(2) *An element $a \in R$ is weakly J-quasipolar if and only if $-a \in R$ is weakly J-quasipolar.*
(3) *Every element in $J(R)$ is weakly J-quasipolar.*
(4) *Boolean rings are weakly J-quasipolar.*
(5) *J-quasipolar rings are weakly J-quasipolar.*

In the sequel, we state elementary properties of weakly J-quasipolar elements and weakly J-quasipolar rings.

Lemma 2.3. *Let R be a ring. If $u \in U(R)$ is weakly J-quasipolar, then 1 is the weakly J-spectral idempotent of u.*

Proof. Let $u \in U(R)$ be weakly J-quasipolar, so $u + p \in J(R)$ or $u - p \in J(R)$ such that $p^2 = p \in comm^2(u)$. If $u - p \in J(R)$, then $u^{-1}u - u^{-1}p = 1 - u^{-1}p \in J(R)$. Hence, $u^{-1}p \in U(R)$ and so $p \in U(R)$. Thus, we have $p = 1$. In case $u + p \in J(R)$, the proof is similar. □

By using the concept of J-quasipolarity, we obtain a characterization for local rings.

Proposition 2.4. *Let R be a weakly J-quasipolar ring. Then R is a local ring if and only if R has only trivial idempotents.*

Proof. Assume that R is a weakly J-quasipolar ring and has only trivial idempotents. Let $a \in R$, so $a + 1 \in J(R)$ or $a - 1 \in J(R)$ or $a \in J(R)$. If $a + 1 \in J(R)$ or $a - 1 \in J(R)$, then $a \in U(R)$. In the last condition, $a \in J(R)$. Consequently, R is a local ring. The converse statement is clear. □

Lemma 2.5. *Let R be a ring. If $a \in R$ and $u \in U(R)$, then a is weakly J-quasipolar if and only if $u^{-1}au$ is weakly J-quasipolar.*

Proof. Assume that a is weakly J-quasipolar. Then there exists $p^2 = p \in comm^2(a)$ such that $a - p \in J(R)$. If q is taken as $q = u^{-1}pu$, then $q^2 = q \in R$ and $u^{-1}au - u^{-1}pu = u^{-1}(a - p)u \in J(R)$. Let $b \in comm(u^{-1}au)$, then $(u^{-1}au)b = b(u^{-1}au)$ and so $a(ubu^{-1}) = (ubu^{-1})a$.

Thus $ubu^{-1} \in comm(a)$. Since $p \in comm^2(a)$, $(ubu^{-1})p = p(ubu^{-1})$. Hence $b(u^{-1}pu) = (u^{-1}pu)b$. Consequently, $u^{-1}pu \in comm^2(u^{-1}au)$ and so $u^{-1}au$ is weakly J-quasipolar. Conversely, assume that $u^{-1}au - q \in J(R)$, so $a - uqu^{-1} \in J(R)$. Also $(uqu^{-1})^2 = uqu^{-1} \in comm^2(a)$. If $a + p \in J(R)$, then proof is similar. □

The proof of Lemma 2.5 reveals that p is weakly J-spectral idempotent of a if and only if $u^{-1}pu$ is the weakly J-spectral idempotent of $u^{-1}au$. We need the following lemma in order to prove Theorem 2.7.

Lemma 2.6. *Let R be a ring. If $a = j_1 - p \in J(R)$ or $a = j_2 + p \in J(R)$ is weakly J-quasipolar decomposition of a in R, then $ann_l(a) \subseteq ann_l(p)$ and $ann_r(a) \subseteq ann_r(p)$.*

Proof. If $r \in ann_l(a)$, then $ra = 0$. Assume that $a + p = j_1 \in J(R)$ such that $p^2 = p \in comm^2(a)$. Then $rp = r(j_1 - a) = rj_1$ and so $rp = rj_1p = rpj_1$. Hence $rp(1-j_1) = rp - rpj_1 = 0$. Since $1-j_1 \in U(R)$, $r \in ann_l(p)$. If $r \in ann_r(a)$, then $ar = 0$. Thus $pr = (j_1-a)r = j_1r$ and so $pr = pj_1r$. Since $a \in comm(a)$ and $p \in comm^2(a)$, $ap = pa$. Hence $(j_1 - p)p = p(j_1 - p)$ and so $j_1p = pj_1$. Therefore $pr = pj_1r = j_1pr$. Also $(1-j_1)pr = pr - j_1pr = 0$. Because of $1-j_1 \in U(R)$, $r \in ann_r(p)$. If $a - p = j_2 \in J(R)$ such that $p^2 = p \in comm^2(a)$, then the proof is similar to above. □

Theorem 2.7. *If R is weakly J-quasipolar, then so is fRf for all $f^2 = f \in R$.*

Proof. For every $a \in fRf$ there exists $p \in comm^2(a)$ such that $a - p \in J(R)$ or $a + p \in J(R)$. Let $a + p = j_1 \in J(R)$ or $a - p = j_2 \in J(R)$. By Lemma 2.6, we have $1 - f \in ann_l(a) \cap ann_r(a) \subseteq ann_l(p) \cap ann_r(p) = R(1-p) \cap (1-p)R = (1-p)R(1-p)$. Then $pf = p = fp$ and so $a = fj_1f - fpf$, $(fpf)^2 = fpf$ and $fj_1f \in fJ(R)f = J(fRf)$. Lastly, let $xa = ax$ and $x \in fRf$, so $x(fpf) = (fpf)x$. If $a - p = j_2 \in J(R)$, then proof is similar. Consequently, a is weakly J-quasipolar in fRf. □

By the definition of weakly J-quasipolar rings, it is clear that every J-quasipolar ring is weakly J-quasipolar. We now investigate under what condition a weakly J-quasipolar ring is J-quasipolar.

Proposition 2.8. *A ring R is J-quasipolar if and only if R is weakly J-quasipolar and $2 \in J(R)$.*

Proof. Let R be a weakly J-quasipolar ring and $2 \in J(R)$. If $a + p \in J(R)$ such that $p^2 = p \in comm^2(a)$, then it is clear. Let $a - p \in J(R)$ and $p^2 = p \in comm^2(a)$. Since $2 \in J(R)$, $a - p + 2p \in J(R)$ and so a is J-quasipolar. The converse is clear. □

The next example illustrates that there are weakly J-quasipolar rings but not J-quasipolar.

Example 2.9. The ring $\mathbb{Z}_6$ is weakly J-quasipolar but not J-quasipolar.

Proof. It is obvious that $\mathbb{Z}_6$ is weakly J-quasipolar. Since $1+1 \notin J(\mathbb{Z}_6) = 0$, by Proposition 2.8, $\mathbb{Z}_6$ is not J-quasipolar. □

In [6], it is shown that every J-quasipolar element is quasipolar. We obtain the following result for this general setting.

Proposition 2.10. *Every weakly J-quasipolar element in a ring R is quasipolar.*

Proof. Let $a \in R$ be weakly J-quasipolar. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p \in J(R)$ or $a - p \in J(R)$. If $a + p \in J(R)$, then a is quasipolar from [6, Proposition 2.4]. If $a - p \in J(R)$ such that $p^2 = p \in comm^2(a)$, then $a + (1-p) \in U(R)$ and also $(a-p)(1-p) = a(1-p) \in J(R) \subseteq R^{qnil}$. Therefore a is a quasipolar element. □

Corollary 2.11. *If R is weakly J-quasipolar, then it is quasipolar.*

The converse statement of Corollary 2.11 is not true in general, i.e., there are quasipolar rings but not weakly J-quasipolar.

Example 2.12. Let $R = \mathbb{Z}_{(5)}$ be the localization ring of $\mathbb{Z}$ at the prime 5. Then R is a local ring and thus quasipolar by [12, Corollary 3.3]. Since $\frac{1}{3} \in \mathbb{Z}_{(5)}$ is not weakly J-quasipolar, $\mathbb{Z}_{(5)}$ is not weakly J-quasipolar.

By Example 2.9, Corollary 2.11 and Example 2.12, it is clear that the class of weakly J-quasipolar rings lies strictly between the class of J-quasipolar rings and the class of quasipolar rings.

Proposition 2.13. *Any weakly J-quasipolar element $a \in R$ has a unique weakly J-spectral idempotent.*

Proof. Assume that p, q are weakly J-spectral idempotents of $a \in R$.
Case 1: If $a + p \in J(R)$ and $a + q \in J(R)$, then $1 - p$ and $1 - q$ are spectral idempotents of $-a$ by the proof of Proposition 2.10. By [6], the spectral idempotent of a and $-a$ is equal. Also by [8, Proposition 2.3], the spectral idempotent of a is unique, so we obtain that $1 - p = 1 - q$. Then p = q.
Case 2: Assume that $a + p \in J(R)$ and $a - q \in J(R)$. Then $1 - p$ is spectral idempotent of $-a$ and $1 - q$ is spectral idempotent of a. The remaining proof is similar to Case 1.
Case 3: Assume that $a - p \in J(R)$ and $a + q \in J(R)$, then similarly $p = q$.

Case 4: Assume that $a - p \in J(R)$ and $a - q \in J(R)$, then similarly $p = q$. □

In [2], an element of a ring is called *strongly J-clean* provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is *strongly J-clean* in case each of its elements is strongly J-clean. From the definition of a strongly J-clean ring, one may suspects that every weakly J-quasipolar ring is strongly J-clean. But the following example erases possibility.

Example 2.14. It is clear that the ring $\mathbb{Z}_3$ is weakly J-quasipolar. Since $2 \notin J(\mathbb{Z}_3)$, it is not strongly J-clean by [2, Proposition 3.1].

Recall that, a ring R is called *periodic* if for each $x \in R$, there exists distinct positive integers m, n depending on x, for which $x^n = x^m$. For an easy reference, we mention Lemma 2.15 which is one of Jacobson's theorem given in [9] relating to periodicity and commutativity of the rings.

Lemma 2.15. *Let R be a ring in which for every $a \in R$ there exists an integer $n(a) > 1$, depending on a such that $a^{n(a)} = a$, then R is commutative.*

We now give a useful result to determine whether R is weakly J-quasipolar.

Theorem 2.16. *If a ring R is weakly J-quasipolar, then $R/J(R)$ is a periodic ring which has three period and $R/J(R)$ is commutative.*

Proof. Let R be weakly J-quasipolar and $r \in R$. So $r + p \in J(R)$ or $r - p \in J(R)$ such that $p^2 = p \in comm^2(a)$. Clearly, $\overline{r} = \overline{p}$ or $\overline{r} = -\overline{p}$ and $\overline{p}^2 = \overline{p}$. If $\overline{r} = \overline{p}$, then $\overline{r}^2 = \overline{r}$ and so $\overline{r}^3 = \overline{r}$. If $\overline{r} = -\overline{p}$, then it is clear that $\overline{r}^3 = \overline{r}$. Hence $R/J(R)$ is a periodic ring which has period three. By Lemma 2.15, $R/J(R)$ is commutative. □

The following example shows that the converse statement of Theorem 2.16 is not true in general.

Example 2.17. It is clear that the ring $\mathbb{Z}$ is commutative, $J(\mathbb{Z}) = 0$ and $\mathbb{Z}/J(\mathbb{Z}) \cong \mathbb{Z}$. But $\mathbb{Z}$ is not weakly J-quasipolar.

By Theorem 2.16, we obtain the following important result for weakly J-quasipolar rings.

Corollary 2.18. *If R is weakly J-quasipolar, then $R/J(R)$ is weakly J-quasipolar.*

Proof. Proof is clear from Lemma 2.2 (1) and (2). □

Recall that a ring R is said to be *clean* if for each $a \in R$ there exists $e^2 = e \in R$ such that $a - e \in U(R)$. According to Nicholson and Zhou [11], a ring R is said to be *uniquely clean* if for each $a \in R$ there exists unique idempotent $e \in R$ such that $a - e \in U(R)$. In [6], it is proved that a ring R is uniquely clean if and only if R is abelian (i.e., each idempotent of R is central) J-quasipolar. In this direction we generalize this result for weakly J-quasipolar rings.

Theorem 2.19. *A ring R is abelian weakly J-quasipolar if and only if R is uniquely clean.*

Proof. Given $a \in R$, then $-a \in R$. Hence $-a + p \in J(R)$ or $-a - p \in J(R)$ such that $p^2 = p \in R$. If $-a + p \in J(R)$, so a is uniquely clean. If $-a - p \in J(R)$, then $a - (1 - p) \in U(R)$.Uniqueness of the idempotent p follows from Proposition 2.13. Therefore R is a uniquely clean ring. The converse is clear by [6, Theorem 2.7]. □

The next example illustrates that "abelian" condition is not superfluous in Teorem 2.19.

Example 2.20. The matrix ring $T_2(\mathbb{Z}_2)$ is weakly J-quasipolar, but not abelian. By [11, Lemma 4], $T_2(\mathbb{Z}_2)$ is not a uniquely clean ring.

In [1], Ungor et al. introduced and studied a new class of reduced rings (i.e., it has no nonzero nilpotent elements). A ring R is called *feckly reduced* if $R/J(R)$ is a reduced ring. In this direction we show that every weakly J-quasipolar ring is feckly reduced.

Theorem 2.21. *If R is a weakly J-quasipolar ring, then it is feckly reduced.*

Proof. Let R be weakly J-quasipolar and $r^2 = 0$. Therefore there exists $p^2 = p \in comm^2(r)$ such that $r + p \in J(R)$ or $r - p \in J(R)$. If $r - p \in J(R)$, then $r(r - p) = r^2 - rp \in J(R)$. Since $r^2 = 0 \in J(R)$, $rp \in J(R)$. Also $(r - p)p = rp - p \in J(R)$. Hence $p \in J(R)$ and so $p = 0$. Thus $r \in J(R)$ and $R/J(R)$ is reduced. If $r + p \in J(R)$, then similarly $r \in J(R)$ and $R/J(R)$ is reduced. Consequently, R is a feckly reduced ring. □

Let $J^\sharp(R)$ denote the subset $\{x \in R \mid \exists n \in \mathbb{N} \textit{ such that } x^n \in J(R)\}$ of R. It is obvious that $J(R) \subseteq J^\sharp(R)$. Weakly J-quasipolar rings play an important role for the reverse inclusion.

Corollary 2.22. *If R is a weakly J-quasipolar ring, then $J(R) = J^\sharp(R)$*

Proof. Let R be a weakly J-quasipolar ring. By Theorem 2.21, R is feckly reduced and so $J(R) = J^\sharp(R)$ from [1, Proposition 2.6]. □

The following result follows from Corollary 2.22.

Corollary 2.23. *If R is a J-quasipolar ring, then $J(R) = J^\sharp(R)$.*

Corollary 2.22 is helpful to show that a ring is not weakly J-quasipolar.

Example 2.24. Let R denote the ring $M_2(\mathbb{Z}_2)$. Then

$$J^\sharp(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

and $J(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$. By Corollary 2.22, R is not weakly J-quasipolar.

Let R be a ring and $a, b \in R$. Then R is called *directly finite*, if $ab = 1$ then $ba = 1$. It is well known that R is directly finite if and only if $R/J(R)$ is directly finite.

Proposition 2.25. *If a ring R is weakly J-quasipolar, then R is directly finite.*

Proof. The proof is clear from [1, Proposition 4.8]. □

Since every J-quasipolar ring is weakly J-quasipolar, the following result follows from Proposition 2.25.

Corollary 2.26. *If R is a J-quasipolar ring, then R is directly finite.*

In [10], strongly clean rings are introduced and studied. A ring R is *strongly clean*, if for every $a \in R$ there exists $e^2 = e \in R$ such that $a - e \in U(R)$ and $ae = ea$. At the end of that paper, the authors ask some open questions. One of them is " Is every strongly clean ring directly finite?". By Proposition 2.25, weakly J-quasipolar rings are both strongly clean and directly finite.

3. Weakly J-quasipolarity of Matrix rings

In this section we study weakly J-quasipolarity of some matrix rings. It is important to determine whether an individual matrix is weakly J-quasipolar. In particular, we investigate necessary and sufficient conditions weakly J-quasipolarity of the matrix ring $T_2(R)$ over a commutative local ring R. We determine under what conditions a single 2×2 matrix over a commutative local ring is weakly J-quasipolar.

We start with the obvious proposition.

Proposition 3.1. (1) *Let R be a commutative local ring. Then $A \in M_2(R)$ is an idempotent if and only if either $A = 0$, or $A = I_2$, or $A = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ where $bc = a - a^2$.*

(2) *Let R be a commutative local ring and $P \in T_2(R)$. Then P is an idempotent if and only if P has a form* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix}$ *for some $x \in R$.*

Proof. (1) is clear from [3, Lemma 16.4.10] and (2) is straightforward. □

Proposition 3.2. *Let R be a commutative local ring. $A= \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}$ is weakly J-quasipolar in $T_2(R)$ if and only if one of the following holds:*

(1) $A \in J(T_2(R))$,
(2) $A \in \pm 1 + J(T_2(R))$,
(3) $A + P$ *or* $A - P \in J(T_2(R))$ *where* $P = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$ *such that* $x = (a_1 - a_3)^{-1} a_2$,
(4) $A - P$ *or* $A + P \in J(T_2(R))$ *where* $P = \begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix}$ *such that* $x = (a_3 - a_1)^{-1} a_2$.

Proof. Assume that A is weakly J-quasipolar.
Case 1: Let $A + P \in J(T_2(R))$ such that $P^2 = P \in comm^2(A)$. Since $A + P = \begin{bmatrix} a_1 + p_1 & a_2 + p_2 \\ 0 & a_3 + p_3 \end{bmatrix} \in J(T_2(R))$, $a_1 + p_1 \in J(R)$ and $a_3 + p_3 \in J(R)$. Besides assume that $B \in comm(A)$ and take $B = \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}$, so
$\begin{bmatrix} b_1 a_1 & b_1 a_2 + b_2 a_3 \\ 0 & b_3 a_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 + a_2 b_3 \\ 0 & a_3 b_3 \end{bmatrix}$. Therefore $a_2(b_1 - b_3) = b_2(a_1 - a_3)$.
(i) If $a_1, a_3 \in J(R)$, then $p_1 = p_3 = 0$. Hence $p_2 = 0$.
(ii) If $a_1, a_3 \in U(R)$, then $p_1 = p_3 = 1$. Hence $p_2 = 0$.
(iii) If $a_1 \in J(R)$, $a_3 \in U(R)$, then $p_1 = 0, p_3 = 1$ and $p_2 = x \in R$. Since $a_1 - a_3 \in U(R)$, $b_2 = (a_1 - a_3)^{-1} a_2 (b_1 - b_3)$. Providing $x = (a_3 - a_1)^{-1} a_2$, then $P \in comm(B)$. Hence $P \in comm^2(A)$.
(iv) If $a_1 \in U(R)$, $a_3 \in J(R)$, then $p_1 = 1, p_3 = 0$ and $p_2 = x \in R$. Because of $a_1 - a_3 \in U(R)$, $b_2 = (a_1 - a_3)^{-1} a_2 (b_1 - b_3)$. Providing $x = (a_1 - a_3)^{-1} a_2$, then $P \in comm(B)$. Therefore $P \in comm^2(A)$.
Case 2: Let $A - P \in J(T_2(R))$ such that $P^2 = P \in comm^2(A)$. Proof is similar to proof of Case 1.

The converse statement is clear. □

The following result is a direct consequence of Proposition 3.2 for J-quasipolar rings.

Corollary 3.3. *Let R be a commutative local ring. $A=\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}$ is J-quasipolar in $T_2(R)$ if and only if one of the following holds:*

(1) $A \in J(T_2(R))$.
(2) $A \in -1 + J(T_2(R))$.
(3) $A + P \in J(T_2(R))$ *where* $P = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$ *such that* $x = (a_1 - a_3)^{-1}a_2$ *or* $x = (a_3 - a_1)^{-1}a_2$.

Corollary 3.4. *Let R be a ring. If $T_n(R)$ with $n \geq 2$ is weakly J-quasipolar, then R is weakly J-quasipolar.*

Proof. Assume that $T_n(R)$ is weakly J-quasipolar. Let f be the unit matrix with $(1,1)$ entry is 1 and the other entries are 0, then $fT_n(R)f \cong R$. By Theorem 2.7, R is weakly J-quasipolar. □

The following example illustrates that the converse statement of Corollary 3.4 is not true in general.

Example 3.5. If $R = \mathbb{Z}_3$, then R is weakly J-quasipolar. For $A=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in U(T_2(R))$, $A+I_2 \notin J(T_2(R))$ and $A-I_2 \notin J(T_2(R))$. Therefore $T_2(R)$ is not weakly J-quasipolar.

Our next endeavor is to find conditions under which an individual matrix in $M_2(R)$ is weakly J-quasipolar.

Lemma 3.6. *Let R be a ring. Then $A \in U(M_2(R))$ and A is weakly J-quasipolar if and only if $A - I_2 \in J(M_2(R))$ or $A + I_2 \in J(M_2(R))$.*

Proof. Let A be weakly J-quasipolar. Since $A \in U(M_2(R))$, weakly J-spectral idempotent of A is I_2. Hence $A + I_2 \in J(M_2(R))$ or $A - I_2 \in J(M_2(R))$. Conversely, if $A - I_2 \in J(M_2(R))$, then $A \in I_2 + J(M_2(R)) \subseteq U(M_2(R))$. If $A + I_2 \in J(M_2(R))$, then it is clear from the proof of [6, Lemma 4.3] that $A \in U(M_2(R))$. □

The following lemma is important to study especially in a matrix ring.

Lemma 3.7. *If R is a weakly J-quasipolar ring, then $6 \in J(R)$.*

Proof. Let R be a weakly J-quasipolar ring, then there exists $p^2 = p \in comm^2(2)$ such that $2 - p \in J(R)$ or $2 + p \in J(R)$. Assume that $2 - p = j \in J(R)$, therefore $2 - j = p$ and $(2 - j)^2 = 2 - j$. Thus

$2 = j(3-j) \in J(R)$. As a consequence $6 \in J(R)$. If $2+p = j_1 \in J(R)$, then $(j_1-2)^2 = (j_1-2)$. So $6 = j_1(5-j_1) \in J(R)$. □

Lemma 3.7 is helpful to show a ring is not weakly J-quasipolar.

Example 3.8. Since $6 \notin J(\mathbb{Z}_{15}) = 0$, by Lemma 3.7, $\mathbb{Z}_{15}$ is not weakly J-quasipolar.

The converse statement of Lemma 3.7 is not true in general, i.e., for a ring R, if $6 \in J(R)$, then R need not be weakly J-quasipolar.

Example 3.9. It is obvious that $6 \in J(T_2(\mathbb{Z}_3))$. By Example 3.5, the ring $T_2(\mathbb{Z}_3)$ is not weakly J-quasipolar.

Proposition 2.8 shows that in case of $2 \in J(R)$, weakly J-quasipolar rings and J-quasipolar rings are the same. The following example indicates that it does not hold in case of $6 \in J(R)$.

Example 3.10. The ring $\mathbb{Z}_9$ is weakly J-quasipolar and $6 \in J(\mathbb{Z}_9)$. Since there is not a J-spectral idempotent for 4 such that $4+p \in J(\mathbb{Z}_9)$, it is not J-quasipolar.

Lemma 3.11. *Let R be a ring with $6 \in J(R)$. If $a \in R$ is weakly J-quasipolar, then $a+5$ or $a-5$ is weakly J-quasipolar.*

Proof. Let $a \in R$ be weakly J-quasipolar. Thus $a+p \in J(R)$ or $a-p \in J(R)$ such that $p^2 = p \in comm^2(a)$. Assume that $a+p \in J(R)$ and $p^2 = p \in comm^2(a)$. Since $6 \in J(R)$, $a-6+p = (a-5)-(1-p) \in J(R)$. So $a-5$ is weakly J-quasipolar. If $a-p \in J(R)$ such that $p^2 = p \in comm^2(a)$, $a+6-p = (a+5)+(1-p) \in J(R)$. □

Proposition 3.12. *Let R be a commutative ring with $6 \in J(R)$ and $A \in M_2(R)$ such that $A \notin J(M_2(R))$. If both $detA$ and trA are in $J(R)$, then A is not weakly J-quasipolar.*

Proof. If A is weakly J-quasipolar, then $A-5$ or $A+5$ weakly J-quasipolar by Lemma 3.11. Note that $det(A-5) = detA - 5(trA + 5) \in U(R)$. Hence weakly J-spectral idempotent of $A-5$ is I_2 by Lemma 2.3. So $A-5-I_2 \in J(M_2(R))$ or $A-5+I_2 \in J(M_2(R))$. If $A-5-I_2 \in J(M_2(R))$, then A is weakly J-quasipolar, which contradicts the assumption. In other condition, let $A-5+I_2 \in J(M_2(R))$ and so $A-4 \in J(M_2(R))$. Therefore $a_{11}-4, a_{22}-4 \in J(R)$, $a_{11}+a_{22}-8 = trA-8 \in J(R)$. Since $trA \in J(R)$, so $8 \in J(R)$ and $8-6 = 2 \in J(R)$. Thus $A-4+4 \in J(M_2(R))$ is a contradiction. As a consequence A is not weakly J-quasipolar. Also in case of $A+5 \in J(M_2(R))$, proof is similar. Finally A is not weakly J-quasipolar. □

Lemma 3.13. *Let R be a commutative local ring. Then $A = \begin{bmatrix} j & 0 \\ 0 & u \end{bmatrix}$ is weakly J-quasipolar in $M_2(R)$ if and only if one of the following holds.*

(1) $A \in J(M_2(R))$.
(2) $A + I_2 \in J(M_2(R))$.
(3) $A - I_2 \in J(M_2(R))$.
(4) $u \in -1 + J(R)$ *and* $j \in J(R)$.
(5) $u \in J(R)$ *and* $j \in -1 + J(R)$.
(6) $u \in J(R)$ *and* $j \in 1 + J(R)$.
(7) $u \in 1 + J(R)$ *and* $j \in J(R)$.

Proof. Let A be weakly J-quasipolar. Then, there exists $P^2 = P \in comm^2(A)$ such that $A + P \in J(M_2(R))$ or $A - P \in J(M_2(R))$. If $A + P \in J(M_2(R))$, then $(1), (2), (4), (5)$ hold by [6, Lemma 4.7]. Assume that $A - P \in J(M_2(R))$. If $P = 0$ or $P = I_2$ it is clear. Let $P \neq 0$ and $P \neq I_2$. By Proposition 3.1, $P = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ where $bc = a - a^2$. Since $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in comm(A)$ and $P \in comm^2(A)$, $FP = PF$. Then, $b = c = 0$. Thus, $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $A - P \in J(M_2(R))$, $u \in J(R)$ and $j \in 1 + J(R)$ or $u \in 1 + J(R)$ and $j \in J(R)$.
Conversely, if $A \in J(M_2(R))$ or $A + I_2 \in J(M_2(R))$ or $A - I_2 \in J(M_2(R))$, then A is weakly J-quasipolar. If $u \in -1 + J(R)$ and $j \in J(R)$ or $u \in J(R)$ and $j \in -1 + J(R)$, then it follows from [6, Lemma 4.7]. Suppose that $u \in J(R)$ and $j \in 1 + J(R)$. Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $P^2 = P$ and $A - P \in J(M_2(R))$. To show that $P^2 = P \in comm^2(A)$, let $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in comm(A)$. Hence $y = z = 0$ and so $PB = BP$. Thus A is weakly J-quasipolar. If $u \in J(R)$ and $j \in 1 + J(R)$, similarly A is weakly J-quasipolar. $\square$

Proposition 3.14. *Let R be a commutative local ring with $6 \in J(R)$ and let $A \in M_2(R)$ such that $A \notin J(M_2(R))$ and $detA \in J(R)$. Then A is weakly J-quasipolar if and only if $x^2 - (trA)x + detA = 0$ has a root in $J(R)$ and a root in $\mp 1 + J(R)$.*

Proof. Let A be weakly J-quasipolar, $A \notin J(M_2(R))$ and $detA \in J(R)$. Then there exists $P^2 = P \in comm^2(A)$ such that $A - P \in J(M_2(R))$ or $A + P \in J(M_2(R))$. Let $A - P \in J(M_2(R))$. So $trA \in U(R)$, by Proposition 3.12. If $x^2 - (trA)x = -detA$, then $x(x(trA)^{-1} - 1) =$

$-detA(trA)^{-1}$. As R is commutative local, $J(R)$ is a prime ideal in R. Hence $x \in J(R)$ or $x(trA)^{-1} - 1 \in J(R)$. We discuss the following cases.
Case 1: If $x \in J(R)$, then $x(trA)^{-1} - 1 \in -1 + J(R)$.
Case 2: If $x(trA)^{-1} - 1 \in J(R)$, then $x \in 1 + J(R)$.
In case of $A + P \in J(M_2(R))$, the proof is similar. Conversely, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assume that γ_1 and γ_2 are roots of characteristic equation of A such that $\gamma_1 \in J(R)$ and $\gamma_2 \in \mp 1 + J(R)$. It is clear that $trA = \gamma_1 + \gamma_2 \in U(R)$. Without loss of generality, we may assume that $a \in U(R)$. Let $W = \begin{bmatrix} b & a - \gamma_1 \\ \gamma_1 - a & c \end{bmatrix} \in M_2(R)$. Then $detW = bc - (a - \gamma_1)(\gamma_1 - a) \in U(R)$ and $W \in U(M_2(R))$. So $W^{-1}AW = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$. By Lemma 3.13, $W^{-1}AW$ is weakly J-quasipolar. Therefore A is weakly J-quasipolar by Lemma 2.5. □

Theorem 3.15. *Let R be a commutative local ring with $6 \in J(R)$. The matrix $A \in M_2(R)$ is weakly J-quasipolar if and only if one of the following holds:*

(1) *Either A or $A - I_2$ or $A + I_2$ is in $J(M_2(R))$.*
(2) *The equation $x^2 - (trA)x + detA = 0$ has a root in $J(R)$ and a root in $\mp 1 + J(R)$.*

Proof. For the sufficiency, in the case (1) clearly A is weakly J-quasipolar. Suppose that (2) holds. Then $A \notin J(M_2(R))$ and $detA \in J(R)$, so A is weakly J-quasipolar, by Proposition 3.14.
For the necessity, suppose that A, $A - I_2$ and $A + I_2$ are not contained in $J(M_2(R))$. Hence $detA \in J(R)$ by Lemma 3.6. Therefore (2) is guaranteed by Proposition 3.14. □

Lemma 3.16. [4, Lemma 1.5] *Let R be a commutative domain. Then $A \in M_2(R)$ is an idempotent if and only if either $A = 0$ or $A = I_2$ or $A = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix}$ where $bc = a - a^2$.*

Proposition 3.17. *$A \in M_2(\mathbb{Z})$ is weakly J-quasipolar if and only if one of the following hold.*

(1) $A = \begin{bmatrix} -a & b \\ c & a - 1 \end{bmatrix}$ *such that $bc = a - a^2$.*
(2) *A is idempotent.*
(3) $A = \begin{bmatrix} -a & -b \\ -c & a - 1 \end{bmatrix}$ *such that $bc = a - a^2$.*

Proof. Assume that A is weakly J-quasipolar. Since $J(M_2(\mathbb{Z})) = 0$, proof is clear. Conversely, If $A=\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ and $bc = a - a^2$, then A is idempotent. So A is weakly J-quasipolar. Let $A=\begin{bmatrix} -a & b \\ c & a-1 \end{bmatrix}$. If idempotent is chosen as $P=\begin{bmatrix} a & -b \\ -c & 1-a \end{bmatrix}$, then it is clear. Lately, let $A=\begin{bmatrix} -a & -b \\ -c & a-1 \end{bmatrix}$. The idempotent is chosen as $P=\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$, it is clear. □

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