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RESULTS ON HILBERT COEFFICIENTS OF A COHEN-MACAULAY MODULE

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ABSTRACT. Let (R, m) be a commutative Noetherian local ring, M a finitely generated R-module of dimension d, and let I be an ideal of definition for M. In this paper, we extend [7, Corollary 10(4)] and also we show that if M is a Cohen-Macaulay R-module and $d = 2$, then $\lambda(\frac{\widetilde{I^n M}}{J\widetilde{I^{n-1} M}})$ does not depend on J for all $n \geq 1$, where J is a minimal reduction of I.

1. Introduction

Throughout this note, we assume that (R, m) is a commutative Noetherian local ring with residue field $k = R/m$, M a finitely generated R-module of dimension d and I an ideal of definition for M; i.e. $\lambda(M/IM)$ is finite. Here $\lambda(-)$ denotes length. Let $G_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and $G_I(M) = \bigoplus_{n\geq 0} I^n M/I^{n+1}M$ be the associated graded ring of R and the associated graded module of M with respect to I, respectively. In [8] the Ratliff-Rush closure of M with respect to I is defined by $\widetilde{IM} = \cup_{k\geq 1}(I^{k+1}M :_M I^k)$ (see also [10] or [9]). Let $\widetilde{G}_I(M) = \bigoplus_{n\geq 0} \widetilde{I^n M}/\widetilde{I^{n+1} M}$ be the associated graded module of the Ratliff-Rush filtration. Recall that an ideal $J \subseteq I$ is said to be a reduction of I if $I^{r+1} = JI^r$ for some $r \geq 0$, and a reduction J of I is called a minimal reduction of I if J is minimal with respect to inclusion. The concepts of reduction and minimal reduction were first introduced by Northcott and Rees [6].

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The formal power series

$$H_M^I(z)=\sum_{n\geq 0}\lambda(I^nM/I^{n+1}M)z^n=\frac{h_M^I(z)}{(1-z)^d},$$

where $h_M^I(z)=h_0^I(M)+h_1^I(M)z+...+h_r^I(M)z^r\in\mathbb{Z}[z]$. This series is called the *Hilbert series* of M and the polynomial $h_M^I(z)$ is called the *h-polynomial* of M.

An element $x\in I$ is called *superficial* for M with respect to I if there exists an integer $k>0$ such that $(I^{n+1}M:_M x)\cap I^kM=I^nM$ for all $n\geq k$. It is known that if $depth M>0$, then every M-superficial element is M-regular (see [4, Lemma 2.1]). Also, if x is superficial and M-regular, then by using the Artin-Rees lemma for M and xM one gets $(I^{n+1}M:_M x)=I^nM$ for all large n (see [12] or [11]).

The aim of this paper is to generalized [7, Corollary 10(4)] and also we extend Proposition 2.3 [2] for a Cohen-Macaulay mdules of dimension 2. We end the paper with some examples. For any unexplained notation or terminology, we refer the reader to [1] and [5].

2. Main results

Lemma 2.1. *Let M be a finitely generated R-module of dimension d. Then $h_0^I(M)=\lambda(M/IM)$ and for all $1\leq i\leq r$ we have $h_i^I(M)=\lambda(I^iM/I^{i+1}M)-\sum_{n=0}^{i-1}\binom{d+n}{d-1}h_{i-1-n}^I(M)$.*

Proof. Since $\sum_{n=0}^{\infty}\lambda(I^nM/I^{n+1}M)z^n=h_M^I(z)/(1-z)^d$, where $h_M^I(z)=\sum_{i=0}^r h_i^I(M)z^i$, by easily calculation the result follows. □

Theorem 2.2. *Let (R,m) be a Noetherian local ring, M a finitely generated R-module of dimension d and I an ideal of definition for M. Let $x\in I$ be both M-superficial and M-regular sequence and $t\in\mathbb{N}_0$. Set $N=M/xM$, $K=I/xI$. Then $h_i^I(M)=h_i^K(N)$ for all $i\leq t$ if and only if $I^{i+1}M:x=I^iM$ for all $i\leq t$.*

Proof. ($\Longrightarrow$). We proceed by induction on t. If $t=0,1$, then by [7, Corollary 10(4)] there is nothing to prove. We assume that the result hold for all $i < t$ and prove it for $i = t$. By using Lemma 2.1, we have

$$\begin{aligned} h_t^K(N) =& \lambda(K^tN/K^{t+1}N) - \sum_{n=0}^{t-1}\binom{n+d-1}{d-2}h_{t-1-n}^K(N) \\ =& \lambda(I^tM + xM/I^{t+1}M + xM) - \sum_{n=0}^{t-1}\binom{n+d-1}{d-2}h_{t-1-n}^I(M) \\ =& \lambda(I^tM/I^{t+1}M + I^tM \cap xM) - \sum_{n=0}^{t-1}\binom{n+d-1}{d-2}h_{t-1-n}^I(M). \end{aligned}$$

Now, by induction hypothesis, we have

$$\begin{aligned} h_t^K(N) =& \lambda(I^tM/I^{t+1}M) - \lambda(xI^{t-1}M/x(I^{t+1}M :_M x)) \\ & - \sum_{n=0}^{t-1}\binom{n+d-1}{d-2}h_{t-1-n}^I(M) \\ =& \lambda(I^tM/I^{t+1}M) - \lambda(I^{t-1}M/(I^{t+1}M :_M x)) \\ & - \sum_{n=0}^{t-1}\binom{n+d-1}{d-2}h_{t-1-n}^I(M) \\ =& \lambda(I^tM/I^{t+1}M) - \lambda(I^{t-1}M/I^tM) + \lambda(I^{t+1}M :_M x/I^tM) \\ & - \sum_{n=0}^{t-1}\binom{n+d-1}{d-2}h_{t-1-n}^I(M) \\ =& \lambda(I^tM/I^{t+1}M) - h_{t-1}^I(M) - \sum_{n=0}^{t-2}\binom{n+d}{d-1}h_{t-2-n}^I(M) \\ & + \lambda(I^{t+1}M :_M x/I^tM) - \sum_{n=0}^{t-1}\binom{n+d-1}{d-2}h_{t-1-n}^I(M) \\ =& \lambda(I^tM/I^{t+1}M) - \sum_{n=0}^{t-1}\binom{n+d}{d-1}h_{t-1-n}^I(M) + \lambda(I^{t+1}M :_M x/I^tM) \\ =& h_t^I(M) + \lambda(I^{t+1}M :_M x/I^tM). \end{aligned}$$

Therefore $I^{t+1}M :_M x = I^tM$, as desired.
($\Longleftarrow$) Again by using induction and the same method the result easily seen. □

Theorem 2.3. *Let M be a finitely generated R-module of dimension 2 and J a minimal reduction of I. Then $\lambda(\frac{\widetilde{I^nM}}{J\widetilde{I^{n-1}M}})$ does not depend on J for all $n \geq 0$.*

Proof. By using [11, Page 28], we have $\widetilde{I^{n+1}M} :_M I = \widetilde{I^nM}$ for all $n \geq 0$. Therefore $depth \widetilde{G_I(M)} \geq 1$. Thus by an argument similar to that used in [13, Corollary 1.2] we have $h^I_M(z) = h^I_0(M) + \sum_{i=1}^{r}(\lambda(\frac{\widetilde{I^iM}}{J\widetilde{I^iM}}) - \lambda(\frac{\widetilde{I^{i+1}M}}{J\widetilde{I^iM}}))z^i$ which is a polynomial with coefficients independent from J. □

The computation of the following examples are performed by using Macaulay 2 and the ground field k is assumed to be characteristic zero (see [3]).

Example 2.4. Let $I = (x^3, y^3, z^3, x^2y, xy^2, yz^2, xyz)$ be an ideal of $R = k[x, y, z]$. The Hilbert series of I is

$$H^I_R(t) = \frac{14 + 7t + 7t^2 - t^3}{(1 - t)^3}.$$

Since $x^3 + y^3 \in I$ is a superficial element and $I^{n+1} : x^3 + y^3 = I^n$ for all $n \geq 0$, we have $H^{I/(x^3+y^3)}_{R/(x^3+y^3)}(t) = \frac{14+7t+7t^2-t^3}{(1-t)^2}$.

Example 2.5. Let $I = (x, y^2, z^2, yw, zw)$ be an ideal of $R = k[x, y, z, w]/(w^3)$. The Hilbert series of I is

$$H^I_R(t) = \frac{6 + 3t + 4t^2 - t^3}{(1 - t)^3}.$$

The element $x \in I$ is a superficial element and $I^{n+1} : x = I^n$ for all $n \geq 0$, so it follows $H^{I/(x)}_{R/(x)}(t) = \frac{6+3t+4t^2-t^3}{(1-t)^2}$.

Example 2.6. Let $R = k[x, y, z, u, v, w]/(z^2, zu.zv, uv, yz - u^3, xz - v^3)$. The Hilbert series of the maximal ideal $m = (x, y, z, u, v, w)$ is

$$H^m_R(t) = \frac{1 + 3t + 3t^3 - t^4}{(1 - t)^3}$$

. Since $m^{n+1} : w = m^n$, it follows $H^{m/(w)}_{R/(w)}(t) = \frac{1+3t+3t^3-t^4}{(1-t)^2}$.

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