Abstract. Here we consider all finite non-abelian 2-generator $p$-groups ($p$ an odd prime) of nilpotency class two and study the probability of having $n^{th}$-roots of them. Also we find integers $n$ for which, these groups are $n$-central.

1. Introduction

Let $n > 1$ be an integer. An element $a$ of group $G$ is said to have an $n^{th}$-root $b$ in $G$, if $a = b^n$. The probability that a randomly chosen element in $G$ has an $n^{th}$-root, is given by

$$P_n(G) = \frac{|G^n|}{|G|}$$

where $G^n = \{a \in G | a = b^n, for some b \in G\} = \{x^n | x \in G\}$. A. Sadeghieh and H. Doostie in [3] computed the probability $P_n(G)$ for Dihedral groups $D_{2m}$ and Quaternion groups $Q_{2m}$ for every integer $m \geq 3$. Also, in [2] the probability that Hamiltonian groups may have $n^{th}$-roots have been calculated.

For $n > 1$, a group $G$ is said to be $n$-central if $[x^n, y] = 1$ for all $x, y \in G$. In [4], some relations between $n$-abelian and $n$-central groups have been investigated.

Suppose that $H \triangleleft G$ and there is subgroup $K$ such that $G = HK$ and $H \cap K = \{e\}$, then $G$ is said to be the semidirect product of $H$ by $K$; in symbol $G = H \rtimes K$. Clearly if $K \triangleleft G$, then $H \rtimes K \cong H \times K$. 


Keywords: $p$-group, $n^{th}$-roots, $n$-central group.

Received: 1 December 2015, Accepted: 30 December 2015.

*Corresponding author.
First, we state the following Lemma without proof.

**Lemma 1.1.** If $G$ is a group and $G' \subseteq Z(G)$, then the following hold for every integer $k$ and $u, v, w \in G$:

(i) $[uv, w] = [u, w][v, w]$ and $[u, vw] = [u, v][u, w]$;
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$;
(iii) $(uv)^k = u^kv^k[u, v]^{k(k-1)/2}$.

The following theorem classifies all finite non-abelian 2-generator $p$-groups of nilpotency class two ($p \neq 2$).

**Theorem 2.1.** Let $G$ be a finite non-abelian 2-generator $p$-group of nilpotency class two ($p$ an odd prime). Then $G$ is isomorphic to exactly one of the following three types of groups:

1. $G \cong \langle (c) \times (a) \rangle \times \langle b \rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$;
2. $G \cong \langle a \rangle \times \langle b \rangle$, where $[a, b] = ap^{\alpha-\gamma}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$;
3. $G \cong \langle (c) \times (a) \rangle \times \langle b \rangle$, where $[a, b] = ap^{\alpha-\gamma}c$, $[c, b] = a^{-p^{2(\alpha-\gamma)}c^{-p^{\alpha-\gamma}}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$.

**Remark 1.3.** By the relators given in each case, every element $x$ of the above classes of groups can be uniquely presented as $x = c^ka^ib^j$ where $0 \leq k < |c|$, $0 \leq i < p^\alpha$ and $0 \leq j < p^\beta$.

In Section 2, we consider all finite nonabelian 2-generator $p$-groups ($p \neq 2$) of nilpotency class two and study the probability of having $n^{th}$-roots of them. Section 3 is devoted to investigating $n$-centrality of these groups.

## 2. The Probability of Having $n^{th}$-Roots

In this section for each class of finite non-abelian 2-generator $p$-groups ($p \neq 2$) of nilpotency class two, we find the probability of having $n^{th}$-roots. Here for $m \in \mathbb{Z}$, by $m^*$ we mean the arithmetic inverse of $m$.

**Theorem 2.1.** Let $G \cong \langle (c) \times (a) \rangle \times \langle b \rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$. Then

$$P_n(G) = \frac{1}{p^{\alpha+\beta+\gamma}}$$

where $(n, p^\alpha) = p^t$, $(n, p^\beta) = p^s$ and $(n, p^\gamma) = p^w$. 

Proof. Let \( x = c^k a^i b^j \) be an element of \( G^n \) where \( 0 \leq k < p^\gamma \), \( 0 \leq i < p^\alpha \) and \( 0 \leq j < p^\beta \). If \( x = (x_1)^n \) when \( x_1 = c^{k_1} a^{i_1} b^{j_1} \in G \), \( 0 \leq k_1 < p^\gamma \), \( 0 \leq i_1 < p^\alpha \) and \( 0 \leq j_1 < p^\beta \), then we must have
\[
  c^k a^i b^j = (c^{k_1} a^{i_1} b^{j_1})^n = c^{nk_1 - \frac{n(n-1)}{2} i_1 j_1} a^{ni_1} b^{nj_1}.
\]
By uniqueness of presentation of elements of \( G \), we obtain
\[
\begin{cases}
  n i_1 \equiv i \pmod{p^\alpha} \\
  n j_1 \equiv j \pmod{p^\beta} \\
  nk_1 - \frac{n(n-1)}{2} i_1 j_1 \equiv k \pmod{p^\gamma}.
\end{cases}
\] (1)
Now let \((n, p^\alpha) = p^s\). The first congruence of the system (1) has the solution
\[
i_1 \equiv \left( \frac{n}{p^s} \right)^* \left( \frac{i}{p^s} \right) \pmod{p^{\alpha-s}}
\]
if and only if \( p^s | i \). Then
\[
i \in \{ p^s, 2p^s, \ldots, p^{\alpha-s}p^s \}.
\]
This means that \( i \) has \( p^{\alpha-s} \) choices. Similarly if \((n, p^\beta) = p^t\), then by the second equation of System (1) we get
\[
j \in \{ p^t, 2p^t, \ldots, p^{\beta-t}p^t \}.
\]
So \( j \) admits \( p^{\beta-t} \) values.
Now suppose \((n, p^\gamma) = p^w\). Since \( p \neq 2 \), clearly for all \( n \in \mathbb{N} \) we have \( p^w | \frac{n(n-1)}{2} \). Hence from the third equation of system (1), we obtain
\[
k_1 \equiv \left( \frac{n}{p^w} \right)^* \left( \frac{n^2 - n}{2p^w} \right)i_1 j_1 + \left( \frac{n}{p^w} \right)^* \left( \frac{k}{p^w} \right) \pmod{p^{\gamma-w}}
\]
provided that
\[
k \in \{ p^w, 2p^w, \ldots, p^{\gamma-w}p^w \}.
\]
Therefore we have \( p^{\gamma-w} \) choices for \( k \). By the above facts, \( | G^n | \) is equal to
\[
| \{ c^k a^i b^j \} | i \in \{ p^s, \ldots, p^{\alpha-s}p^s \}, j \in \{ p^t, \ldots, p^{\beta-t}p^t \}, k \in \{ p^w, \ldots, p^{\gamma-w}p^w \} |.
\]
Thus
\[
| G^n | = p^{\alpha-s} \times p^{\beta-t} \times p^{\gamma-w} = p^{\alpha+\beta+\gamma-s-t-w}
\]
and
\[
| G | = | a | \times | b | \times | c | = p^{\alpha+\beta+\gamma}.
\]
So
\[
P_n(G) = \frac{| G^n |}{| G |} = \frac{1}{p^{s+t+w}}.
\]
\(\square\)
To continue, we find the probability of having \( n^{th} \)-root for second class of groups of Theorem 1.2.

**Theorem 2.2.** Let \( G \cong \langle a \rangle \rtimes \langle b \rangle \), where \([a, b] = a^{p^{\alpha-\gamma}}, |a| = p^\alpha, |b| = p^\beta, |[a, b]| = p^\gamma\), \( \alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma, \beta \geq \gamma \). Then

\[
P_n(G) = \frac{1}{p^{s+t}}
\]

where \((n, p^\alpha) = p^s\) and \((n, p^\beta) = p^t\).

**Proof.** Let \( x = a^i b^j \in G^n \) where \( 0 \leq i < p^\alpha \) and \( 0 \leq j < p^\beta \). If \( x_1 = a^{i_1} b^{j_1} \in G \), \( 0 \leq i_1 < p^\alpha \) and \( 0 \leq j_1 < p^\beta \) such that \( x = (x_1)^n \), then by uniqueness of presentation of elements of \( G \) (See Remark 1.3) we must have

\[
a^i b^j = (a^{i_1} b^{j_1})^n = a^{n i_1 - \frac{n(n-1)}{2} i_1 j_1} b^{n j_1}.
\]

So

\[
\begin{cases}
  nj_1 \equiv j \pmod{p^\beta} \\
  ni_1 - \frac{n(n-1)}{2} i_1 j_1 \equiv i \pmod{p^\alpha}.
\end{cases}
\] (2)

Now, we consider two cases:

**Case 1.** Suppose \( p^\beta \mid n \). Then the above system changes to

\[
\begin{cases}
  j = 0 \\
  ni_1 \equiv i \pmod{p^\alpha}.
\end{cases}
\]

If \((n, p^\alpha) = p^s\), then

\[
i_1 \equiv \left( \frac{n}{p^\beta} \right)^* \left( \frac{i}{p^\beta} \right) \pmod{p^{\alpha-s}}
\]

is the solution of system (2) if and only if \( p^s \mid i \). So

\[
i \in \{p^s, 2p^s, \ldots, p^{\alpha-s}p^s\}.
\]

Therefore in this case

\[
P_n(G) = \frac{|G^n|}{|G|} = \frac{|\{(i, j) \mid i \in \{p^s, 2p^s, \ldots, p^{\alpha-s}p^s\}, j = 0\}|}{|a| \times |b|} = \frac{p^{\alpha-s}}{p^{\alpha+s}} = \frac{1}{p^{s+t}}.
\]
Case 2. Let $p^β \nmid n$ and $(n, p^β) = p^t$. Then the first equation of the System (2) has solution

$$j_1 \equiv \left(\frac{n}{p^t}\right)^* \left(\frac{j}{p^t}\right) \pmod{p^{β-t}} \quad (3)$$

if $p^t \mid j$. Then

$$j \in \{p^t, 2p^t, \ldots, p^{β-t}p^t\}.$$ 

Now let $(n, p^α) = p^s$. For finding the number of choices of $i$, we have to consider two subcases:

Subcase 2.a. Let $n$ be an even integer, then in second congruence of system (2) we have

$$\frac{n}{2}i_1(2 - (n - 1)j_1) \equiv i \pmod{p^α}.$$ 

Since $(p^α, \frac{n}{2}) = p^s,$

$$i_1(2 - (n - 1)j_1) \equiv \left(\frac{n}{2p^s}\right)^*(\frac{i}{p^s}) \pmod{p^{α-s}}.$$ 

Now by replacing $j = p^{t+1}$ in Congruence (3), we get

$$j_1 \equiv p\left(\frac{n}{p^t}\right)^* \pmod{p^{β-t}}.$$ 

Then $2 - (n - 1)j_1$ and $p^{α-s}$ are prime to each other. So we can write

$$i_1 \equiv \left(\frac{n}{2p^s}\right)^*(2 - (n - 1)j_1)^*(\frac{i}{p^s}) \pmod{p^{α-s}}$$

provided that

$$i \in \{p^s, 2p^s, \ldots, p^{α-s}p^s\}.$$ 

This means that there are $p^{α-s}$ solutions for $i$.

Subcase 2.b. Let $n$ be an odd integer, then

$$ni_1(1 - \frac{(n - 1)}{2}j_1) \equiv i \pmod{p^α}.$$ 

So by considering $j = p^{t+1}$, we get that

$$j_1 \equiv p\left(\frac{n}{p^t}\right)^* \pmod{p^{β-t}}.$$ 

Hence we can write

$$i_1 \equiv \left(\frac{n}{p^s}\right)^*(1 - \frac{(n - 1)}{2}j_1)^*(\frac{i}{p^s}) \pmod{p^{α-s}}.$$
This obtained $i_1$ is a solution of the second equation of system (2) if and only if

$$i \in \{p^s, 2p^s, \dotsc, p^{\alpha-s}p^s\}.$$ 

Now since in both subcases we have $p^{\alpha-s}$ choices for $i$, we get

$$P_n(G) = \frac{|G^n|}{|G|} = \frac{|\{(i, j) \mid i \in \{p^s, \dotsc, p^{\alpha-s}p^s\}, j \in \{p^t, \dotsc, p^{\beta-t}p^t\}\}|}{|a| \times |b|} = \frac{p^{\alpha+\beta-s-t}}{p^{\alpha+\beta}} = \frac{1}{p^{s+t}}.$$ 

Finally for third class of groups of Theorem 1.2, we have the following theorem.

**Theorem 2.3.** Let $G \cong ((c) \times (a)) \rtimes (b)$, where $[a, b] = a^{p^{\alpha-\gamma}}c$, $[c, b] = a^{-p^{\beta(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\sigma$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, $\beta \geq \gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$. Then

$$P_n(G) = \frac{1}{p^{s+t+u}}$$

where $(n, p^\alpha) = p^s$, $(n, p^\beta) = p^t$ and $(n, p^\sigma) = p^\alpha$.

**Proof.** Let $x = c^ka^ib^j$ be an element of $G^n$ where $0 \leq k < p^\sigma$, $0 \leq i < p^\alpha$ and $0 \leq j < p^\beta$. If $x_1 = c^ka^ib^j \in G$ where $0 \leq k_1 < p^\sigma$, $0 \leq i_1 < p^\alpha$, $0 \leq j_1 < p^\beta$ and $x = (x_1)^n$, then we must have

$$c^ka^ib^j = (c^ka^ib^j)^n = c^{nk_1 - \frac{n(n-1)}{2}i_1j_1 + \frac{n(n-1)}{2}p^{\alpha-\gamma}k_1j_1}a^{ni_1 - \frac{n(n-1)}{2}p^{\alpha-\gamma}i_1j_1 + \frac{n(n-1)}{2}p^{2(\alpha-\gamma)}k_1j_1}b^{nj_1}.$$ 

So by uniqueness of presentation of elements of $G$ (See Remark 1.3), we obtain

$$\begin{cases}
nj_1 \equiv j \pmod{p^\beta} \\
nk_1 - \frac{n(n-1)}{2}i_1j_1 + \frac{n(n-1)}{2}p^{\alpha-\gamma}k_1j_1 \equiv k \pmod{p^\sigma} \quad (4) \\
ni_1 - \frac{n(n-1)}{2}p^{\alpha-\gamma}i_1j_1 + \frac{n(n-1)}{2}p^{2(\alpha-\gamma)}k_1j_1 \equiv i \pmod{p^\alpha}.
\end{cases}$$

For solution of this system, we consider two cases:

**Case I.** Let $(n, p^\beta) = p^t$ and $t \neq \beta$. Then the first congruence of System (4) has the solution

$$j_1 \equiv \left(\frac{n}{p^t}\right)^*(-\frac{j}{p^t}) \pmod{p^{\beta-t}}.$$
if and only if \( p^i \mid j \). So

\[
j \in \{p^i, 2p^i, \ldots, p^{\beta-i}p^i\}
\]

and consequently we have \( p^{\beta-i} \) choices for \( j \). Now let \((n, p^\alpha) = p^s\) and \((n, p^\sigma) = p^u\). For solving congruences, we consider two cases. First let \( n \) be an even integer, then we can write

\[
\frac{n}{2} \left(2k_1 - (n - 1)i_1j_1 + (n - 1)p^{\alpha-\gamma}k_1j_1\right) \equiv k \pmod{p^\gamma}
\]

Since \( p \neq 2 \), we have \((\frac{n}{2}, p^\sigma) = p^u\). Therefore

\[
2k_1 - (n - 1)i_1j_1 + (n - 1)p^{\alpha-\gamma}k_1j_1 \equiv \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* \pmod{p^{\sigma-u}}
\]

provided that \( p^u \mid k \). So

\[
k_1(2 + (n - 1)p^{\alpha-\gamma}j_1) \equiv \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* + (n - 1)i_1j_1 \pmod{p^{\sigma-u}} \quad (5)
\]

if

\[
k \in \{p^u, 2p^u, \ldots, p^{\sigma-u}p^u\}.
\]

Hence there are at most \( p^{\sigma-u} \) choices for \( k \). On the other hand, we write

\[
\frac{n}{2} \left(2i_1 - (n - 1)p^{\alpha-\gamma}i_1j_1 + (n - 1)p^{2(\alpha-\gamma)}k_1j_1\right) \equiv i \pmod{p^\alpha}.
\]

Since \((\frac{n}{2}, p^\alpha) = p^s\), we obtain

\[
2i_1 - (n - 1)p^{\alpha-\gamma}i_1j_1 + (n - 1)p^{2(\alpha-\gamma)}k_1j_1 \equiv \left(\frac{n}{2p^s}\right)^* \frac{i}{p^s} \pmod{p^{\alpha-s}}.
\]

provided that \( p^s \mid i \). By replacing the obtained \( k_1 \), in the above congruence we get

\[
2i_1 - (n - 1)p^{\alpha-\gamma}i_1j_1 + (n - 1)p^{2(\alpha-\gamma)}j_1(2 + (n - 1)p^{\alpha-\gamma}j_1)^* \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* + (n - 1)^2p^{2(\alpha-\gamma)}i_1j_1^2(2 + (n - 1)p^{\alpha-\gamma}j_1)^* \equiv \left(\frac{n}{2p^s}\right)^* \frac{i}{p^s} \pmod{p^{\alpha-s}}.
\]

Therefore

\[
i_1(2 - (n - 1)p^{\alpha-\gamma}j_1 + (n - 1)^2p^{2(\alpha-\gamma)}j_1^2(2 + (n - 1)p^{\alpha-\gamma}j_1)^*) \equiv \left(\frac{n}{2p^s}\right)^* \frac{i}{p^s} - (n - 1)p^{2(\alpha-\gamma)}j_1(2 + (n - 1)p^{\alpha-\gamma}j_1)^* \frac{k}{p^u}\left(\frac{n}{2p^u}\right)^* \pmod{p^{\alpha-s}}.
\]
Since \( p \mid (n-1)p^{\alpha-\gamma} j_1 \) and \( p \mid (n-1)^2 p^{2(\alpha-\gamma)} j_1^2 \), we can write
\[ i_1 \equiv (2 - (n-1)p^{\alpha-\gamma} j_1 + (n-1)p^{2(\alpha-\gamma)} j_1^2 (2 + (n-1)p^{\alpha-\gamma} j_1)^* \equiv \frac{i}{p^s} \times \left( \frac{n}{2p^s} \right)^* \]
\[ \times \left( 2 - (n-1)p^{\alpha-\gamma} j_1 + (n-1)p^{2(\alpha-\gamma)} j_1^2 (2 + (n-1)p^{\alpha-\gamma} j_1)^* \equiv \frac{k}{p^u} \left( \frac{n}{2p^u} \right)^* \right) \pmod{p^{\alpha-s}} \]
provided that \( p^s \mid i \). Now clearly \( i_1 \) is a solution of this system if and only if
\[ i \in \{ p^s, 2p^s, \ldots, p^{\alpha-s} p^s \}. \]
Hence we must have exactly \( p^{\alpha-s} \) choices for \( i \). By replacing \( i_1 \) in congruence (5), we get
\[ k_1 \equiv (2 + (n-1)p^{\alpha-\gamma} j_1)^* \frac{k}{p^u} \left( \frac{n}{2p^u} \right)^* + (2 + (n-1)p^{\alpha-\gamma} j_1)^* (n-1) j_1 \]
\[ \times (2 - (n-1)p^{\alpha-\gamma} j_1 + (n-1)^2 p^{2(\alpha-\gamma)} j_1^2 (2 + (n-1)p^{\alpha-\gamma} j_1)^* \equiv \frac{n}{2p^s}\right)^* \]
\[ \times \left( 2 + (n-1)p^{\alpha-\gamma} j_1 + (n-1)^2 p^{2(\alpha-\gamma)} j_1^2 (2 + (n-1)p^{\alpha-\gamma} j_1)^* \equiv \frac{k}{p^u} \left( \frac{n}{2p^u} \right)^* \right) \pmod{p^{\alpha-s}} \]
\[ \times \frac{k}{p^u} \left( \frac{n}{2p^u} \right)^* \]
So we conclude that \( k \) can be chosen in exactly \( p^{s-u} \) ways. Therefore
\[ |G^n| = p^{\alpha-s} \times p^{\beta-t} \times p^{\sigma-u} = p^{\alpha+\beta+\sigma-s-t-u} \]
and
\[ |G| = |a| \times |b| \times |c| = p^{\alpha+\beta+\sigma}. \]
Then we get the desired result. When \( n \) is an odd integer, the theorem can be proved similarly.

Case II. Let \((n, p^\beta) = p^t\). Then clearly \( p^\beta \mid j \) and since \( 0 \leq j < p^\beta \), \( j = 0 \). Then the second and third congruence of System (4) will be proved similar to the proof of Case I. In this case we obtain
\[ |G^n| = |\{(i, j, k) \mid i \in \{p^s, \ldots, p^{\alpha-s} p^s\}, j = 0, k \in \{p^u, \ldots, p^{\sigma-u} p^u\}\}|. \]
Hence
\[ P_n(G) = \frac{|G^n|}{|G|} = \frac{p^{\alpha+s-u}}{p^{\alpha+\beta+\sigma}} = \frac{1}{p^{\beta+s+u}} = \frac{1}{p^{s+\ell+u}}. \]
\(\square\)
3. $n$-CENTRALITY

In this section, we again consider all finite non-abelian 2-generator $p$-groups ($p \neq 2$) of nilpotency class two and this time we investigate $n$-centrality for them.

**Theorem 3.1.** Let $G$ be a finite non-abelian 2-generator $p$-group of nilpotency class two. Then for $n > 1$, the group $G$ is $n$-central if and only if $p^n \mid n$.

**Proof.** According to the Theorem 1.2, we consider three cases:

Case 1. Let $G \cong \langle a \rangle \times \langle b \rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq \beta \geq \gamma$. Also let $x = c^{k_1}a^{i_1}b^{j_1}$ and $y = c^{k_2}a^{i_2}b^{j_2}$ be two elements of $G$ where $0 \leq k_1, k_2 < p^\gamma$, $0 \leq i_1, i_2 < p^\alpha$ and $0 \leq j_1, j_2 < p^\beta$. Then by Lemma 1.1, we get
\[
x^n = c^{nk_1}a^{ni_1}b^{nj_1},
\]
and
\[
x^n y = c^{nk_1+k_2-n(n-1)/2}i_1j_1-ni_2j_1 a^{ni_1+i_2b^{nj_1+j_2}}.
\]
Also we obtain
\[
yx^n = c^{nk_1+k_2-n(n-1)/2}i_1j_1-ni_2j_1 a^{ni_1+i_2b^{nj_1+j_2}}.
\]
We know that $G$ is $n$-central if and only if $x^n y = y x^n$, for all $x, y \in G$. Furthermore by uniqueness of presentation of $x^n y$ and $y x^n$, we see that $x^n y = y x^n$ if and only if
\[
nk_1+k_2-n(n-1)/2i_1j_1-ni_2j_1 \equiv nk_1+k_2-n(n-1)/2i_1j_1-ni_2j_1 (mod p^\gamma).
\]
This is equivalent to
\[
n(i_1j_2-i_2j_1) \equiv 0 (mod p^\gamma).
\]
Now since this holds for all $x, y \in G$, $p^\gamma \mid n$.

Case 2. Let $G \cong \langle a \rangle \times \langle b \rangle$, where $[a, b] = a^{p^\alpha \gamma}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$, $\beta \geq \gamma$. Also, let $x = a^{i_1}b^{j_1}$, $y = a^{i_2}b^{j_2}$ be two elements of $G$, where $0 \leq i_1, i_2 < p^\alpha$ and $0 \leq j_1, j_2 < p^\beta$. By using Lemma 1.1, we get
\[
x^n y = a^{n(i_1+i_2-n(n-1)/2)\gamma i_1j_1-np^\alpha \gamma i_2j_1 b^{nj_1+j_2}}
\]
and
\[
yx^n = a^{n(i_1+i_2-n(n-1)/2)\gamma i_2j_1-np^\alpha \gamma i_1j_2 b^{nj_1+j_2}}.
\]
Hence by uniqueness of presentation of \(x^ny\) and \(yx^n\), the statement \(x^ny = yx^n\) is equal to
\[
n(i_1j_2 - i_2j_1) \equiv 0 \pmod{p^\gamma}
\]
for all \(x, y \in G\). So, we get the desired result.

Case 3. Let \(G \cong (\langle c \rangle \times \langle a \rangle) \times \langle b \rangle\), where \([a, b] = a^{p^{\alpha - \gamma}}c, [c, b] = a^{-p^{2(\alpha - \gamma)}}c^{-p^{\alpha - \gamma}}, |a| = p^\alpha, |b| = p^\beta, |c| = p^\sigma, |[a, b]| = p^\gamma, \alpha, \beta, \gamma, \sigma \in \mathbb{N}, \beta \geq \gamma > \sigma \geq 1, \alpha + \sigma \geq 2\gamma\). By the presentation of elements of \(G\), we have \(x = c^{k_1}a^{i_1}b^{j_1}\) and \(y = c^{k_2}a^{i_2}b^{j_2}\) where \(0 \leq k_1, k_2 < p^\sigma, 0 \leq i_1, i_2 < p^\alpha\) and \(0 \leq j_1, j_2 < p^\beta\).

\[
x^ny = c^{nk_1 + k_2 + \frac{n(n-1)}{2}p^{\alpha - \gamma}k_1j_1 - \frac{n(n-1)}{2}i_1j_1 + np^{\alpha - \gamma}k_2j_1 - ni_2j_1}
\]
\[
\times a^{ni_1 + i_2 + \frac{n(n-1)}{2}p^{2(\alpha - \gamma)}k_1j_1 - \frac{n(n-1)}{2}p^{\alpha - \gamma}i_1j_1 + np^{2(\alpha - \gamma)}k_2j_1 - ni_2j_1}
\]
\[
\times b^{nj_1 + j_2}.
\]

and

\[
yx^n = c^{nk_1 + k_2 + \frac{n(n-1)}{2}p^{\alpha - \gamma}k_1j_1 - \frac{n(n-1)}{2}i_1j_1 + np^{\alpha - \gamma}k_2j_1 - ni_2j_1}
\]
\[
\times a^{ni_1 + i_2 + \frac{n(n-1)}{2}p^{2(\alpha - \gamma)}k_1j_1 - \frac{n(n-1)}{2}p^{\alpha - \gamma}i_1j_1 + np^{2(\alpha - \gamma)}k_2j_1 - ni_2j_1}
\]
\[
\times b^{nj_1 + j_2}.
\]

By the above facts, we see that for all \(x, y \in G; x^ny = yx^n\) if and only if the following system holds
\[
\begin{cases}
n(p^{\alpha - \gamma}(k_1j_2 - k_2j_1) + i_2j_1 - i_1j_2) \equiv 0 \pmod{p^\sigma} \\
n(p^{\alpha - \gamma}(k_1j_2 - k_2j_1) + i_2j_1 - i_1j_2) \equiv 0 \pmod{p^\gamma}.
\end{cases}
\] (6)

Now let \(p^\gamma|n\), then surely \(p^\sigma|n\) and the above congruence system holds. Hence \(G\) will be \(n\)-central.

Conversely let \(G\) be an \(n\)-central group. So the system (6) must hold for all \(x, y \in G\) such as \(x = c^2ab\) and \(y = c^2a^2b\). Then we get
\[
\begin{cases}
n(p^{\alpha - \gamma} - 1) \equiv 0 \pmod{p^\sigma} \\
n(p^{\alpha - \gamma} - 1) \equiv 0 \pmod{p^\gamma}.
\end{cases}
\]

Hence \(p^\gamma|n\).
REFERENCES


Mansour Hashemi
Faculty of mathematical sciences, University of Guilan, P.O.Box 41335-19141, Rasht, Iran.
Email: m_hashemi@guilan.ac.ir

Mikhak Polkouei
Faculty of mathematical sciences, University of Guilan, P.O.Box 41335-19141, Rasht, Iran.
Email: mikhakp@yahoo.com