Journal of Algebra and Related Topics
Vol. 4, No 1, (2016), pp 13-20

HOMOTOPY APPROXIMATION OF MODULES

M. ROUTARAY * AND A. BEHERA

Abstract. Deleanu, Frei, and Hilton have developed the notion of generalized Adams completion in a categorical context. In this paper, we have obtained the Postnikov-like approximation of a module, with the help of a suitable set of morphisms.

1. Introduction

The notion of (generalized) Adams completion arose from a general categorical completion process, suggested by Adams ([1],[2]). Originally, this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [7] where an arbitrary category and an arbitrary set of morphisms of the category are considered.

We emphasize that many algebric and geometrical constructions in algebric topology can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to carefully chosen sets of morphisms. The current work is also in the same direction. The central idea of this note is to approximate a module (Postnikov-like decomposition) in terms of Adams completion.

Let $\mathcal{C}$ be a category and S be a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect to S and $F : \mathcal{C} \to \mathcal{C}[S^{-1}]$, the canonical functor. Let S denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$, $\mathcal{C}[S^{-1}](\text{-}, Y) : \mathcal{C} \to S$ defines

MSC(2010): Primary: 16D10, ; Secondary: 55P60
Keywords: Category of fractions, calculus of left fractions, Adams completion, Grothedieck universe, homotopy theory of modules.
Received: 8 December 2015, Accepted: 25 April 2016.
*Corresponding author .

a contravariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, i.e., $\mathcal{C}[S^{-1}](\text{-}, Y) \cong \mathcal{C}(\text{-}, Y_S)$, then Y_S is called the *generalized Adams completion of* Y with respect to the set of morphisms S or simply the S-*completion* of Y. We shall often refer to Y_S as the *completion* of Y [8].

2. The category $\tilde{\mathcal{M}}$

Behera and Nanda [4] have obtained the Postnikov approximation of a 1-connected based CW-complex, with the help of a suitable set of morphisms. They have obtained this decomposition by introducing a Serre class $\mathcal{C}$ of groups. This note contains a Postnikov-like decomposition of a module over a ring with unity.

The relative homotopy theory of modules, including the (module) homotopy exact sequence was introduced by Peter Hilton ([11], Chapter 13). In fact he has developed homotopy theory in module theory, parallel to the existing homotopy theory in topology. Unlike homotopy theory in topology, there are two types of homotopy theory in module theory, the injective theory and projective theory. They are dual but not isomorphic [16]. Using injective theory we have obtained, by considering a Serre class $\mathcal{C}$ of modules [6], the Postnikov-like factorzation of a module. The narrative may be recalled from [11]. We briefly describe some of the concepts towards notational view-points.

Let Λ be a ring with unity. Let A and B be right Λ-modules and $f : A \to B$ a Λ-homomorphism in the category $\mathcal{M}$. The map f is *i-nullhomotopic,* denoted $f \simeq_i 0$, if f can be extended to some injective module $\bar{A}$ containing A. Also if $g : A \to B$ then $f \simeq_i g$, if $f - g \simeq_i 0$ [11]. The i-homotopy class of f is denoted by $[f]_i$.

Let A and B be right Λ modules and $f : A \to B$. A *mapping cylinder* of f is the module $\bar{A} \oplus B$ together with maps $\lambda : A \to \bar{A} \oplus B$, given by $\lambda(a) = i(a) + f(a)$ where $i : A \to \bar{A}$ is the inclusion, and $\kappa : \bar{A} \oplus B \to B$ is defined by $\kappa(\bar{a} + b) = b$ [11].

Let $\mathcal{U}$ be a fixed Grothendieck universe [11]. Let $\mathcal{M}$ denote the category of all Λ-modules and Λ-homomorphisms and let $\tilde{\mathcal{M}}$ be the corresponding i-homotopy category, that is, the objects of $\tilde{\mathcal{M}}$ are all Λ-modules and the morphisms of $\tilde{\mathcal{M}}$ are i-homotopic classes of Λ-homomorpisms.

We assume that the underlying sets of the elements of $\mathcal{M}$ are elements of $\mathcal{U}$. We fix a suitable set of morhisms in $\tilde{\mathcal{M}}$. For $n \geq 1$, let S_n denote the set of all maps $\alpha : A \to B$ such that for any module M, $\alpha_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, B)$ is a $\mathcal{C}$-isomorphism for $m \leq n$ and a $\mathcal{C}$-epimorphism for $m = n + 1$.

We will show that the set of morphisms S_n of the category $\tilde{\mathcal{M}}$ admits a calculus of left fractions [12].

Proposition 2.1. *S_n admits a calculus of left fractions.*

Proof. Clearly S_n is closed under composition. We shall verify conditions (i) and (ii) of ([7], Theorem 1.3, p.67). Let $\beta\alpha \in S_n$ and $\alpha \in S_n$ where $\alpha : X \to Y$ and $\beta : Y \to Z$. Since $\beta\alpha, \alpha \in S_n$, for any module M in $\tilde{\mathcal{M}}$, $(\beta\alpha)_* = \beta_*\alpha_* : \bar{\pi}_m(M, X) \to \bar{\pi}_m(M, Z)$ and $\alpha_* : \bar{\pi}_m(M, X) \to \bar{\pi}_m(M, Y)$ are $\mathcal{C}$-isomorphisms for $m \leq n$ and $\mathcal{C}$-epimorphisms for $m = n + 1$. It is to be shown that $\beta_* : \bar{\pi}_m(M, Y) \to \bar{\pi}_m(M, Z)$ is $\mathcal{C}$-isomorpism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n + 1$. Since $\beta_*\alpha_*$ and α_* are $\mathcal{C}$-isomorphism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n + 1$, $\beta_* : \bar{\pi}_m(M, Y) \to \bar{\pi}_m(M, Z)$ is a $\mathcal{C}$-epimorphism for $m \leq n + 1$. It is enough to show that β_* is a $\mathcal{C}$-monomorphism for $m \leq n$. This is obvious for any $[b]\,, \left[\tilde{b}\right] \in \bar{\pi}_m(M, Y)$ with $\beta_*[b] = \beta_*\left[\tilde{b}\right]$ there exist $[a]\,, [\tilde{a}] \in \bar{\pi}_m(M, X)$ such that $\alpha_*[a] = [b]$ and $\alpha_*[\tilde{a}] = \left[\tilde{b}\right]$, since α_* is a $\mathcal{C}$-isomorphism for $m \leq n$; hence $(\beta\alpha)_*[a] = \beta_*\alpha_*[a] = \beta_*[b] = \beta_*\left[\tilde{b}\right] = \beta_*\alpha_*[\tilde{a}] = (\beta\alpha)_*[\tilde{a}]$ giving $[a] = [\tilde{a}]$ as $(\beta_*\alpha_*)$ is a $\mathcal{C}$-isomorphism for $m \leq n$. Hence $\beta \in S_n$.

In order to prove condition (ii) of ([7], Theorem 1.3, p. 67) consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\gamma}\downarrow & & \\ C & & \end{array}$$

in $\tilde{\mathcal{M}}$ with $\gamma \in S_n$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ C & \xrightarrow[\beta]{} & D \end{array}$$

in $\tilde{\mathcal{M}}$ with $\delta \in S_n$. Let $\alpha = [f]_i$ and $\gamma = [s]_i$. Let $\bar{A}$ be an injective module containing A and $\iota : A \to \bar{A}$ be the inclusion. The map $\iota_f : A \to \bar{A} \oplus B$ is defined by $\iota_f(a) = i(a) + f(a)$, and $r : \bar{A} \oplus B \to B$ is defined by $r(\bar{a} + b) = b$. Clearly $r \circ \iota_f = f$; this implies ι_f is cofibration [17]. Let $j : B \to \bar{A} \oplus B$ be defined by $j(b) = 0 + b = b$. Clearly $r \circ j = 1_B$. We need to show that $j \circ r \simeq 1_{\bar{A}\oplus B}$, i.e., $1_{\bar{A}\oplus B} - jr \simeq_i 0$. We have $j \circ r(\bar{a} + b) = j(b) = b$ and $(1_{\bar{A}\oplus B} - jr)\,(\bar{a} + b) - jr(\bar{a} + b) = \bar{a}$.

Let $t : \bar{A} \oplus B \to \bar{A}$ be defined by $t(\bar{a} + b) = \bar{a}$ and $s : \bar{A} \to \bar{A} \oplus B$ be defined by $s(\bar{a}) = \bar{a}$. We have $s \circ t : \bar{A} \oplus B \to \bar{A} \oplus B$ and $st(\bar{a} + b) = s(t(\bar{a}+b)) = s(\bar{a}) = \bar{a}$. Clearly $1_{\bar{A}\oplus B} - jr = s \circ t$. Since $\bar{A}$ is injective it follows that $1_{\bar{A}\oplus B} - jr \simeq_i 0$. Thus $1_{\bar{A}\oplus B} \simeq_i jr$. We consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & C \\ {\scriptstyle \iota_f}\downarrow & & \downarrow{\scriptstyle u} \\ \bar{A} \oplus B & \underset{v}{\dashrightarrow} & Q \end{array}$$

and form its push-out in $\mathcal{M}$ where $Q = \left(\bar{A} \oplus B \oplus C\right)/L$ is the factor module and $L = \{i(a) + f(a) + s(a) : a \in A\}$ is a Λ-submodule of $\bar{A} \oplus B \oplus C$. Define $u : C \to Q$ by $u(c) = (0 + 0 + c) + L$ and $v : \bar{A} \oplus B \to Q$ by $v(\bar{a} + b) = (\bar{a} + b + 0) + L$. Clearly, the two maps are well defined and Λ-module homomorphisms. For any $a \in A$, $us(a) = u(s(a)) = (0 + 0 + s(a)) + L = (s(a)) + L = L$. On the other hand $v\iota_f(a) = v\left(\iota(a) + f(a)\right) = (\iota(a) + f(a) + 0) + L = L$. Thus $us = v\iota_f$. Hence the above diagram is commutative.

Since ι_f is co-fibration, so is u [17], we therefore have the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{\iota_f} & \bar{A} \oplus B & \xrightarrow{p} & X \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle v} & & \Vert \\ C & \xrightarrow[u]{} & Q & \xrightarrow[q]{} & X \end{array}$$

where X is the co-kernel of ι_f, as well as of u; p and q are the usual projections. We consider the exact homotopy sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \bar{\pi}_{m+1}(M, X) & \longrightarrow & \bar{\pi}_m(M, A) & \longrightarrow & \bar{\pi}_m(M, \bar{A} \oplus B) & \longrightarrow \\ & & \Vert & & \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle v_*} & \\ \cdots & \longrightarrow & \bar{\pi}_{m+1}(M, X) & \longrightarrow & \bar{\pi}_m(M, C) & \longrightarrow & \bar{\pi}_m(M, Q) & \longrightarrow \end{array}$$

$$\begin{array}{ccccc} \bar{\pi}_m(M, X) & \longrightarrow & \bar{\pi}_{m-1}(M, A) & \longrightarrow & \cdots \\ \Vert & & \downarrow{\scriptstyle s_*} & & \\ \bar{\pi}_m(M, X) & \longrightarrow & \bar{\pi}_{m-1}(M, C) & \longrightarrow & \cdots \end{array}$$

From Five lemma [5] it follows that $v_* : \bar{\pi}_m(M, \bar{A} \oplus B) \to \bar{\pi}_m(M, Q)$ is $\mathcal{C}$-isomorphism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n + 1$. Since j is a i-null homotopy equivalence, $(vj)_* : \bar{\pi}_m\left(M, B\right) \to \bar{\pi}_m\left(M, Q\right)$ is a $\mathcal{C}$-isomorphism for $m \leq n$ and a $\mathcal{C}$-epimorphism for $m = n+1$. We consider the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{s} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow[vj]{} & Q \end{array}$$

Let $\beta = [u]_i$ and $\delta = [vj]_i$. Taking the corresponding i-homotopic classes, we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{r=[s]_i} & C \\ {\scriptstyle \alpha=[f]_i}\downarrow & & \downarrow{\scriptstyle \beta=[u]_i} \\ B & \xrightarrow[\delta=[vj]_i]{} & Q \end{array}$$

in $\tilde{\mathcal{M}}$ with $\delta \in S_n$. This indeed is a weak push-out diagramin $\tilde{\mathcal{M}}$. This completes the proof of the proposition. □

Proposition 2.2. *Let $s_j : A_j \to B_j$ lie in S_n, for each $j \in J$, where the index set J is an element of $\mathcal{U}$. Then $\bigvee_{j \in J} s_j : \bigvee_{j \in J} A_j \to \bigvee_{j \in J} B_j$ lies in S_n.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{j\in J} \bar{\pi}(M, A_j) & \xrightarrow[\simeq]{\{\alpha_{j*}\}} & \bar{\pi}(M, \bigvee_{j\in J} A_j) \\ {\scriptstyle \bigoplus_{j\in J} s_{j*}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \left(\bigvee_{j\in J} s_j\right)_*} \\ \bigoplus_{j\in J} \bar{\pi}(M, B_i) & \xrightarrow[\{\beta_{j*}\}]{\simeq} & \bar{\pi}(M, \bigvee_{j\in J} A_j) \end{array}$$

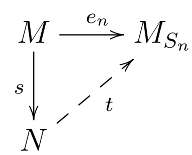
where $\alpha_j : A_j \to \bigvee_{j\in J} A_j$ and $\beta_j : B_j \to \bigvee_{j\in J} B_j$ are the canonical inclusions. Note that each horizontal row is an isomorphism, hence a $\mathcal{C}$-isomorphism. Since each s_{j_*} is a $\mathcal{C}$-isomorphism in dim $\leq n$ and a $\mathcal{C}$-epimorphism in dimension $n+1$, so is $\bigoplus_{j\in J} s_{j*}$, and from the commutative diagram it follows that $\left(\bigvee_{j\in J} s_j\right)_*$ is also a $\mathcal{C}$-isomorphism in dim $\leq n$ and a $\mathcal{C}$-epimorphism in dim $n+1$. Thus $\bigvee_{j\in J} s_j \in S_n$. This competes the proof. □

The following result is well known.

Proposition 2.3. *The category $\tilde{\mathcal{M}}$ is cocomplete.*

From Propositions 2.1, 2.2 and 2.3 we see that all the conditions of ([4], p. 528) are satisfied and hence we have the following result.

Theorem 2.4. *Every object M of the category $\tilde{\mathcal{M}}$ has an Adams completion M_{S_n} with respect to the set S_n of Λ-module homorphisms. Furthermore, there exists a Λ- module homorphism $e_n : M \to M_{S_n}$ in $\bar{S}_n$ which is couniversal with respect to the Λ-module homomorphisms in S_n: given a Λ- module homorphism $s : M \to N$ in S_n there exists a unique Λ-module homorphism $t : N \to M_{S_n}$ in $\bar{S}_n$ such that $ts = e_n$. In other words the following diagram is commutative:*



Theorem 2.5. *The Λ-module homorphism $e_n : M \to M_{S_n}$ is in S_n.*

Proof. Let S_n^1 be the set of all morphisms $f : A \to B$ in the category $\mathcal{M}$ such that $f_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, B)$ is a $\mathcal{C}$-monomorphism for $m \leq n$ and S_n^2 be the set of all morphisms $f : A \to B$ in the category $\tilde{\mathcal{M}}$ such that $f_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, B)$ is a $\mathcal{C}$-epimorphism for $m \leq n + 1$. Clearly (i) $S_n = S_n^1 \cap S_n^2$, (ii) S_n^1 and S_n^2 satisfy all the conditions of ([4], P.533). Therefore, $e_n \in S_n$. This completes the proof. □

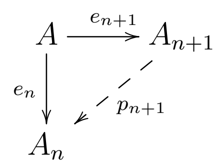
3. A Postnikov-like approximation

We obtain a decomposition of a module with the help of the sets of morphisms S_n.

Theorem 3.1. *For any Λ-module A, for $n \geq 1$, there exist modules A_n, maps $e_n : A \to A_n$ and maps $p_{n+1} : A_{n+1} \to A_n$ such that*

(i) *$e_{n_*} : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, A_n)$ is $\mathcal{C}$- isomorphism for $m \leq n$ and $\bar{\pi}_m(M, A_n) = 0$, for $m > n$,*
(ii) *$e_n = p_{n+1} \circ e_{n+1}$.*

Proof. For each integer $n \geq 1$, let A_n be the S_n-completion of A and $e_n : A \to A_n$ be the canonical map as stated in Theorem 2.5 Since $e_{n+1} \in S_{n+1}$, it follows that $e_{n+1} \in S_n$; hence by the couniversal property of e_{n+1}, there exists a map $p_{n+1} : A_{n+1} \to A_n$ making the following diagram commutative, i.e., $p_{n+1} \circ e_{n+1} = e_n$



Since $e_n \in S_n$, $e_{n_*} : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, A_n)$ is a $\mathcal{C}$-isomorphism for $m \leq n$. We show that $\bar{\pi}_m(M, A_n) = 0, \quad m > n$. Every Λ-module M has an injective resolution [5]. So we can take an injective resolution of M as $M \to \overline{M} \to \overline{SM} \to \cdots \to \overline{S^m M} \to \cdots$ with successive cokernels $SM, S^2M \cdots, S^{m+1}M, \cdots$. We break the exact sequence into short exact sequences:

$$
\begin{aligned}
&0 \to M \to \overline{M} \to SM \to 0\\
&0 \to SM \to \overline{SM} \to S^2M \to 0\\
&\qquad\vdots\\
&0 \to S^{m-1}M \to \overline{S^{m-1}M} \to S^mM \to 0\\
&\qquad\vdots
\end{aligned}
$$

Applying $\mathrm{Ext}^j_\Lambda(M, -)$ to the short exact sequence $0 \to S^{m-1}M \to \overline{S^{m-1}M} \to S^mM \to 0$ of Λ-modules, we get the exact sequence

$$0 \to \mathrm{Ext}^j_\Lambda\left(M, S^{m-1}M\right) \to \mathrm{Ext}^j_\Lambda\left(M, \overline{S^{m-1}M}\right) \to \mathrm{Ext}^j_\Lambda\left(M, S^mM\right) \to 0$$

for any $j > 0$. Since $\overline{S^{m-1}M}$ is injective, $\mathrm{Ext}^j_\Lambda(M, \overline{S^{m-1}M}) = 0$ for each $j > 0$ [5]. It is clear that $\mathrm{Ext}^j_\Lambda(M, S^mM) = 0$ and S^mM is injective [11]. Hence $\bar{\pi}_m(M, A_n) = 0$ for all $m > n$. Thus we get Postnikov-like approximation of a module in $\tilde{\mathcal{M}}$. This completes the proof.

□

Acknowledgments

The author would like to thank the unknown reviewers for the comments which resulted in an improved presentation of the paper.

References

1. J. F. Adams, *Idempotent functors in homotopy theory*, Manifolds, Conf. Univ. of Tokyo Press, Tokyo, 1973.

2. J. F. Adams, *Stable homotopy and generalized homology*, The Univ. of Chicago Press, Chicago and London, 1974.
3. J. F. Adams, *Localization and completion*, Lecture Notes in Mathematics, Univ. of Chicago, 1975.
4. A. Behera and S. Nanda, *Mod -$\mathcal{C}$ Postnikov approximation of a 1-connected space*, Canad. J. Math., (39) **3** (1987), 527-543.
5. P. E. Bland, *Rings and their Modules*, De Gruyter, 2011.
6. S. Cormack, *Extensions relative to a serre class*, University of Edinburgh, 1974.
7. A. Deleanu, A. Frei, and P. Hilton, *Generalized Adams Completions*, Cahiers de Top. et Geom. Diff., (15) **1** (1974), 61-82.
8. A. Deleanu, *Existence of the Adams completions for objects of cocomplete categories*, J. Pure and Appl. Alg., **6** (1975), 31-39.
9. D. S. Dummit and R.M. Foote, *Abstract algebra*, John Wiley and Sons, 2004.
10. P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer Verlag, New York, 1967.
11. P. Hilton, *Homotopy theory and duality*, Grdon and Breach, 1965.
12. H. Schubert, *Categories*, Springer-Verlag, New York, 1972.
13. E. H. Spanier, *Algebric topology*, McGraw, 1966.
14. C. J. Su, *The category of long exact sequences and the homotopy exact sequence of modules*, IJMMS, **22** (2003).
15. C. J. Su, *On long exact ($\bar{\pi}, \mathrm{Ext}_\Lambda$)*, IJMMS, **26** (2004), 1347-1361.
16. C. J. Su, *On relative homotopy groups of modules*, IJMMS, Article ID 27626, 2007.
17. K.Varadarajan, *Numerical invariants in homotopical algebra, I*, Can. J. Math., (27) **4** (1975), 901-934.
18. K. Varadarajan, *Numerical invariants in homotopical algebra, II- applications*, Can. J. Math., (27) **4** (1975), 935-960.
19. J. Abaffy, C. G. Broyden and E. Spedicato, *A class of direct methods for linear equations*, Numer. Math, **45** (1984), 361–376.
20. H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese Journal of Mathematics, (4) **11** (2007), 1189-1201.
21. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.

M. Routaray
Department of Mathematics, National Institute of Technology, Rourkela
ROURKELA - 769 008 (India).
Email: `mitaray8@gmail.com`

A. Behera
Department of Mathematics, National Institute of Technology, Rourkela
ROURKELA - 769 008 (India).
Email: `abehera@nitrkl.ac.in`