ON ZERO-DIVISOR GRAPHS OF QUOTIENT RINGS
AND COMPLEMENTED ZERO-DIVISOR GRAPHS

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Abstract. For an arbitrary ring $R$, the zero-divisor graph of $R$, denoted by $\Gamma(R)$, is an undirected simple graph that its vertices are all nonzero zero-divisors of $R$ in which any two vertices $x$ and $y$ are adjacent if and only if either $xy = 0$ or $yx = 0$. It is well-known that for any commutative ring $R$, $\Gamma(R) \cong \Gamma(T(R))$ where $T(R)$ is the (total) quotient ring of $R$. In this paper we extend this fact for certain noncommutative rings, for example, reduced rings, right (left) self-injective rings and one-sided Artinian rings. The necessary and sufficient conditions for two reduced right Goldie rings to have isomorphic zero-divisor graphs is given. Also, we extend some known results about the zero-divisor graphs from the commutative to noncommutative setting: in particular, complemented and uniquely complemented graphs.

1. Introduction

Throughout the paper, $R$ denotes a ring with identity element (not necessarily commutative) and a zero-divisor in $R$ is an element of $R$ which is either a left or a right zero-divisor. We denote the set of all zero-divisors of $R$ and the set of all regular elements of $R$ by $Z(R)$ and $C_R$, respectively. Also, the set of all minimal prime ideals of $R$ is denoted by $\text{minSpec}(R)$. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is an undirected simple graph with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$ in which any two distinct vertices $x$ and $y$ are adjacent if and only if either $xy = 0$ or $yx = 0$. The notion of zero-divisor graph of a commutative

Keywords: Quotient ring, zero-divisor graph, reduced ring, complemented graph.
Received: 4 March 2016, Accepted: 8 June 2016.
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A right Ore set. For a right Ore ring $R$, $a/s$ is an equivalence relation and so we write $s(a)$ or $a/s$. The set of all equivalence classes is denoted by $RC^{-1}$. Moreover if $\Gamma(R)$ contains a cycle, then the girth of $\Gamma(R)$ is at most 4.

A multiplicative set $S \subseteq R$ is called a right Ore set if for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$. We say that $R$ is a right Ore ring if $CR$ is a right Ore set. For a right Ore ring $R$, we define a relation “$\sim$” on $R \times CR$ as follow: $(a_1, s_1) \sim (a_2, s_2)$ if and only if there exist $b_1, b_2 \in R$ such that $s_1b_1 = s_2b_2 \in CR$ and $a_1b_1 = a_2b_2 \in R$. It can be seen that the relation “$\sim$” is an equivalence relation and so we write $a/s$ or $a^{-1}s$ for the equivalence class $(a, s)$. The set of all equivalence classes is denoted by $RC^{-1}$. For any $a_1/s_1, a_2/s_2 \in RC^{-1}$, there exist $s, s' \in CR$ and $r, r' \in R$ such that $s_1s = s_2r \in CR$ and $a_2s' = s_1r'$. Thus we define $a_1/s_1 + a_2/s_2 = (a_1s + a_2r)/t$, where $t = s_1s = s_2r$ and $(a_1/s_1)(a_2/s_2) = a_1r'/s_2s'$. It is well-known that the addition and the multiplication defined on $RC^{-1}$ are binary operations and under these operations $RC^{-1}$ becomes a ring (for more details see [8, p. 301-302]). The ring $RC^{-1}$ is usually called the (classical) right quotient ring of $R$.

In Section 1, we prove that for any reduced right Ore ring $R$, $\Gamma(R)$ and $\Gamma(RC^{-1})$ are isomorphic (Theorem 2.2). Also it is shown that if $R$ is a von Neumann regular ring, a right (left) self-injective ring or a right (left) Artinian ring, then $\Gamma(R)$ and $\Gamma(RC^{-1})$ are isomorphic. We show that if $R$ is a reduced right Goldie ring, then $\Gamma(R) \cong \Gamma(D_1 \times D_2 \times \cdots \times D_n)$ for suitable division rings $D_1, D_2, \ldots, D_n$ and integer number $n$ (Proposition 2.4). In Section 2, first complemented and uniquely complemented are introduced and then we give some results about them. For example, it is shown that for any reduced ring $R$, $\Gamma(R)$ is complemented if and only if $\Gamma(R)$ is uniquely complemented (Proposition 3.3). Also we prove that for any reduced right Ore ring $R$, if $RC^{-1}$ is von Neumann regular, then $\Gamma(R)$ is uniquely complemented and while $\Gamma(R)$ is complemented, then every prime ideal of $RC^{-1}$ is maximal (Proposition 3.6). Next we show that for an Artinian ring $R$ with $\text{Nil}_v(R)$ nonzero:

1. If $\Gamma(R)$ is complemented, then either $|R| = 8$, $|R| = 9$, or $|R| > 9$ and $\text{Nil}_v(R) = \{0, x\}$, for some $0 \neq x \in R$.
2. If $\Gamma(R)$ is uniquely complemented and $|R| > 9$, then any complement of the nonzero nilpotent element of $R$ is an end (Theorem 3.7).
2. Zero-divisor graphs of quotient rings

Remark 2.1. For a reduced right Ore ring $R$, the right quotient ring $RC_R^{-1}$ is also reduced. To see this, suppose that $xy = 0$, where $x, y \in R$. First we show that $xs^{-1}y = 0$, for any $s \in C_R$. Since $R$ is a right Ore ring, there exist $r_1 \in R$ and $s_1 \in C_R$ such that $sr_1 = ys_1$ and so $y = sr_1s_1^{-1}$. Thus $0 = xy = xsr_1s_1^{-1}$ and hence $xsr_1 = 0$. Since $R$ is reduced, we have $r_1xs = 0$. It follows that $r_1x = 0$, and so $xsr_1 = 0$. Thus $0 = xsr_1s_1^{-1} = xs^{-1}y$. Now suppose that $xs^{-1}yt^{-1} = 0$, where $x, y \in R$ and $s, t \in C_R$. Then $xs^{-1}y = 0$ and so $xsr_1s_1^{-1} = 0$ (note that $s^{-1}y = r_1s_1^{-1}$). Thus $xsr_1 = 0 = r_1x$ because $R$ is reduced. By the first part of the proof, $0 = r_1(t_1s_1)^{-1}x = r_1s_1^{-1}t^{-1}x$. Therefore $s^{-1}yt^{-1}x = 0$ and so $yt^{-1}xs^{-1} = 0$. Thus $RC_R^{-1}$ is reduced.

Let $G$ be an undirected simple graph. As in [9], for every two vertices $a$ and $b$ of $G$, we define $a \leq b$ if $a$ and $b$ are not adjacent and each vertex of $G$ adjacent to $b$ is also adjacent to $a$. We write $a \sim b$ if both $a \leq b$ and $b \leq a$. It is easy to see that $\sim$ is an equivalence relation on $G$. We denote the equivalence class of a vertex $x$ of $G$ by $[x]$. Note that for any ring $R$ with $a, b \in Z(R)^*$, we have $a \sim b$ in $\Gamma(R)$ if and only if $(\text{ann}(a) \cup \text{ann}(b)) \setminus \{a\} = (\text{ann}(b) \cup \text{ann}(a)) \setminus \{b\}$. If $R$ is a right Ore ring and $A \subseteq R$, then the set $\{a/s \mid a \in A, s \in C_R\}$ is denoted by $A_{CR}$.

In [4], the authors proved that for any commutative ring $R$, $\Gamma(R) \cong \Gamma(T(R))$ where $T(R)$ is the quotient ring of $R$. Here, by the same method as [4], we extend this fact to the reduced right Ore rings.

Theorem 2.2. Let $R$ be a reduced right Ore ring with right quotient ring $RC_R^{-1}$. Then the graphs $\Gamma(R)$ and $\Gamma(RC_R^{-1})$ are isomorphic.

Proof. Let $S = C_R$ and $T = RS^{-1}$. Denote the equivalence relations defined above on $Z(R)^*$ and $Z(T)^*$ by $\sim_R$ and $\sim_T$, respectively, and denote their respective equivalence classes by $[a]_R$ and $[a]_T$. Since $R$ and $RS^{-1}$ are reduced, we note that $\text{ann}_T(x/s) = \text{ann}_R(x)_S$ and $\text{ann}_T(x/s) \cap R = \text{ann}_R(x)$; thus $x/s \sim_T x/t$, $x \sim_R y \iff x/s \sim_T y/s$, $([x]_R)_S = [x/1]_T$ and $[x/s]_T \cap R = [x]_R$ for all $x, y \in Z(R)^*$ and $s, t \in S$. Since $Z(T) = Z(R)_S$, by the above comments, we have $Z(R)^* = \bigcup_{a \in A} [a]_R$ and $Z(T)^* = \bigcup_{a \in A} [a/1]_T$ (both disjoint unions) for some $\{a_\alpha\}_\alpha \subseteq R$.

We next show that $|[a]|_R = |[a/1]|_T$ for each $a \in Z(R)^*$. First assume that $[a]_R$ is finite. Then it is clear $[a]_R \subseteq [a/1]_T$. For the inverse inclusion, let $x \in [a/1]_T$. Then $x = b/s$ with $b \in [a]_R$ and $s \in S$. Since $\{bs^n \mid n \geq 1\} \subseteq [a]_R$ is finite, $b = bs^i$ for some
integer $i > 1$, and hence $b/s = bs^i/s = bs^{i-1} \in [a]_R$. Now suppose that $[a]_R$ is infinite. Clearly $|[a]_R| \leq |[a/1]_T|$. Define an equivalence relation $\approx$ on $S$ by $s \approx t$ if and only if $sa = ta$. Then $s \approx t$ if and only if $sb = tb$ for all $b \in [a]_R$. It is easily verified that the map $[a]_R \times S/\approx \rightarrow [a/1]_T$, given by $(b, [s]) \rightarrow b/s$, is well-defined and surjective. Hence $|[a/1]_T| \leq |[a]_R| |S/\approx|$. Also the map $S/\approx \rightarrow [a]_R$, given by $[s] \rightarrow sa$, is clearly well-defined and injective. Hence $|S/\approx| \leq |[a]_R|$, and so $|[a/1]_T| \leq |[a]_R|^2 = |[a]_R|$ since $|[a]_R|$ is infinite. Thus $|[a]_R| = |[a/1]_T|$. Therefore there is a bijection $\phi : [a_\alpha] \rightarrow [a_\alpha/1]$ for each $\alpha \in A$. Define $\phi : Z(R)^* \rightarrow Z(T)^*$ by $\phi(x) = \phi_\alpha(x)$ if $x \in [a_\alpha]$. Thus we need only show that $x$ and $y$ are adjacent in $\Gamma(R)$ if and only if $\phi(x)$ and $\phi(y)$ are adjacent in $\Gamma(T)$; i.e., $xy = 0$ if and only if $\phi(x)\phi(y) = 0$. Let $x \in [a]_R, y \in [b]_R, w \in [a/1]_T$ and $z \in [b/1]_T$. It is sufficient to show that $xy = 0$ if and only if $zw = 0$. Note that $\text{ann}_T(x) = \text{ann}_T(a) = \text{ann}_T(w)$ and $\text{ann}_T(y) = \text{ann}_T(b) = \text{ann}_T(z)$. Thus $xy = 0 \iff y \in \text{ann}_T(x) = \text{ann}_T(w) \iff yw = 0 \iff w \in \text{ann}_T(y) = \text{ann}_T(z) \iff wz = 0$. Hence $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic as graphs.

Let $R$ be a ring. We denote the group of unit elements of $R$ by $U(R)$. By [8, Proposition 11.4], the right quotient ring of $R$ exists and $R \cong RC^{-1}_R$ if and only if $C_R = U(R)$. In this case, we say that $R$ is a classical ring and it is clear that $\Gamma(R) \cong \Gamma(RC^{-1}_R)$. Recall that $R$ is von Neumann regular if for each $x \in R$, there exists $y \in R$ such that $x = xyx$. In the following we give some examples of noncommutative rings $R$ for which $\Gamma(R) \cong \Gamma(RC^{-1}_R)$.

**Example 2.3.** (a) For any von Neumann regular ring $R$, we have $\Gamma(R) \cong \Gamma(RC^{-1}_R)$. To see this, let $q \in C_R$. Then there exists $q' \in R$ such that $q = qq'q$. So $q(1 - q'q) = 0 = (1 - qq')q$. Since $q$ is regular, $qq' = q'q = 1$ and hence $q \in U(R)$. Thus $C_R = U(R)$ and this implies that $R$ is a classical ring. Therefore $\Gamma(R) \cong \Gamma(RC^{-1}_R)$.

(b) Let $R$ be a ring in which for any $q \in R$, the chain $qR \supseteq q^2R \supseteq \ldots$ stabilizes. Then $R$ is a classical ring. Indeed, if $q \in C_R$, then by hypothesis, there exists $n \geq 1$ such that $q^nR = q^{n+1}R$. Thus $q^n = q^{n+1}q'$, for some $q' \in R$. Since $q$ is regular, $qq' = 1$. Also $q(1 - q'q) = 0$ and hence $q'q = 1$. Thus $C_R = U(R)$ and we conclude that $\Gamma(R) \cong \Gamma(RC^{-1}_R)$. In particular if $R$ is a right (left) Artinian ring, then $\Gamma(R) \cong \Gamma(RC^{-1}_R)$.

(c) Let $V$ be a vector space over a division ring $K$. Then $R = \text{End}(V_R)$ is a classical ring. To see this, we note that $V$ is a semisimple $K$-module.
and by [7, Proposition 4.27], \( R \) is a von Neumann regular ring. Now
the assertion is obtained from (a).

(d) Every left (right) self-injective ring is a classical ring. Suppose that
\( R \) is a left self-injective ring and \( a \) is a regular element in \( R \). We show
that \( a \in U(R) \). Define \( R \)-monomorphism \( f : R \to R \) by \( f(r) = ra \). Since
\( R \) is self-injective, there exists \( R \)-homomorphism \( g : R \to R \) such
that \( gf = 1 \). Now \( a = 1(a) = gf(a) = g(a^2) = a^2g(1) \). Since \( a \) is
a regular element, we have \( 1 = ag(1) \). Thus \( a = ag(1)a \) and hence
\( 1 = g(1)a \) because again \( a \) is regular. Therefore \( a \in U(R) \) and so \( R \)
is a classical ring. This implies that \( \Gamma(R) \cong \Gamma(RC^{-1}_R) \).

(e) Let \( R \) be a right Ore ring such that \( RC^{-1}_R \) is a Noetherian right \( R \)-
module. Then \( \Gamma(R) \cong \Gamma(RC^{-1}_R) \). Clearly, the natural homomorphism
\( \phi : R \to RC^{-1}_R \), given by \( \phi(r) = r/1 \), is injective. We show that \( \phi \)
is an isomorphism. Let \( rs^{-1} \in RC^{-1}_R \). Then the chain \( s^{-1}R \subseteq s^{-2}R \subseteq \ldots
\) stabilizes because \( RC^{-1}_R \) is Noetherian as right \( R \)-module. Thus there
exists \( n \geq 1 \) such that \( s^{-n}R = s^{-n-1}R \), and so \( s^{-n} = s^{-n}r_1 \) for some
\( r_1 \in R \). Hence \( s^{-1} = r_1 \in R \) and so \( \phi(rr_1) = rr_1 = rs^{-1} \). This implies
that \( \phi \) is epimorphism; thus \( \Gamma(R) \cong \Gamma(RC^{-1}_R) \).

**Proposition 2.4.** Let \( R \) be a reduced right Goldie ring. Then \( \Gamma(R) \cong \Gamma(D_1 \times D_2 \times \cdots \times D_n) \) for suitable division rings \( D_1, D_2, \ldots, D_n \)
and integer number \( n \).

**Proof.** By Goldie’s Theorem [8, Theorem 11.13], \( RC^{-1}_R \) is a semisimple
ring. Also by Remark 2.1, \( RC^{-1}_R \) is reduced. Using the Weddernborn-
Artin Theorem, we conclude that \( RC^{-1}_R \cong D_1 \times D_2 \times \cdots \times D_n \) for
suitable division rings \( D_1, D_2, \ldots, D_n \) and integer number \( n \). Now by
Theorem 2.2, \( \Gamma(R) \cong \Gamma(D_1 \times D_2 \times \cdots \times D_n) \).

Let \( x \) be a vertex of a graph \( G \). We say that \( x \) is a **primitive vertex**, if it is a minimal element in the ordering \( \leq \).

**Theorem 2.5.** Let \( \{A_i\}_{i \in I} \) and \( \{B_j\}_{j \in J} \) be two families of domains
and let \( A = \prod_{i \in I} A_i \) and \( B = \prod_{j \in J} B_j \). Then \( \Gamma(A) \cong \Gamma(B) \) if and only
if there exists a bijection \( \psi : I \to J \) such that \( |A_i| = |B_{\psi(i)}| \) for each
\( i \in I \).

**Proof.** One direction of the proof is clear. For the other direction, suppose that \( \phi : \Gamma(A) \to \Gamma(B) \) is an isomorphism. We note that
each primitive vertex in \( \Gamma(A) \) has exactly one nonzero component. Let
\( x = (x_i)_{i \in I} \) be a primitive vertex in \( \Gamma(A) \). Then there is \( i_0 \in I \) such
that \( x_{i_0} \neq 0 \) and \( x_i = 0 \) for each \( i \neq i_0 \in I \). Thus the set \( \{[z] \mid z \) is a
primitive vertex of \( \Gamma(A) \) has cardinality \( |I| \). Similarly, the set \( \{ [z] \mid z \) is a primitive vertex of \( \Gamma(B) \} \) has cardinality \( |J| \). One can easily see that \( z \) is a primitive vertex of \( \Gamma(A) \) if and only if \( \phi(z) \) is a primitive vertex of \( \Gamma(B) \). Also \( [z] = [z'] \) if and only if \( [\phi(z)] = [\phi(z')] \). Thus we have \( |I| = |J| \). On the other hand, \( z \in [x] \) if and only if \( \phi(z) \in [\phi(x)] \) and hence \( |[x]| = |[\phi(x)]| \). Moreover \( |[x]| = |A_{i_0}| \) and \( |[\phi(x)]| = |B_{j}| \) for some \( j \in J \). Clearly \( \phi \) induces the required bijection \( \psi \).

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\text{Corollary 2.6. Let } A \text{ and } B \text{ be two reduced right Goldie rings which are not domains. Then } \Gamma(A) \cong \Gamma(B) \text{ if and only if there exists a bijection } \phi : \minSpec(A) \to \minSpec(B) \text{ such that } |A/P| = |B/\phi(P)| \text{ for each } P \in \minSpec(A).
\]

\[
\text{Proof. Set } T(A) = AC^{-1}_A \text{ and } T(B) = BC^{-1}_B. \text{ Since } A \text{ and } B \text{ are reduced right Goldie rings, by [8, Proposition 11.22], we may assume that } \minSpec(A) = \{ P_1, P_2, \ldots, P_m \}, \minSpec(B) = \{ Q_1, Q_2, \ldots, Q_n \} \text{ and } T(A) \cong K_1 \times \ldots \times K_m, T(B) \cong L_1 \times \ldots \times L_n, \text{ where division rings } K_i \text{ and } L_j \text{ are the quotient rings of } A/P_i \text{ and } B/Q_j, \text{ respectively for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n. \text{ By Theorem 2.2, } \Gamma(A) \cong \Gamma(K_1 \times K_2 \times \cdots \times K_m) \text{ and } \Gamma(B) \cong \Gamma(L_1 \times L_2 \times \cdots \times L_n). \text{ Now suppose that } \Gamma(A) \cong \Gamma(B). \text{ By Theorem 2.5, we conclude that } m = n \text{ and there exists a permutation } \rho \text{ of } \{ 1, \ldots, n \} \text{ such that } |A/P_i| = |K_i| = |L_{\rho(i)}| = |B/Q_{\rho(i)}| \text{ for } 1 \leq i \leq n. \text{ It is clear that } \rho \text{ induces the required bijection } \phi. \text{ Conversely, if there exists such a bijection } \phi, \text{ then by Theorem 2.5, } \Gamma(K_1 \times \cdots \times K_m) \cong \Gamma(L_1 \times \cdots \times L_n) \text{ and hence } \Gamma(A) \cong \Gamma(B). \]

3. Complemented zero-divisor graphs

Let \( G \) be an undirected simple graph. As in [9], for distinct vertices \( a \) and \( b \) of \( G \), we say that \( a \) and \( b \) are orthogonal, written by \( a \perp b \), if \( a \) and \( b \) are adjacent and there is no vertex \( c \) of \( G \) which is adjacent to both \( a \) and \( b \), i.e., the edge \( a - b \) is not part of any triangle of \( G \). Thus for \( a, b \in Z(R)^* \), we have \( a \perp b \) in \( \Gamma(R) \) if and only if \( ab = 0 \) or \( ba = 0 \) and

\[
(\text{ann}_l(a) \cup \text{ann}_r(a)) \cap (\text{ann}_l(b) \cup \text{ann}_r(b)) \subseteq \{ 0, a, b \}.
\]

Finally, we say that \( G \) is a complemented graph if for each vertex \( a \) of \( G \), there exists a vertex \( b \) of \( G \) (called a complement of \( a \)) such that \( a \perp b \), and that \( G \) is uniquely complemented if it is complemented and whenever \( a \perp b \) and \( a \perp c \), then \( b \sim c \).

In this section, we first show that for any reduced ring \( R \), \( \Gamma(R) \) is complemented if and only if \( \Gamma(R) \) is uniquely complemented. Next we prove that if \( R \) is a reduced and von Neumann regular ring, then \( \Gamma(R) \) is complemented. In the end of this section, we show that if \( R \) is not
reduced, then under certain conditions, $\Gamma(R)$ is complemented or $\Gamma(R)$ is uniquely complemented. In order to show these results, we need the following two lemmas which translate the above graph-theoretic concepts into ring-theoretic terms.

**Lemma 3.1.** Consider the following statements for a ring $R$ and $a, b \in Z(R)^*$.
(1) $a \sim b$.
(2) $aR = bR$.
(3) $\text{ann}_l(a) = \text{ann}_l(b)$.
(a) If $R$ is reduced, then (1) and (3) are equivalent.
(b) If $R$ is a reduced von Neumann regular ring, then all three statements are equivalent.

*Proof.* (a). If $R$ is reduced, then $\text{ann}_l(x) = \text{ann}_r(x)$ for each $x \in Z(R)^*$. Thus we have $a \sim b$ if and only if $\text{ann}_l(a) = \text{ann}_l(b)$.
(b). Since $R$ is a reduced ring, it is enough to show that (2) and (3) are equivalent. (2) $\Rightarrow$ (3) is clear. To show (3) $\Rightarrow$ (2), let $a = aca$ for some $c \in R$. Thus $a(1 - ca) = 0$ and so $1 - ca \in \text{ann}_r(a) = \text{ann}_l(a)$. Since $\text{ann}_l(a) = \text{ann}_l(b)$, we have $(1 - ca)b = 0$ and hence $b(1 - ca) = 0$. Therefore $b = bca \in Ra$. This implies that $Rb \subseteq Ra$. Similarly, $Ra \subseteq Rb$ and so $Ra = Rb$. \hfill $\square$

**Lemma 3.2.** Let $R$ be a reduced ring and $a, b \in Z(R)^*$. Then the following statements are equivalent.
(1) $a \perp b$.
(2) $ab = 0$ and $a + b$ is a regular element of $R$.

*Proof.* (1) $\Rightarrow$ (2). Since $a \perp b$ and $R$ is reduced, we have $ab = 0$. Suppose that $(a + b)c = 0$ for some $c \in Z(R)^*$. Let $y = ac = -bc$. Then $by = ay = 0$. Since $a \perp b$, we conclude that $y \in \{0, a, b\}$. If $y = a$, then $a^2 = ay = 0$ which a contradiction. Similarly, $y = b$ implies that $b^2 = 0$, again a contradiction. Hence $y = 0$ and so $ac = bc = 0$. It follows that $c \in \{0, a, b\}$ because $a \perp b$. If $c = a$, then $a^2 = 0$, a contradiction. Similarly, $c \neq b$ and hence $a + b$ is regular.
(2) $\Rightarrow$ (1). Suppose that $ca = cb = 0$ for some $c \in Z(R)^*$. Then $c(a + b) = 0$, a contradiction because $a + b$ is regular. Since $ab = 0$, we have $a \perp b$. \hfill $\square$

**Proposition 3.3.** Let $R$ be a reduced ring and $a, b, c \in Z(R)^*$. If $a \perp b$ and $a \perp c$, then $b \sim c$. Consequently, $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is complemented.

*Proof.* Since $a \perp b$ and $a \perp c$, we have $ab = ac = 0$. We first show that $bc \neq 0$. If $bc = 0$, then $c \in \{0, a, b\}$ because $ac = 0$ and $a \perp b$. \hfill $\square$
By our assumption, $c = a$ or $c = b$. If $c = a$, then $ac = a^2 = 0$ and hence $a = 0$, a contradiction. Similarly $c \neq b$. Thus $bc \neq 0$. Now suppose that $db = 0$ for some $d \in Z(R)^*$. Then $0 = (ac)d = a(cd)$ and $0 = (db)c = c(db) = (cd)b$. It follows that $cd \in \{0, a, b\}$ because $a \perp b$. If $cd \neq 0$, then $cd = a$ or $cd = b$ and hence $a^2 = 0$ or $b^2 = 0$, which a contradiction. Therefore $cd = 0$ and so $c \leq b$. Similarly $b \leq c$, and thus $b \sim c$. 

**Remark 3.4.** Let $R$ be a reduced von Neumann regular ring. Then for any $a \in Z(R)^*$, we have $a = ue$ where $u \in U(R)$ and $e \in R$ is idempotent. To see this, let $a \in R$. Since $R$ is von Neumann regular, there exists $b \in R$ such that $a = aba$. Then $a(1 - ba) = 0$ and hence $(1 - ba)a = 0$ because $R$ is reduced. Thus $a = ba^2$. Similarly, $a = a^2b$. We set $x = b^2a$, $e = ax$ and $u = (1 - e + a)$. Then $e^2 = axax = ab^2a^2b^2a = ab^2a^2b^2a = ab^2a^2b^2a = ab^2a^2 = e$. Also, since $a = a^2b^2a$, and $0 = ab^2 - a^2b^2ab^2 = a(b^2 - ab^2a^2)$, we have $(b^2 - ab^2a^2)ab^2 = 0$ and so $b^2a = ab^2a$. This implies that $u(1 - e + x) = (1 - e + a)(1 - e + x) = 1$. On the other hand, $ab^2a^2 = a$ and $a^2b^2 - ab = 0$. Hence $a(ab^2 - b) = 0 = (ab^2 - b)a = 0$ and so $ab^2a = ba = b^2a^2$. Also $ab^2 - abab^2 = 0$ implies that $a(b^2 - bab^2a) = 0$ and hence $(b^2 - bab^2)a = 0$. Thus $b^2a = bab^2a = b^2a^2b^2a$. Now, we conclude that $(1 - e + x)u = (1 - e + x)(1 - e + a) = 1$.

**Corollary 3.5.** If $R$ is a reduced von Neumann regular ring, then $\Gamma(R)$ is a uniquely complemented graph.

**Proof.** By Remark 3.4, for any $a \in Z(R)^*$, there exist $u \in U(R)$ and idempotent $e \in R$ such that $a = ue$. Clearly, $a(1 - e) = 0$. Suppose that $ax = 0$ and $(1 - e)x = 0$, for some $x \in R$. Then $x = ex$ and hence $ux = uex = 0$. Since $u \in U(R)$, we conclude that $x = 0$. Thus $a \perp (1 - e)$. 

**Proposition 3.6.** Let $R$ be a reduced right Ore ring. Then:

(a) If $RC_R^{-1}$ is von Neumann regular, then $\Gamma(R)$ is uniquely complemented.

(b) If $\Gamma(R)$ is complemented, then every prime ideal of $RC_R^{-1}$ is maximal.

**Proof.** (a). Since $R$ is reduced, by Theorem 2.2, $\Gamma(R) \cong \Gamma(RC_R^{-1})$. Also by Corollary 3.5, $\Gamma(RC_R^{-1})$ is uniquely complemented. Thus $\Gamma(R)$ is uniquely complemented.

(b). Let $P$ and $Q$ be two prime ideals of $RC_R^{-1}$ such that $P \subsetneq Q$. Thus there exists $xs^{-1} \in Q$ such that $xs^{-1} \notin P$. Then $x \in Z(R)^*$ because $P \neq R$. Since $\Gamma(R)$ is complemented, there exists $y \in Z(R)^*$ such that
if $R$ other vertex adjacent to it. We say that a ring $R$ is an Artinian ring if $R$ is both a left and a right Artinian ring. Let $R$ be a ring. The prime radical of $R$, denoted by $\text{Nil}_*(R)$, is the intersection of all prime ideals in $R$ and the Jacobson radical of $R$, denoted by $\text{Rad}(R)$, is the intersection of all maximal right ideals of $R$. We conclude the paper with the following theorem which gives the necessary conditions for an Artinian ring $R$ with $\text{Nil}_*(R) \neq 0$, such that $\Gamma(R)$ is a complemented or uniquely complemented graph.

**Theorem 3.7.** Let $R$ be an Artinian ring with $\text{Nil}_*(R)$ nonzero.

(a) If $\Gamma(R)$ is complemented, then either $|R| = 8$, $|R| = 9$, or $|R| > 9$ and $\text{Nil}_*(R) = \{0, x\}$ for some $0 \neq x \in R$.

(b) If $\Gamma(R)$ is uniquely complemented and $|R| > 9$, then any complement of the nonzero nilpotent element of $R$ is an end.

**Proof.** (a). Suppose that $\Gamma(R)$ is complemented and let $a \in \text{Nil}_*(R)$ have index of nilpotence $n \geq 3$. Let $y \in Z(R)^*$ be a complement of $a$. Then $a^{n-1}y = 0 = a^{n-1}a$; so $y = a^{n-1}$, because $a \perp y$. Thus $a \perp a^{n-1}$ and this implies that $\text{ann}_l(a) \cup \text{ann}_r(a) = \{0, a^{n-1}\}$. Similarly, $a^i \perp a^{n-1}$ for each $1 \leq i \leq n-2$. Suppose that $n > 3$. Then $a^{n-2}a^{n-1}$ kills both $a^{n-2}$ and $a^{n-1}$, a contradiction, because $a^{n-2} \perp a^{n-1}$ and $a^{n-2} + a^{n-1} \notin \{0, a^{n-2}, a^{n-1}\}$. Thus if $R$ has a nilpotent element with index $n \geq 3$, then $n = 3$. In this case, $Ra^2 = \{0, a^2\}$ because any $z \in Ra^2$ kills both $a$ and $a^2$ and $a \perp a^2$. Also if $za^2 = 0$, then $za \in \text{ann}_l(a) = \{0, a^2\}$ and so either $za = 0$ or $za = a^2$. If $za = 0$, then $z = 0$ or $z = a^2$ while if $za = a^2$, then $(z-a)a = 0$ and hence $z = a$ or $z = a + a^2$. Therefore $\text{ann}_l(a^2) = \{0, a, a^2, a + a^2\}$. Thus the $R$-epimorphism $r \mapsto ra^2$, from $R$ onto $Ra^2$ implies that $R$ is a local ring with $|R| = 8$, $\text{Nil}_*(R) = \text{ann}_l(a^2)$ its maximal ideal and $\Gamma(R)$ is a star graph with center $a^2$ and two edges.

Now suppose that each nonzero nilpotent element of $R$ has index of nilpotence 2. Let $y \in \text{Nil}_*(R)$ have complement $z \in Z(R)^*$ and assume that $2y \neq 0$. Without loss of generality, we can assume that $yz = 0$. Note that $(ry)y = 0 = (ry)z$ for all $r \in R$. Thus $Ry \subseteq \{0, y, z\}$. Then necessarily $2y = z$ since $2y \in Ry \subseteq \{0, y, z\}$. Also $\text{ann}_r(y) = \{0, y, 2y\}$ since $y \perp 2y$. Thus $Ry = \{0, y, 2y\}$; so we have $|R| = 9$. In this case,
$R$ is local with maximal ideal $\text{Nil}_s(R) = \text{ann}_s(y)$ and $\Gamma(R)$ is a star graph with one edge. 

Next suppose that each nonzero nilpotent element of $R$ has index nilpotence 2 and $|R| \neq 9$. By above, we must have $2y = 0$. We show that $\text{Nil}_s(R) = \{0, y\}$. Suppose that $z$ is another nonzero nilpotent element of $R$; so $z^2 = 0$. Then $y + z \in \text{Nil}_s(R)$ and hence $(y + z)^2 = 0$. Suppose that $y'$ and $z'$ are complements of $y$ and $z$, respectively. Then we have $yy' = 0$ or $y'y = 0$ and $zz' = 0$ or $z'z = 0$. We proceed by cases.

Case 1. $yy' = 0$ and $zz' = 0$. Since $y \perp y'$ and $z \perp z'$, $Ry \subseteq \{0, y, y'\}$ and $Rz \subseteq \{0, z, z'\}$. We claim that $yz = zy = 0$. Note that $yz \in Rz \subseteq \{0, z, z'\}$. If $yz \neq 0$, then either $yz = z$ or $yz = z'$. If $yz = z$, then $0 = y(yz) = yz$, a contradiction. Thus $yz = z'$. It follows that $z' \in \text{Nil}_s(R)$ and so $z'^2 = 0$. Since $z \perp z'$ and $z(z + z') = (z + z')z' = 0$, we conclude that $z + z' = 0$; so $z' = -z = z$, a contradiction (by the definition of complement). Thus $yz = 0$. Similarly, $zy = 0$. Let $w$ be a complement of $y + z$. Then $w(y + z) = 0$ or $(y + z)w = 0$. We note that $w \neq y$. For if $w = y$, then $(y + z)z = 0$ and $wz = 0$ and hence $z \in \{0, w, y + z\} = \{0, y, y + z\}$, a contradiction. Similarly, $w \neq z$. We claim that $(y + z)w = 0$. Otherwise $(w(y + z) = 0$. Then $wy = wz \in Ry \cap Rz$. Thus $wy = wz = 0$ or $wy = wz = y' = z'$. If $wy = 0$, then since $(y + z)y = 0$ and $(y + z) \perp w$, we conclude that $y \in \{0, y + z, w\}$, a contradiction. If $wy = wz = y' = z'$, then $y' \in \text{Nil}_s(R)$ which again is a contradiction (similar to what was described above for $z' \in \text{Nil}_s(R)$). Thus $(y + z)w = 0$. On the other hand, $z'y \in Ry \subseteq \{0, y, y'\}$. If $z'y = 0$, then since $yz = 0$ and $z \perp z'$, we have $y \in \{0, z, z'\}$ and hence $y = z'$. Thus $z' \in \text{Nil}_s(R)$, a contradiction. If $z'y = y'$, then $y' \in \text{Nil}_s(R)$ which again is a contradiction. Thus $z' = y = z$. Similarly, $y'z = z$. Also $y'y \in Ry \subseteq \{0, y, y'\}$. We claim that $y'y = y$. If $y'y = y'$, then $y' \in \text{Nil}_s(R)$, a contradiction. If $y'y = 0$, then $y' = y + z$ and $z = z \in \{0, y + z, w\}$ which is a contradiction (because $zw = 0$ and $(y + z)z = 0$). Thus $y'y = y$ and so $y'^2 \neq 0$. Since $R$ is an Artinian ring, every right zero-divisor of $R$ is a left zero-divisor. Thus $y't = 0$ for some nonzero $t \in R$. Now $ty \in R$ and so $ty = 0$, $ty = y$ or $ty = y'$. If $ty = 0$, then since $y't = 0$ and $y \perp y'$, we have $t \in \{0, y, y'\}$. Then $0 = y't = y'^2$ or $0 = y't = y'y = y$, which is a contradiction. Thus $ty = y$ or $ty = y'$ and we conclude that $0 = y'ty = y'y = y$ or $0 = y'ty = y'^2$, a contradiction.

Case 2. $yy' = 0$ and $zz' = 0$. Then $yR \subseteq \{0, y, y'\}$ and $Rz \subseteq \{0, z, z'\}$ and so $yy' \in \{0, y, y'\}$. If $yy' = y'$, then $y' \in \text{Nil}_s(R)$, a contradiction. If $yy' = 0$, then by Case 1, we are done. Thus we have $yy' = y$. Now since
$R$ is an Artinian ring and $y'y = 0$, $ty' = 0$ for some nonzero $t \in R$. On the other hand, $yt \in \{0, y, y'\}$. If $yt = 0$, then $t \in \{0, y, y'\}$ (because $ty' = 0$ and $y \perp y'$). Therefore either $t = y$ or $t = y'$. This implies that $0 = ty' = yy' = y$ or $0 = ty' = y'^2$, again a contradiction. Thus we have $yt = y$ or $yt = y'$. Then $0 = yty' = yy' = y$ or $0 = y'yt = y'^2$, a contradiction.

**Case 3.** $y'y = 0$ and $z'z = 0$. It is similar to Case 1.

**Case 4.** $yy' = 0$ and $z'z = 0$. It is similar to Case 2.

(b). Suppose that $\Gamma(R)$ is uniquely complemented and $|R| > 9$. Let $0 \neq x \in \text{Nil}_*(R)$. By part (a), $\text{Nil}_*(R) = \{0, x\}$. Let $y$ be a complement of $x$. Then $xy = 0$ or $yx = 0$. Without loss of generality, we may assume that $xy = 0$. Clearly $x(x+y) = 0$, since $x^2 = 0$. We claim that $x \perp (x+y)$. Suppose $w \in Z(R)^*$ such that $x-w$ and $(x+y)-w$ are two edges of $\Gamma(R)$. Now we proceed by cases.

**Case 1.** $wx = 0$ and $(x+y)w = 0$. Then $yw = 0$ and since $xw = 0$ and $x \perp y$, we conclude that $w \in \{0, x, y\}$. If $w = y$, then $y^2 = 0$ and hence $x(x+y) = 0$ and $(x+y)y = 0$. This contradicts that $x \perp y$. Thus $w = x$ and we are done.

**Case 2.** $wx = 0$ and $(x+y)w = 0$. Then $xw = yw$ (note that $x = -x$) and since $xw \in \{0, x\}$, either $xw = 0$ or $xw = x$. If $xw = 0$, then $xw = yw = 0$ and similar to Case 1, $w = x$. Thus suppose that $xw = yw = x$. Since $xy = 0$ and $R$ is an Artinian ring, $yt = 0$ for some $t \in Z(R)^*$. Now if $tx = 0$, then $t \in \{0, x, y\}$ (note that $x \perp y$) and we deduce $t = x$ (since $y^2 \neq 0$). If $tx = x$, then $0 = ytx = yx$. Thus in any case, we have $yx = 0$. On the other hand, $(wy)x = 0$ and $y(wy) = 0$ and hence $wy \in \{0, x, y\}$. Now we continue the proof by subcases;

Subcase 1. $wy = 0$. Then since $wx = 0$ and $x \perp y$ and $w \in \{0, x, y\}$; so $w = x$ and we are done.

Subcases 2. $wy = y$. Then $(w-1)y = 0$ and also $x(w-1) = 0$. Thus $(w-1) \in \{0, x, y\}$. Clearly $w \neq 1$. If $w-1 = x$, then $w = 1+x$ is invertible (note that $x \in \text{Rad}(R) = \text{Nil}_*(R)$), a contradiction. Therefore $w-1 = y$ and this implies that $0 = wx = x+yx = x$, which again is a contradiction.

Subcases 3. $wy = x$. Then $0 = wx = w(yw) = (wy)w = wx = x$, a contradiction.

**Case 3.** $xw = 0$ and $w(x+y) = 0$. It is similar to Case 2.

**Case 4.** $wx = 0$ and $w(x+y) = 0$. It is similar to Case 1.

Thus in any case, we conclude that $x \perp (x+y)$. Since $\Gamma(R)$ is uniquely complemented and $x \perp y$, we have that $(x+y) \sim y$. Suppose that there exists $z \in Z(R)^* \setminus \{x\}$ such that $zy = 0$ or $yz = 0$. Without loss of generality, we can assume that $zy = 0$. Then since $(x+y) \sim y$, we
have \( z(x + y) = 0 \) or \((x + y)z = 0\). If \( z(x + y) = 0 \), Then \( zx = 0 \), a contradiction. Thus \((x + y)z = 0\). Then \( xz + yz = 0 \) and since \( xz \neq 0 \), we have \( x + yz = 0 \). Now \( zx = yz = 0 \), which again is a contradiction. Thus no such \( z \) can exist; so \( y \) is an end.  

**Acknowledgments.** This work was partially supported by Islamic Azad University, Khorramabad Branch, Khorramabad, Iran.

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