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WEAKLY IRREDUCIBLE IDEALS

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ABSTRACT. Let R be a commutative ring. The purpose of this article is to introduce a new class of ideals of R called weakly irreducible ideals. This class could be a generalization of the families quasi-primary ideals and strongly irreducible ideals. The relationships between the notions primary, quasi-primary, weakly irreducible, strongly irreducible and irreducible ideals, in different rings, has been given. Also the relations between weakly irreducible ideals of R and weakly irreducible ideals of localizations of the ring R are also studied.

1. INTRODUCTION

Throughout this article, R denotes a commutative ring with identity. About a quarter of a century before, in [3] the notion of quasi-primary ideals as a generalization of the notion primary ideals was introduced. Indeed, a proper ideal q of R is called quasi-primary if $rs \in q$, for $r, s \in R$, implies that either $r \in \sqrt{q}$ or $s \in \sqrt{q}$. Equivalently, q is quasi-primary if and only if $\sqrt{q}$ is prime [3, Definition 2, p. 176].

In [5], a proper ideal I of a ring R is called strongly irreducible if for ideals A and B of R, the inclusion $A \cap B \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$. Strongly irreducible ideals over commutative rings have been extensively studied in [2] and [5]. It is easy to see that every prime ideal is strongly irreducible. Also every strongly irreducible ideal is irreducible and hence strongly irreducible ideals over a Noetherian ring are primary [5, Lemma 2.2(1),(2)]. Over a commutative ring, it is

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therefore natural to pursue the analogues of this property. This leads us to the following definition as a generalization of the notion strongly irreducible ideals.

Definition 1.1. *We say that a proper ideal I of R is weakly irreducible provided that for each pair of ideals A and B of R, $A \cap B \subseteq I$ implies that either $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$.*

Clearly every quasi-primary ideal of R is weakly irreducible. But the converse is not true in general. For example, let R be a Noetherian local ring with maximal ideal m having more than one minimal prime. Let $E = E(R/m)$ denote the injective envelope of the residue field R/m of R as an R-module. In [5, Example 2.4], it has been shown that the zero ideal of the idealization $A = R + E$ [6, page 2] is strongly irreducible, and hence weakly irreducible. But the zero ideal in A is not quasi-primary.

We begin with a few well-known results about strongly irreducible ideals. Recall that a ring R is called arithmetical provided that for all ideals I, J and K of R, $I + (J \cap K) = (I + J) \cap (I + K)$ (See [4]).

Proposition 1.2. *Let I be an ideal in a ring R. Then:*

1. *If R is an arithmetical ring, I is irreducible if and only if I is strongly irreducible* [5, Lemma 2.2(3)]
2. *Let R be an arithmetical ring. If I is a primary ideal of R, then I is an irreducible ideal of R* [4, Theorem 6].
3. *If R is a Laskerian ring(i.e. every proper ideal of R has a primary decomposition) or R is a unique factorization domain (UFD), then every strongly irreducible ideal of R is a primary ideal* [2, Theorem 2.1(iii)] *and* [2, Theorem 2.2(iv)].
4. *If R is an absolutely flat ring or R is a Zerlegung Primideale ring (ZPI-ring; i.e. every proper ideal of R can be written as a product of prime ideals of R), I is strongly irreducible if and only if I is primary* [2, Theorem 2.1(iv)] *and* [2, Theorem 3.7].

In this paper, we characterize the notion weakly irreducible ideals over different rings. Moreover, the relationships between the notions primary, quasi-primary, weakly irreducible, strongly irreducible and irreducible ideals, in different rings, has been given. The relations between weakly irreducible ideals of a ring and weakly irreducible ideals of localizations of the ring also studied. In the following, some of these results has been mentioned.

Theorem 1.3. *Let R be a ring.*

(1) *If R is a UFD or a Laskerian ring, then an ideal is weakly irreducible ideal if and only if it is a quasi-primary ideal (Theorem 2.1*(4) *and Theorem 2.3*(2)*).*

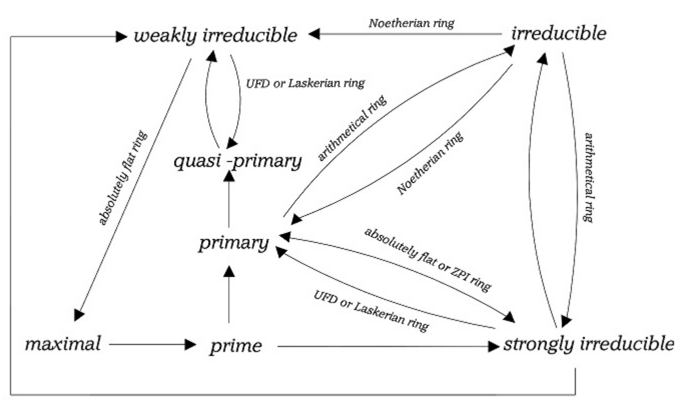
(2) *For an absolutely flat ring, the notions maximal, prime, primary, quasi-primary, strongly irreducible and weakly irreducible ideals are equal (Proposition 1.2*(4) *and Theorem 2.1*(5)*).*

(3) *(Theorem 3.3) If R is a ring and S is a multiplicatively closed subset of R, then the following are equivalent:*
   (i) *Every weakly irreducible ideal A of R which $A = I^c$, the contraction of an ideal I of $S^{-1}R$ is quasi-primary;*
   (ii) *Every weakly irreducible ideal B of $S^{-1}R$ is quasi-primary.*

(4) *(Corollary 3.4) Let I be an ideal of a ring R and p a prime ideal of R containing I. The following are equivalent:*
   (i) *If I is a weakly irreducible ideal of R such that $I = J^c$ for some ideal J of R_p, then I is quasi-primary.*
   (ii) *If I_p is a weakly irreducible ideal of R_p, then I_p is quasi-primary.*

(5) *(Theorem 3.5) For a ring R, the following are equivalent.*
   (i) *Every proper ideal of R is weakly irreducible;*
   (ii) *The radicals of every two ideals of R are comparable;*
   (iii) *Every proper ideal of R is quasi-primary.*
   (iv) *The prime ideals of R form a chain with respect to inclusion.*

Some of the main interrelations of the above mentioned types of ideals can be summarized in the following chart.



2. Weakly irreducible ideals

Theorem 2.1. *Let R be a ring.*

(1) *Let I be a proper ideal of R. Then the following are equivalent:*
   (i) *I is a quasi-primary ideal;*
   (ii) *$\sqrt{I}$ is a weakly irreducible ideal;*
   (iii) *$\sqrt{I}$ is a prime ideal.*

(2) *Let I be a weakly irreducible ideal of R. Then I is a prime ideal if and only if $I = \sqrt{I}$.*

(3) *If R is a Laskerian ring, then every weakly irreducible ideal of R is a quasi-primary ideal.*

(4) *For any ideal I of an absolutely flat ring R, the following are equivalent:*
   (i) *I is a maximal ideal;*
   (ii) *I is a quasi-primary ideal;*
   (iii) *I is a weakly irreducible ideal.*

(5) *If I is weakly irreducible and if A is an ideal contained in I, then I/A is weakly irreducible in R/A.*

Proof. (1) Suppose I is a proper ideal of R. $(i) \Leftrightarrow (iii)$ follows from [3, Definition 2 p. 176]. $(i) \Rightarrow (ii)$. Let I be a quasi-primary ideal of R. Then $\sqrt{I}$ is a prime and hence a weakly irreducible ideal of R. $(ii) \Rightarrow (i)$. Let $ab \in I$ for $a, b \in R$. Then $Ra \cap Rb \subseteq \sqrt{Ra \cap Rb} = \sqrt{Rab} \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a weakly irreducible ideal, we have either $Ra \subseteq \sqrt{I}$ or $Rb \subseteq \sqrt{I}$.
(2) If I is a prime ideal of R, then clearly $I = \sqrt{I}$. Conversely, assume that I is a weakly irreducible ideal of R such that $I = \sqrt{I}$. By (1), $I = \sqrt{I}$ is a prime ideal of R.
(3) Let $I = \bigcap_{i=1}^{n} q_i$ be a minimal primary decomposition of the weakly irreducible ideal I of R. Then, for some $1 \leq j \leq n$, $q_j \subseteq \sqrt{I} = \sqrt{\bigcap_{i=1}^{n} q_i} \subseteq \sqrt{q_j}$ and hence $p_j = \sqrt{q_j} = \sqrt{I}$. Thus I is a quasi-primary ideal of R.
(4) Suppose R is an absolutely flat ring and I an ideal of R. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. $(ii) \Rightarrow (i)$. Let I be a quasi-primary ideal of R and $x \in R \backslash \sqrt{I}$. Since R is absolutely flat, [1, p. 37 Exersice 27] follows the principal ideal Rx is idempotent and hence there exists $a \in R$ such that $x(ax-1) = 0 \in I$. Thus $(ax-1)^n \in I$ for some positive integer n; i.e. $\overline{ax-1}$ is a nilpotent element of $R/\sqrt{I}$ and therefore $\overline{ax}$ is a unit. This implies that $\overline{x}$ is unit and thus $R/\sqrt{I}$ is a field and $\sqrt{I}$ is maximal. On the other hand, since every principal ideal of R is

idempotent, it follows that $\sqrt{I} = I$ and hence I is a maximal ideal of R. $(iii) \Rightarrow (ii)$. Let $ab \in I$ for some $a, b \in R$. Hence the ideals Ra and Rb are idempotent. Thus there exists $t, s \in R$ such that $a = ra^2$ and $b = tb^2$. Let $k \in Ra \cap Rb$, then $k = ak_1 = bk_2$ for some $k_1, k_2 \in R$. Now $k = ak_1 = ra^2k_1 = ak_1ra = bk_2ra \in Rab$, then $Ra \cap Rb \subseteq Rab \subseteq I$. Since I is weakly irreducible, $Ra \subseteq \sqrt{I}$ or $Rb \subseteq \sqrt{I}$; i.e. I is a quasi-primary ideal.
(5). Let J and K be ideals in R such that $(J/A) \cap (K/A) \subseteq I/A$. Then $(J + A) \cap (K + A) \subseteq I + A = I$, since $A \subseteq I$. Since I is weakly irreducible it follows that either $J \subseteq \sqrt{I}$ or $K \subseteq \sqrt{I}$, hence either $J/A \subseteq \sqrt{I/A}$ or $K/A \subseteq \sqrt{I/A}$, so I/A is weakly irreducible. □

It is easy to see that every two elements of a unique factorization domain (UFD) R have a least common multiple. We denote the least common multiple of every two elements $x, y \in R$ by $[x, y]$.

Lemma 2.2. *Let R be a UFD and I a proper ideal of R.*

1. *I is weakly irreducible if and only if for each $x, y \in R$, $[x, y] \in I$ implies that $x \in \sqrt{I}$ or $y \in \sqrt{I}$.*
2. *I is weakly irreducible if and only if $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k} \in I$, where p_i are distinct prime elements of R and n_i are natural numbers, implies that $p_j \in \sqrt{I}$, for some j, $1 \leq j \leq k$.*

Proof. (1) Let I be a weakly irreducible ideal and for $x, y \in R$, $[x, y] \in I$. If $[x, y] = c$, then obviously $Rx \cap Ry = Rc \subseteq I$. So $Rx \subseteq \sqrt{I}$ or $Ry \subseteq \sqrt{I}$.

Conversely, if $Rx \cap Ry \subseteq I$ for $x, y \in R$, then $[x, y] \in Rx \cap Ry \subseteq I$, so by our assumption $x \in \sqrt{I}$ or $y \in \sqrt{I}$.
(2) If I is weakly irreducible, then the result is clear by part (1). Conversely, let $[x, y] \in I$ for $x, y \in R\backslash 0$, and

$$\begin{aligned} x &= p_1^{n_1}p_2^{n_2}p_3^{n_3}\cdots p_k^{n_k}q_1^{m_1}q_2^{m_2}q_3^{m_3}\cdots q_s^{m_s}, \\ y &= p_1^{t_1}p_2^{t_2}p_3^{t_3}\cdots p_k^{t_k}r_1^{l_1}r_2^{l_2}r_3^{l_3}\cdots r_u^{l_u} \end{aligned}$$

be prime decompositions for x and y, respectively. Therefore,

$$[x, y] = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}q_1^{m_1}q_2^{m_2}q_3^{m_3}\cdots q_s^{m_s}r_1^{l_1}r_2^{l_2}r_3^{l_3}\cdots r_u^{l_u},$$

where $\alpha_i = max\{n_i, t_i\}$ for each i. Since $[x, y] \in I$, by the assumption, we have one of following:

(a) for some i, $p_i \in \sqrt{I}$;
(b) for some i, $q_i \in \sqrt{I}$;
(c) for some i, $r_i \in \sqrt{I}$.

If (a) holds, then clearly $x, y \in \sqrt{I}$. For the case (b), $x \in \sqrt{I}$. If c satisfies, then $y \in \sqrt{I}$. □

Theorem 2.3. *Let R be a UFD and I a proper ideal of R.*

1. *If I is a nonzero principal ideal, then I is weakly irreducible if and only if the generator of I is a power of a prime element of R.*
2. *The two classes weakly irreducible ideals and quasi-primary ideals are equal.*

Proof. (1) Let $I = Ra$ be a nonzero weakly irreducible ideal of R, and $a = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ be a prime decomposition for a. By Lemma 2.2(2), for some i, $p_i \in \sqrt{I}$. Hence

$$p_i \in \sqrt{R(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k})} = \sqrt{Rp_1^{\alpha_1} \cap Rp_2^{\alpha_2} \cap Rp_3^{\alpha_3} \cap \cdots \cap Rp_k^{\alpha_k}} = \sqrt{Rp_1^{\alpha_1}} \cap \sqrt{Rp_2^{\alpha_2}} \cap \sqrt{Rp_3^{\alpha_3}} \cap \cdots \cap \sqrt{Rp_k^{\alpha_k}} = Rp_1 \cap Rp_2 \cap Rp_3 \cap \cdots \cap Rp_k.$$

Thus $p_j \mid p_i$ for $1 \leq j \leq k$ and hence $p_i = p_j$ for every $1 \leq j \leq k$. It means that $I = Rp_i^{\alpha_i}$.

Conversely, let $I = Rp^n$ for a prime element p of R. Suppose that $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \in I = Rp^n$ for some distinct prime elements $p_1, p_2, \cdots, p_k$ of R and natural numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$. Then $p^n \mid p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$. So, for some j, $1 \leq j \leq k$, we have $p = p_j$ and $n \leq n_j$. Therefore, $p_j^{n_j} \in \sqrt{I}$. Thus, by Lemma 2.2(2), I is a weakly irreducible ideal.

(2) Let I be a weakly irreducible ideal and $xy \in I$, where $x, y \in R \backslash 0$, and let

$$\begin{aligned} x &= p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} q_3^{m_3} \cdots q_s^{m_s}, \\ y &= p_1^{t_1} p_2^{t_2} p_3^{t_3} \cdots p_k^{t_k} r_1^{l_1} r_2^{l_2} r_3^{l_3} \cdots r_u^{l_u} \end{aligned}$$

be prime decomposition for x and y, respectively. Since $xy \in I$, by part (2), we have one of the following:

(a) for some i, $p_i \in \sqrt{I}$;
(b) for some i, $q_i \in \sqrt{I}$;
(c) for some i, $r_i \in \sqrt{I}$.

If (a) holds, then clearly $x, y \in \sqrt{I}$. For the case (b), $x \in \sqrt{I}$. If c satisfies, then $y \in \sqrt{I}$. Thus I is a quasi-primary ideal of R. □

3. LOCALIZATION AND WEAKLY IRREDUCIBLE IDEALS

Let R be a ring and let S be a multiplicatively closed subset of R. For each ideal I of the ring $S^{-1}R$, we consider

$I^c = \{x \in R \mid x/1 \in I\} = I \cap R$, and $C = \{I^c \mid I \text{ is an ideal of } S^{-1}R\}$.

Theorem 3.1. *Let R be a ring and S be a multiplicatively closed subset of R. Then there is a one-to-one correspondence between the weakly irreducible ideals of $S^{-1}R$ and weakly irreducible ideals of R contained in C.*

Proof. Let I be a weakly irreducible ideal of $S^{-1}R$. Obviously, $I^c \neq R$, $I^c \in C$ and $I^c \cap S = \emptyset$. Let $A \cap B \subseteq I^c$, where A and B are ideals of R. Then we have $S^{-1}A \cap S^{-1}B = S^{-1}(A \cap B) \subseteq S^{-1}(I^c) = I$. Hence, $S^{-1}(A) \subseteq \sqrt{I}$ or $S^{-1}(B) \subseteq \sqrt{I}$, and so $A \subseteq (S^{-1}(A))^c \subseteq (\sqrt{I})^c = \sqrt{I^c}$ or $B \subseteq (S^{-1}(B))^c \subseteq (\sqrt{I})^c = \sqrt{I^c}$. Thus I^c is a weakly irreducible ideal of R.

Conversely, let I be a weakly irreducible ideal of R, $I \cap S = \emptyset$ and $I \in C$. Since $I \cap S = \emptyset$, $S^{-1}I \neq S^{-1}R$. Let $A \cap B \subseteq S^{-1}I$, where A and B are ideals of $S^{-1}R$. Then $A^c \cap B^c = (A \cap B)^c \subseteq (S^{-1}I)^c$. Now since $I \in C$, $(S^{-1}I)^c = I$. So $A^c \cap B^c \subseteq I$. Consequently, $A^c \subseteq \sqrt{I}$ or $B^c \subseteq \sqrt{I}$. Thus $A = S^{-1}A^c \subseteq S^{-1}(\sqrt{I}) \subseteq \sqrt{S^{-1}(I)}$ or $B = S^{-1}B^c \subseteq S^{-1}(\sqrt{I}) \subseteq \sqrt{S^{-1}(I)}$. Therefore, $S^{-1}(I)$ is a weakly irreducible ideal of $S^{-1}R$. □

Lemma 3.2. *Let S be a multiplicatively closed subset of a ring R and p a prime ideal of R such that $p \cap S = \emptyset$. Then*

(1) *If q is a p-quasi-primary ideal of R, then $S^{-1}q$ is a $S^{-1}p$-quasi-primary ideal of $S^{-1}R$.*
(2) *If $S^{-1}q$ is a quasi-primary ideal of $S^{-1}R$ such that $\sqrt{S^{-1}q} = S^{-1}p$ and q is contained in C, then q is a p-quasi-primary ideal of R.*

Proof. (1) Let q be a quasi-primary ideal of R with $\sqrt{q} = p$. Since $p \cap S = \emptyset$, $S^{-1}p$ is a prime ideal of $S^{-1}R$ and so that $\sqrt{S^{-1}q} = S^{-1}(\sqrt{q}) = S^{-1}p$ implies that $S^{-1}q$ is a $S^{-1}p$-quasi-primary ideal of R.
(2) Let $S^{-1}q$ be a quasi-primary ideal of $S^{-1}R$ such that $\sqrt{S^{-1}q} = S^{-1}p$ and $q \in C$. It is clear that $(S^{-1}q)^c = q$, since $q \in C$. It follows that $\sqrt{q} = \sqrt{(S^{-1}q)^c} = (\sqrt{S^{-1}q})^c = (S^{-1}p)^c = p$ and hence q is a p-quasi-primary ideal of R.

□

Theorem 3.3. *If R is a ring and S is a multiplicatively closed subset of R, then the following are equivalent:*

(1) *Every weakly irreducible ideal A of R which $A \in C$ is quasi-primary;*
(2) *Every weakly irreducible ideal of $S^{-1}R$ is quasi-primary;*

Proof. (1) ⇒ (2) Let B be a weakly irreducible ideal of $S^{-1}R$. Then by the proof of Theorem 3.1, B^c is a weakly irreducible ideal of R and, by our assumption, B^c is a quasi-primary ideal of R. Now, by Lemma 3.2(1), $B = S^{-1}(B^c)$ is a quasi-primary ideal of $S^{-1}R$.

(2) ⇒ (1) Let A be a weakly irreducible ideal of R such that $A \in C$. By Theorem 3.1, $S^{-1}A$ is a weakly irreducible ideal of R. Since $A \in C$, we have $\sqrt{(S^{-1}A)^c} = (S^{-1}\sqrt{A})^c = \sqrt{A}$. Thus $\sqrt{(S^{-1}A)^c} \cap S = \emptyset$. Now, by our assumption, $S^{-1}A$ is a quasi-primary ideal of $S^{-1}R$ and so A is a quasi-primary ideal of R by Lemma 3.2(2). □

Corollary 3.4. *Let I be an ideal of a ring R and p a prime ideal of R containing I. The following are equivalent:*

1. *If I is a weakly irreducible ideal of R such that $I = J^c$ for some ideal J of R_p, then I is quasi-primary.*
2. *If I_p is a weakly irreducible ideal of R_p, then I_p is quasi-primary.*

Theorem 3.5. *For a ring R, the following are equivalent.*

1. *Every proper ideal of R is weakly irreducible;*
2. *The radicals of every two ideals of R are comparable;*
3. *Every proper ideal of R is quasi-primary.*
4. *The prime ideals of R form a chain with respect to inclusion.*

Proof. (1) ⇒ (2) By our assumption, $I \cap J$ is a weakly irreducible ideal of R where I and J are two ideals of R. Thus $I \cap J \subseteq I \cap J$ implies that $I \subseteq \sqrt{I \cap J} = \sqrt{IJ}$ or $J \subseteq \sqrt{I \cap J} = \sqrt{IJ}$. On the other hand, $\sqrt{IJ} \subseteq \sqrt{I}$ and $\sqrt{IJ} \subseteq \sqrt{J}$ and so $\sqrt{I} \cap \sqrt{J} = \sqrt{IJ} = \sqrt{I}$ or $\sqrt{I} \cap \sqrt{J} = \sqrt{IJ} = \sqrt{J}$. It means that $\sqrt{I} \subseteq \sqrt{J}$ or $\sqrt{J} \subseteq \sqrt{I}$.

(2) ⇒ (3) Let I be a proper ideal of R and $ab \in I$ for $a, b \in R$. By our assumption, $\sqrt{Ra} \subseteq \sqrt{Rb}$ or $\sqrt{Rb} \subseteq \sqrt{Ra}$. Therefore $\sqrt{Ra} \cap \sqrt{Rb} = \sqrt{Rab} \subseteq \sqrt{I}$ and hence $Ra \subseteq \sqrt{Ra} \subseteq \sqrt{I}$ or $Rb \subseteq \sqrt{Rb} \subseteq \sqrt{I}$; i.e. I is a quasi-primary ideal of R.

(3) ⇒ (1) is clear.

(4) ⇒ (3) is trivial, since by (4) radical of every ideal is prime.

(2) ⇒ (4) is clear. □

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