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A NOTE ON A GRAPH ASSOCIATED TO A COMMUTATIVE RING

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Abstract. The rings considered in this article are commutative with identity. This article is motivated by the work on comaximal graphs of rings. In this article, with any ring R, we associate an undirected graph denoted by $G(R)$, whose vertex set is the set of all elements of R and distinct vertices x, y are joined by an edge in $G(R)$ if and only if $Rx \cap Ry = Rxy$. In Section 2 of this article, we classify rings R such that $G(R)$ is complete and we also consider the problem of determining rings R such that $\chi(G(R)) = \omega(G(R)) < \infty$. In Section 3 of this article, we classify rings R such that $G(R)$ is planar.

1. Introduction

The rings considered in this article are commutative with identity. Inspired by the research work of I. Beck in [5], several algebraists associated a graph with certain subsets of a ring and investigated the interplay between the ring-theoretic properties of a ring and the graph-theoretic properties of the graph associated with it (See for example, [1, 2, 6, 7, 10, 12, 13, 14]). The present article is motivated by the research work of P.K. Sharma and S.M. Bhatwadekar in [14]. Let R be a ring. The authors of [14] introduced an undirected graph on R, whose vertex set is the set of all elements of R and distinct vertices a, b are joined by an edge in this graph if and only if $Ra + Rb = R$ and in [14], they investigated mainly on the coloring of the graph introduced

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by them. The work presented in [14] inspired a lot of research work in this area. In [12], H.R. Maimani, M. Salimi, A. Sattari, and S. Yassemi called the graph introduced in [14] on the set of all elements of R as the *comaximal graph* of the ring R and used the notation $\Gamma(R)$ for this graph. The authors of [12] investigated several other properties of $\Gamma(R)$ and in addition, they explored some subgraphs of $\Gamma(R)$. For a ring R, we denote the set of all units of R by $U(R)$ and the set of all nonunits of R by $NU(R)$, and the Jacobson radical of R by $J(R)$. The subgraphs of $\Gamma(R)$ investigated in [12] are $\Gamma_1(R), \Gamma_2(R)$, and $\Gamma_2(R)\backslash J(R)$, where $\Gamma_1(R)$ is the subgraph induced on $U(R)$, $\Gamma_2(R)$ is the subgraph induced on $NU(R)$, and $\Gamma_2(R)\backslash J(R)$ is the subgraph induced on $NU(R)\backslash J(R)$.

Let R be a ring. In this article, we introduce a graph structure on R, denoted by $G(R)$, is an undirected graph whose vertex set is the set of all elements of R and distinct vertices x, y are joined by an edge in $G(R)$ if and only if $Rx \cap Ry = Rxy$. The aim of this article is to study the interplay between the graph-theoretic properties of $G(R)$ and the ring-theoretic properties of R.

It is useful to recall the following definitions from graph theory before we describe the results that are proved in Section 2 of this article. The graphs considered in this article are simple and undirected. Let $G = (V, E)$ be a graph. Let $a, b \in V, a \neq b$. Recall that the *distance* between a and b, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b if there exists such a path in G; otherwise we define $d(a, b) = \infty$. We define $d(a, a) = 0$. The *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b)|a, b \in V\}$ [4]. A graph $G = (V, E)$ is said to be *connected* if for any distinct $a, b \in V$, there exists a path in G between a and b [4]. A simple graph $G = (V, E)$ is said to be *complete* if every pair of distinct vertices of G are adjacent in G [4, Definition 1.1.11]. For a graph G, we denote the set of all vertices of G and the set of all edges of G by $V(G)$ and $E(G)$ respectively. A subgraph H of G is said to be a *spanning subgraph* of G if $V(G) = V(H)$.

Let $G = (V, E)$ be a graph. Recall from [4, Definition 1.2.2] that a *clique* of G is a complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$ is defined as the largest integer $n \geq 1$ such that G contains a clique on n vertices [4, page 185]. We set $\omega(G) = \infty$, if G contains a clique on n vertices for all $n \geq 1$. Recall from [4, page 129] that a *vertex coloring* of G is a map $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper*, if adjacent vertices of G receive different colors of S; that is, if a and b are adjacent vertices of G, then $f(a) \neq f(b)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed for a proper vertex coloring of G [4, Definition 7.1.2]. It is well-known that for any graph G, $\omega(G) \leq \chi(G)$.

We next recall some definitions and results from commutative ring theory that are used in this article. Let R be a ring. We denote the nilradical of R using the notation $nil(R)$. A ring R is said to be *reduced* if $nil(R) = (0)$. Recall from [9, Exercise 16, page 111] that a ring R is said to be *von Neumann regular* if for each $a \in R$, there exists $b \in R$ such that $a = a^2b$. A principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the unique prime ideal of a special principal ideal ring R, then $\mathfrak{m}$ is nilpotent. If $\mathfrak{m}$ is the only prime ideal of a SPIR R, then we denote it using the notation $(R, \mathfrak{m})$ is a SPIR. Suppose that $(R, \mathfrak{m})$ is a SPIR which is not a field. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $(iii) \Rightarrow (i)$ of [3, Proposition 8.8] that $\{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R. A ring R which has a unique maximal ideal is referred to as a *local* ring. We denote the cardinality of a set A using the notation $|A|$. If A and B are sets such that A is properly contained in B, then we denote it using the notation $A \subset B$.

We now give a brief summary of the results that are proved in Section 2 of this article. Let R be a ring. It is observed in Lemma 2.1 that $\Gamma(R)$ is a spanning subgraph of $G(R)$. It is noted in Lemma 2.2 that $G(R)$ is connected and $diam(G(R)) \leq 2$. In Section 2, we first focus on classifying rings R such that $G(R)$ is complete. Let R be a reduced ring. It is shown in Proposition 2.4 that $G(R)$ is complete if and only if R is von Neumann regular. Let R be a ring such that R is not reduced. It is proved in Theorem 2.9 that $G(R)$ is complete if and only if $(R, \mathfrak{m})$ is local with $\mathfrak{m}^2 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$. It was proved in [14, Proposition 2.2] that for a ring R, $\omega(\Gamma(R)) < \infty$ if and only if R is finite. For any finite ring R, it was shown in the moreover part of [14, Theorem 2.3] that $\chi(\Gamma(R)) = \omega(\Gamma(R)) = t + l$, where t is the number of maximal ideals of R and l is the number of units of R. Since $\Gamma(R)$ is a spanning subgraph of $G(R)$, it follows that $\omega(G(R)) < \infty$ implies that R is a finite ring. Motivated by [14, Proposition 2.2 and Theorem 2.3], in Section 2 of this article, we try to classify finite rings R in order that $\chi(G(R)) = \omega(G(R))$. Let R be a finite reduced ring. Then it is shown in Proposition 2.10 that $\chi(G(R)) = \omega(G(R)) = |R|$. Let R be a finite ring which is not reduced. In Section 2 of this article, we try to classify such rings R in order that $\chi(G(R)) = \omega(G(R))$. We provide some sufficient conditions on the ring R in order that $\chi(G(R)) = \omega(G(R))$. Let $(R, \mathfrak{m})$ be a finite local ring such that $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. It is proved in Proposition 2.12 that $\chi(G(R)) = \omega(G(R)) = |U(R)| + k + 1$, where k is the number of minimal ideals of R. In Example 2.14, it is verified that the ring R provided by D.D. Anderson and M. Naseer in [1] which answered a conjecture made by I. Beck in [5] in the negative is such that $\omega(G(R)) = 20 < \chi(G(R)) = 21$. Let $(R, \mathfrak{m})$ be a finite SPIR which is not a field. It is shown in Proposition 2.15 that $\chi(G(R)) = \omega(G(R)) = |U(R)| + 2$.

In Section 3 of this article, we discuss on the planarity of $\Gamma(R)$ (respectively, $G(R)$). We try to classify rings R such that $\Gamma(R)$ (respectively, $G(R)$) is planar. It is useful to recall the following definitions and results from graph theory before we give a brief summary of the results that are proved in Section 3. Let $n \in \mathbb{N}$. A complete graph on n vertices is denoted by K_n. A graph $G = (V, E)$ is said to be *bipartite* if V can be partitioned into two nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. Let $m, n \in \mathbb{N}$. Let $G = (V, E)$ be a complete bipartite graph with $V = V_1 \cup V_2$. If $|V_1| = m$ and $|V_2| = n$, then G is denoted by $K_{m,n}$ [4, Definition 1.1.12].

Let $G = (V, E)$ be a graph. Recall from [4, Definition 8.1.1] that G is said to be *planar* if G can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G. Recall that two adjacent edges are said to be in *series* if their common end vertex is of degree two [8, page 9]. Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by insertion of vertices of degree two or by the merger of edges in series [8, page 100]. It is useful to note from [8, page 93] that the graph K_5 is referred to as *Kuratowski's first graph* and the graph $K_{3,3}$ is referred to as *Kuratowski's second graph*. The celebrated theorem of Kuratowski states that a graph G is planar if and only if G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [8, Theorem 5.9].

It is convenient to name the following conditions satisfied by a graph $G = (V, E)$ so that it can be used throughout Section 3 of this article.

(i) We say that G *satisfies* (Ku_1) if G does not contain K_5 as a subgraph (that is, equivalently, if $\omega(G) \leq 4$).

(ii) We say that G *satisfies* (Ku_1^*) if G satisfies (Ku_1) and moreover, G does not contain any subgraph homeomorphic to K_5.

(iii) We say that G *satisfies* (Ku_2) if G does not contain $K_{3,3}$ as a subgraph.

(iv) We say that G *satisfies* (Ku_2^*) if G satisfies (Ku_2) and moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

Note that a graph $G = (V, E)$ is planar if and only if G satisfies both (Ku_1^*) and (Ku_2^*) [8, Theorem 5.9]. Thus if G is planar, then G satisfies both (Ku_1) and (Ku_2). It is interesting to note that a graph G can be nonplanar even if it satisfies both (Ku_1) and (Ku_2). For an example of this type, refer [8, Figure 5.9(a), page 101] and the graph given in this example does not satisfy (Ku_2^*). It is not hard to construct an example of a graph such that G satisfies (Ku_1) but G does not satisfy (Ku_1^*).

For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$. Let p be a prime number and $n \geq 1$. We denote the finite field containing exactly p^n elements by $\mathbb{F}_{p^n}$. Let R be a ring. It is observed in Lemma 3.1 that if $\Gamma(R)$ satisfies (Ku_1), then $|Max(R)| \leq 3$ and $|U(R)| \leq 3$. Let R be a ring. If $|Max(R)| = 3$, then it is proved in Theorem 3.2 that $\Gamma(R)$ is planar if and only if $\Gamma(R)$ satisfies (Ku_1) if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. If $|Max(R)| = 2$, then it is shown in Theorem 3.8 that $\Gamma(R)$ is planar if and only if $\Gamma(R)$ satisfies both (Ku_1) and (Ku_2) if and only if R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3\}$. For a local ring $(R, \mathfrak{m})$, it is proved in Theorem 3.10 that $\Gamma(R)$ is planar if and only if $\Gamma(R)$ satisfies (Ku_1) if and only if R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$. Let R be a ring. It is shown in Theorem 3.11 that $G(R)$ is planar if and only if $G(R)$ satisfies (Ku_1) if and only if R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$.

2. Some basic properties of $G(R)$

Lemma 2.1. *Let R be a ring. Then $\Gamma(R)$, the comaximal graph of R is a spanning subgraph of $G(R)$.*

Proof. Note that the vertex set of $\Gamma(R)$ = the vertex set of $G(R)$ = the set of all elements of R. Let x, y be distinct elements of R such that x and y are adjacent in $\Gamma(R)$. Hence, $Rx + Ry = R$. This implies by [3, Proposition 1.10(i)] that $Rx \cap Ry = Rxy$. Therefore, x and y are adjacent in $G(R)$. This proves that $\Gamma(R)$ is a spanning subgraph of $G(R)$. □

Lemma 2.2. *Let R be a ring. Then $G(R)$ is connected and moreover, $diam(G(R)) \leq 2$.*

Proof. Let $x, y \in R$ be such that $x \neq y$. Suppose that x and y are not adjacent in $G(R)$. Observe that $x - 0 - y$ is a path in $G(R)$ of length two between x and y. This proves that $G(R)$ is connected and $diam(G(R)) \leq 2$. □

We next try to classify rings R such that $G(R)$ is complete (that is, equivalently, we try to classify rings R such that $diam(G(R)) = 1$). For any ring R, we denote the Krull dimension of R by $dim R$ and we denote the set of all prime ideals of R by $Spec(R)$.

Lemma 2.3. *If $G(R)$ is complete, then* $\dim R = 0$.

Proof. Assume that $G(R)$ is complete. Hence, any distinct $x, y \in R$ are adjacent in $G(R)$ and so, $Rx \cap Ry = Rxy$. Suppose that $dim R > 0$. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in Spec(R)$ be such that $\mathfrak{p}_1 \subset \mathfrak{p}_2$. Let $r \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$. We claim that $Rr^m \neq Rr^n$ for all distinct $m, n \in \mathbb{N}$. Suppose that $Rr^m = Rr^n$ for some distinct $m, n \in \mathbb{N}$. We can assume without loss of generality that $m < n$. Observe that $r^m = sr^n$ for some $s \in R$. This implies that $r^m(1 - sr^{n-m}) = 0 \in \mathfrak{p}_1$. Since $\mathfrak{p}_1 \in Spec(R)$ and $r \notin \mathfrak{p}_1$, we obtain that $1 - sr^{n-m} \in \mathfrak{p}_1 \subset \mathfrak{p}_2$. As $r \in \mathfrak{p}_2$, it follows that $1 = 1 - sr^{n-m} + sr^{n-m} \in \mathfrak{p}_2$. This is impossible. Therefore, $Rr^m \neq Rr^n$ for all distinct $m, n \in \mathbb{N}$. Hence, in particular, $r \neq r^2$. Observe that $Rr \cap Rr^2 = Rr^2$, whereas $Rrr^2 = Rr^3$. From $Rr^2 \neq Rr^3$, it follows that r and r^2 are not adjacent in $G(R)$. This is in contradiction to the assumption that $G(R)$ is complete. Therefore, we get that $dim R = 0$. □

We prove in Proposition **??** that for a reduced ring R, $G(R)$ is complete if and only if R is von Neumann regular.

Proposition 2.4. *Let R be a reduced ring. Then the following statements are equivalent:*
(i) $G(R)$ is complete.
(ii) R is von Neumann regular.

Proof. $(i) \Rightarrow (ii)$ Assume that $G(R)$ is complete. We know from Lemma 2.3 that $dim R = 0$. As R is reduced and $dim R = 0$, we obtain from $(d) \Rightarrow (a)$ of [9, Exercise 16, page 111] that R is von Neumann regular.
$(ii) \Rightarrow (i)$ Assume that R is von Neumann regular ring. Let $x, y \in R$ with $x \neq y$. It follows from $(1) \Rightarrow (3)$ of [9, Exercise 29, page 113] that there are units u, v in R and idempotent elements e, f of R such that $x = ue$ and $y = vf$. Note that $xy = uvef$, $Rx = Re$, and $Ry = Rf$. Thus $Rx \cap Ry = Re \cap Rf = Ref = Rxy$. This shows that x and y are adjacent in $G(R)$. Therefore, $G(R)$ is complete. □

Let R be a ring such that R is not reduced. We next proceed to classify such rings R in order that $G(R)$ be complete.

Lemma 2.5. *Let R be a ring. If $G(R)$ is complete, then $(nil(R))^2 = (0)$.*

Proof. Assume that $G(R)$ is complete. Hence, any distinct $x, y \in R$ are adjacent in $G(R)$ and so, $Rx \cap Ry = Rxy$. Let $x \in nil(R)$ be such that $x \neq 0$. Observe that $x \neq xy$ for any $y \in nil(R)$. For if $x = xy$ for some $y \in nil(R)$, then $x(1-y) = 0$. This implies that $x = 0$, since $1 - y$ is a unit in R. This is a contradiction. Hence, $x \neq xy$ for any $y \in nil(R)$. In particular, $x \neq x^2$. As we are assuming that $G(R)$ is complete, we get that $Rx \cap Rx^2 = Rx^3$. Therefore, $Rx^2 = Rx^3$ and so, $x^2 = rx^3$ for some $r \in R$. Hence, $x^2(1-rx) = 0$. Since $1 - rx$ is a unit in R, we obtain that $x^2 = 0$. Let $x, y \in nil(R)$. We claim that $xy = 0$. This is clear if $x = 0$. Hence, we can assume that $x \neq 0$. It is already noted that $x \neq xy$. Since $G(R)$ is complete, we obtain that $Rxy = Rx \cap Rxy = Rx^2y$. As $x^2 = 0$, it follows that $Rxy = (0)$ and so, $xy = 0$. This proves that $(nil(R))^2 = (0)$. □

Lemma 2.6. *Let R_1, R_2 be rings and let $R = R_1 \times R_2$. Suppose that R is not reduced. Then $G(R)$ is not complete.*

Proof. As R is not reduced, either R_1 is not reduced or R_2 is not reduced. Without loss of generality, we can assume that R_1 is not reduced. Let $x \in R_1$ be such that $x \neq 0$ but $x^2 = 0$. Consider the elements $a, b \in R$ given by $a = (x, 0)$ and $b = (x, 1)$. It is clear that $a \neq b$. From $x^2 = 0$, it follows that $ab = (0, 0)$. Observe that $a = (x, 0) = (1, 0)(x, 1) = (1, 0)b$. Hence, $Ra \subseteq Rb$ and so, $Ra \cap Rb = Ra \neq ((0, 0))$. Therefore, $Ra \cap Rb \neq Rab$. This shows that a and b are not adjacent in $G(R)$ and hence, we obtain that $G(R)$ is not complete. □

Lemma 2.7. *Let R be a ring such that R is not reduced. If $G(R)$ is complete, then R is local.*

Proof. Assume that $G(R)$ is complete. We know from Lemma 2.3 that $dim R = 0$. Suppose that R is not local. Let us denote the ring $\frac{R}{nil(R)}$ by T. Observe that $dim T = 0$ and T is reduced. Hence, we obtain from $(d) \Rightarrow (a)$ of [9, Exercise 16, page 111] that T is von Neumann regular. Since we are assuming that R is not local, it follows that T is not local. As T is von Neumann regular, we obtain that T has at least one nontrivial idempotent. Let $t = r + nil(R)$ be a nontrivial idempotent of T. Since $nil(R)$ is a nil ideal of R (indeed, we know from Lemma 2.5 that $(nil(R))^2 = (0)$), it follows from [11, Proposition 1, page 72] that there exists a nontrivial idempotent e of R such that $t = e + nil(R)$. Observe that the mapping $f : R \to Re \times R(1-e)$

defined by $f(x) = (xe, x(1-e))$ is an isomorphism of rings. Let us denote the ring $Re \times R(1-e)$ by S. Since $R \cong S$ as rings, we obtain that $G(S)$ is complete. But we know from Lemma 2.6 that $G(S)$ is not complete. This is a contradiction. Therefore, R is local. □

Let $(R, \mathfrak{m})$ be a local ring such that R is not reduced. In Proposition 2.8, we classify such rings R in order that $G(R)$ be complete.

Proposition 2.8. *Let $(R, \mathfrak{m})$ be a local ring such that R is not reduced. Then the following statements are equivalent:*
(i) $G(R)$ is complete.
(ii) $\mathfrak{m}^2 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$.

Proof. $(i) \Rightarrow (ii)$ Assume that $G(R)$ is complete. Hence, any distinct $x, y \in R$ are adjacent in $G(R)$ and so, $Rx \cap Ry = Rxy$. We know from Lemma 2.3 that $dim R = 0$. Hence, we obtain that $\mathfrak{m}$ is the only prime ideal of R and therefore, it follows from [3, Proposition 1.8] that $nil(R) = \mathfrak{m}$. Since $G(R)$ is complete, we obtain from Lemma 2.5 that $\mathfrak{m}^2 = (0)$. We next verify that $|\frac{R}{\mathfrak{m}}| = 2$. Let $x \in \mathfrak{m}, x \neq 0$. Let $r \in R \backslash \mathfrak{m}$. Since r is a unit in R and $x \neq 0$, it follows that $rx \neq 0$. We claim that $x = rx$. Suppose that $x \neq rx$. Then $Rrx = Rx \cap Rrx = Rrx^2$. Since $x^2 = 0$, we obtain that $rx = 0$. This is impossible. Therefore, $x = rx$. This implies that $x(1-r) = 0$. As $x \neq 0$, we get that $1 - r \in \mathfrak{m}$. This shows that $\frac{R}{\mathfrak{m}} = \{0 + \mathfrak{m}, 1 + \mathfrak{m}\}$ and so, $|\frac{R}{\mathfrak{m}}| = 2$.
$(ii) \Rightarrow (i)$ Let $x, y \in R$ be such that $x \neq y$. We want to show that $Rx \cap Ry = Rxy$. If either $x = 0$ or $y = 0$, then it is clear that $(0) = Rx \cap Ry = Rxy$. Hence, we can assume that $x \neq 0$ and $y \neq 0$. If at least one between x and y is a unit in R, then it is clear that $Rx \cap Ry = Rxy$. Therefore, we can assume that $x, y \in \mathfrak{m}$. As $\mathfrak{m}^2 = (0)$, we get that $xy = 0$ and so, $Rxy = (0)$. Let $z \in Rx \cap Ry$. Then $z = rx = sy$ for some $r, s \in R$. We claim that $z = 0$. Suppose that $z \neq 0$. As $\mathfrak{m}^2 = (0)$, it follows that r, s are units in R. Since $|\frac{R}{\mathfrak{m}}| = 2$, we obtain that $r = 1 + m_1$ and $s = 1 + m_2$ for some $m_1, m_2 \in \mathfrak{m}$. From $\mathfrak{m}^2 = (0)$, we obtain that $rx = (1 + m_1)x = x$ and $sy = (1 + m_2)y = y$. Thus $z = rx = sy$ implies that $x = y$. This is a contradiction and so, $z = 0$. Therefore, $(0) = Rx \cap Ry = Rxy$. This proves that $G(R)$ is complete. □

Theorem 2.9. *Let R be a ring. The following statements are equivalent:*
(i) $G(R)$ is complete.
(ii) Either R is von Neumann regular or $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$.

Proof. $(i) \Rightarrow (ii)$ Assume that $G(R)$ is complete. If R is reduced, then we know from $(i) \Rightarrow (ii)$ of Proposition 2.4 that R is von Neumann regular. Suppose that R is not reduced. Then it follows from Lemma 2.7 and $(i) \Rightarrow (ii)$ of Proposition 2.8 that $(R, \mathfrak{m})$ is local which satisfies $\mathfrak{m}^2 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$. Since R is not reduced, it is clear that $\mathfrak{m} \neq (0)$.
$(ii) \Rightarrow (i)$ If R is von Neumann regular, then it follows from $(ii) \Rightarrow (i)$ of Proposition 2.4 that $G(R)$ is complete. Suppose that $(R, \mathfrak{m})$ is local with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$. Then we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.8 that $G(R)$ is complete. □

Let R be a ring. We know from Lemma 2.1 that $\Gamma(R)$ is a spanning subgraph of $G(R)$. Thus if $\omega(G(R)) < \infty$, then it follows that $\omega(\Gamma(R)) < \infty$. Therefore, we obtain from [14, Proposition 2.2] that R is a finite ring. We are interested to know the status of the moreover part of **??**Theorem 2.3]14 in the case of $G(R)$. In Proposition 2.10, we classify reduced rings R such that $\chi(G(R)) < \infty$.

Proposition 2.10. *Let R be a reduced ring. Then the following statements are equivalent:*
(i) $\chi(G(R)) < \infty$.
(ii) $\omega(G(R)) < \infty$.
(iii) There exist $n \in \mathbb{N}$ and finite fields $F_1, \ldots, F_n$ such that $R \cong F_1 \times \cdots \times F_n$ as rings.

Moreover, if either (i), (ii) or (iii) holds (and hence, all the three hold), then $\chi(G(R)) = \omega(G(R)) = |R|$.

Proof. $(i) \Rightarrow (ii)$ This is clear.
$(ii) \Rightarrow (iii)$ It is observed in Lemma 2.1 that $\Gamma(R)$ is a spanning subgraph of $G(R)$. Therefore, we obtain that $\omega(\Gamma(R)) < \infty$. Hence, we obtain from [14, Proposition 2.2] that R is finite. (This part of the proof does not make use of the fact that R is reduced). Since we are assuming that R is reduced, it follows that there exist $n \in \mathbb{N}$ and finite fields $F_1, \ldots, F_n$ such that $R \cong F_1 \times \cdots \times F_n$ as rings.
$(iii) \Rightarrow (i)$ If (iii) holds, then R is a finite ring and so, $\chi(G(R)) < \infty$.

We next verify the moreover part of this Proposition. Assume that (iii) holds. Then R is a finite von Neumann regular ring. Hence, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.4 that $G(R)$ is complete. Thus $G(R)$ is a complete graph on $|R|$ vertices. Therefore, $\chi(G(R)) = \omega(G(R)) = |R|$. □

Proposition 2.10 gives us the status of [14, Theorem 2.3] for $G(R)$ in the case of reduced rings R. We are interested in determining the status of [14, Theorem 2.3] for $G(R)$ in the case of rings R such that R is not reduced.

Remark 2.11. Let R be a ring such that R is not reduced. If $\omega(G(R)) < \infty$, then it is already noted in the proof of $(ii) \Rightarrow (iii)$ of Proposition 2.10 that R is finite. Let $\{\mathfrak{m}_i | i \in \{1, \ldots, n\}\}$ denote the set of all maximal ideals of R. Observe that $nil(R) = \cap_{i=1}^{n} \mathfrak{m}_i$ is nilpotent. Let $t \in \mathbb{N}$ be least with the property that $(\cap_{i=1}^{n} \mathfrak{m}_i)^t = (0)$. Since R is not reduced, it is clear that $t \geq 2$. As $\mathfrak{m}_i^t + \mathfrak{m}_j^t = R$ for all distinct $i, j \in \{1, \ldots, n\}$, it follows from [3, Proposition 1.10 (i)] that $\cap_{i=1}^{n} \mathfrak{m}_i^t = \prod_{i=1}^{n} \mathfrak{m}_i^t$ and so, $\cap_{i=1}^{n} \mathfrak{m}_i^t = (0)$. Moreover, we obtain from the Chinese remainder theorem [3, Proposition 1.10 (ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1^t} \times \cdots \times \frac{R}{\mathfrak{m}_n^t}$ given by $f(r) = (r + \mathfrak{m}_1^t, \ldots, r + \mathfrak{m}_n^t)$ is an isomorphism of rings. Let $i \in \{1, \ldots, n\}$ and let us denote the ring $\frac{R}{\mathfrak{m}_i^t}$ by R_i. Note that R_i is a finite local ring with $\frac{\mathfrak{m}_i}{\mathfrak{m}_i^t}$ as its unique maximal ideal and $R \cong R_1 \times \cdots \times R_n$ as rings.

Let $(R, \mathfrak{m})$ be a finite local ring which is not reduced. In Proposition 2.12, we provide a sufficient condition on $\mathfrak{m}$ in order that $\chi(G(R)) = \omega(G(R))$.

Proposition 2.12. *Let $(R, \mathfrak{m})$ be a finite local ring which is not reduced. If $\mathfrak{m}^2 = (0)$, then $\chi(G(R)) = \omega(G(R)) = |U(R)| + k + 1$, where k is the number of minimal ideals of R.*

Proof. Let $\{u_i | i \in \{1, \ldots, m\}\}$ denote the set of all units of R and let $\{Rx_j | j \in \{1, \ldots, k\}\}$ denote the set of all minimal ideals of R. We claim that the subgraph of $G(R)$ induced on $\{0, u_1, \ldots, u_m, x_1, \ldots, x_k\}$ is a clique. It is clear that 0 is adjacent to all the vertices v of $G(R)$ such that $v \neq 0$ and if $u \in U(R)$, then u is adjacent to all the vertices w of $G(R)$ such that $w \neq u$. Let $j_1, j_2 \in \{1, \ldots, k\}$ be such that $j_1 \neq j_2$. Since Rx_{j_1} and Rx_{j_2} are distinct minimal ideals of R, it follows that $Rx_{j_1} \cap Rx_{j_2} = (0)$ and so, $Rx_{j_1} \cap Rx_{j_2} = (0) = Rx_{j_1}x_{j_2}$. This shows that x_{j_1} and x_{j_2} are adjacent in $G(R)$. From the above given arguments, we obtain that the subgraph of $G(R)$ induced on $\{0, u_1 \ldots, u_m, x_1 \ldots, x_k\}$ is a clique. Therefore, $\omega(G(R)) \geq m + k + 1$. We next verify that the vertices of $G(R)$ can be properly colored using a set of $m + k + 1$ distinct colors. Let $\{c_0, c_1, \ldots, c_m, c_{m+1}, \ldots, c_{m+k}\}$ be a set consisting of $m + k + 1$ distinct colors. Let us assign the color c_0 to 0, the color c_i to u_i for each $i \in \{1, \ldots, m\}$, and the color c_{m+j} to x_j for each $j \in \{1, \ldots, k\}$. Let $x \in R \backslash \{0\}$ be such that x is not a unit of R. We are assuming that $\mathfrak{m}^2 = (0)$. Hence, Rx is a minimal ideal of R and so,

$Rx = Rx_j$ for a unique $j \in \{1, \ldots, k\}$. Let us assign the color c_{m+j} to x. We claim that the above assignment of colors to the vertices of $G(R)$ is a proper vertex coloring of $G(R)$. Let $x, y \in \mathfrak{m}\backslash\{0\}$ be such that x and y are adjacent in $G(R)$. Since $\mathfrak{m}^2 = (0)$, it follows that $Rx \cap Ry = Rxy = (0)$. Therefore, Rx and Ry are distinct minimal ideals of R. Let $j_1, j_2 \in \{1, \ldots, k\}$ be such that $Rx = Rx_{j_1}$ and $Ry = Rx_{j_2}$. Observe that $j_1 \neq j_2$, x is assigned the color c_{m+j_1}, and y is assigned the color c_{m+j_2}. This shows that the above assignment of colors is a proper vertex coloring of $G(R)$. Hence, we get that $\chi(G(R)) \leq m + k + 1$. Therefore, $m + k + 1 \leq \omega(G(R)) \leq \chi(G(R)) \leq m + k + 1$ and so, $\chi(G(R)) = \omega(G(R)) = m + k + 1$. □

Remark 2.13. Let $G = (V, E)$ be a finite graph. Assume that G is simple. Let V_1, V_2 be nonempty subsets of V such that $V_1 \cap V_2 = \emptyset$ and $V = V_1 \cup V_2$. Let G_i be the subgraph of G induced on V_i for each $i \in \{1, 2\}$. Suppose that for each $a \in V_1$ and $b \in V_2$, a, b are adjacent in G. That is, $G = G_1 \vee G_2$. Then it is not hard to verify that $\omega(G) = \omega(G_1) + \omega(G_2)$ and $\chi(G) = \chi(G_1) + \chi(G_2)$. Let R be a finite ring. Note that the subgraph of $G(R)$ induced on $U(R)$ is a clique and if $u \in U(R), r \in NU(R)$, then u and r are adjacent in $G(R)$. Let us denote the subgraph of $G(R)$ induced on $U(R)$ by $G_1(R)$ and the subgraph of $G(R)$ induced on $NU(R)$ by $G_2(R)$. Observe that $\chi(G_1(R)) = \omega(G_1(R)) = |U(R)|$. Hence, to determine $\omega(G(R))$ (respectively, $\chi(G(R))$), it is enough to determine $\omega(G_2(R))$ (respectively, $\chi(G_2(R))$). In Example 2.14, we mention an example of a finite local ring $(R, \mathfrak{m})$ such that $\omega(G(R)) < \chi(G(R))$ and this illustrates that the hypothesis $\mathfrak{m}^2 = (0)$ cannot be omitted in Proposition 2.12. The example mentioned in Example 2.14 is an interesting and inspiring example due to Anderson and Nasser [1] which answered a conjecture of I. Beck [5] in the negative.

Example 2.14. Let $T = \mathbb{Z}_4[X, Y, Z]$ be the polynomial ring in three variables X, Y, Z over $\mathbb{Z}_4$. Let I be the ideal of T generated by $\{X^2 - 2, Y^2 - 2, Z^2, XY, YZ - 2, XZ, 2X, 2Y, 2Z\}$. Let $R = \frac{T}{I}$. Then $\omega(G(R)) = 20 < \chi(G(R)) = 21$.

Proof. It was already noted in [1] that R is a finite local ring with $\mathfrak{m} = \frac{TX+TY+TZ}{I}$ as its unique maximal ideal, $|U(R)| = 16$, and $|R| = 32$. Moreover, it was noted in [1] that $\mathfrak{m}^2 = R(2 + I) = \{0 + I, 2 + I\}$ and $\mathfrak{m}^3 = (0 + I)$. It is convenient to denote $X + I$ by x, $Y + I$ by y, and $Z + I$ by z. Note that $NU(R) = \mathfrak{m} = \{0+I, x, y, z, 2+I, x+y, y+z, z+x, x+y+z, x+2, y+2, z+2, x+y+2, y+z+2, z+x+2, x+y+z+2\}$. It was already observed in the proof of [7, Proposition 2.1] that the set of all nonzero proper ideals of R equals $\{R(2 + I), Rx, Ry, Rz, R(x +$

$y), R(y+z), R(z+x), R(x+y+z), Rx+Ry, Ry+Rz, Rz+Rx, Rx+R(y+z), Ry+R(z+x), Rz+R(x+y), Rx+Ry+Rz\}$. Observe that $\mathfrak{m}^2 = R(2+I)$ is the unique minimal ideal of R and hence, for any $m \in \mathfrak{m}\backslash\mathfrak{m}^2$, $Rm = R(m+2+I)$. In view of Remark 2.13, to determine $\omega(G(R))$ (respectively, $\chi(G(R))$), it is enough to determine $\omega(G_2(R))$ (respectively, $\chi(G_2(R))$), where $G_2(R)$ is the subgraph of $G(R)$ induced on $\mathfrak{m}$. As $\mathfrak{m}^3 = (0+I)$, it follows that $2+I$ is not adjacent to m in $G(R)$ for any $m \in \mathfrak{m}\backslash\mathfrak{m}^2$. It is clear from the above given arguments that to determine $\omega(G_2(R))$ (respectively, $\chi(G_2(R))$), we need to determine $\omega(H)$ (respectively, $\chi(H)$), where H is the subgraph of $G(R)$ induced on $\{0+I, x, y, z, x+y, y+z, z+x, x+y+z\}$. Note that $R(2+I) \subseteq Rm_1 \cap Rm_2$ for any $m_1, m_2 \in \mathfrak{m}\backslash\mathfrak{m}^2$. Since $xy = xz = x(y+z) = 0+I$, it follows that x is not adjacent to any member of $\{y, z, y+z\}$ in $G(R)$. It is clear that if $m_1, m_2 \in \mathfrak{m}\backslash\mathfrak{m}^2$ are such that $Rm_1 \neq Rm_2$, then $Rm_1 \cap Rm_2 \subseteq \mathfrak{m}^2$. As $x(x+y) = x(x+z) = x(x+y+z) = 2+I$, we obtain that x is adjacent to each member of $\{x+y, x+z, x+y+z\}$ in $G(R)$. Observe that $(x+y)(x+z) = (x+z)(x+y+z) = 0+I$ and so, $x+z$ is not adjacent to any of the member of $\{x+y, x+y+z\}$ in $G(R)$. The clique of H of largest size containing x is the subgraph of H induced on $\{0+I, x, x+y, x+y+z\}$. Similarly, it can be verified that the clique of H of largest size containing y is the subgraph of H induced on $\{0+I, y, z, x+y\}$. Note that the clique of H of largest size containing z is the subgraph of H induced on $\{0+I, z, y, x+y\}$ and the subgraph of H induced on $\{0+I, z, y+z, x+y+z\}$. It is easy to verify that the clique of H of largest size containing $x+y$ is the subgraph of H induced on $\{0+I, x+y, x, x+y+z\}$; the subgraph of H induced on $\{0+I, x+y, z, x+y+z\}$, and the subgraph of H induced on $\{0+I, x+y, y, z\}$. Observe that the clique of H of largest size containing $y+z$ is the subgraph of H induced on $\{0+I, y+z, z, x+y+z\}$; the clique of H of largest size containing $x+z$ is the subgraph of H induced on $\{0+I, x+z, x\}$ and the subgraph of H induced on $\{0+I, x+z, y\}$; and the clique of H of largest size containing $x+y+z$ is the subgraph of H induced on $\{0+I, x+y+z, x, x+y\}$; the subgraph of H induced on $\{0+I, x+y+z, z, x+y\}$, and the subgraph of H induced on $\{0+I, x+y+z, z, y+z\}$. From the above discussion, it is now clear that $\omega(H) = 4$. Hence, $\chi(H) \geq 4$. We claim that $\chi(H) > 4$. Suppose that $\chi(H) = 4$. This implies that the vertex set of H can be properly colored using a set of four distinct colors. Note that $V(H) = \{0+I, x, y, z, x+y, y+z, x+z, x+y+z\}$. Let $\{c_1, c_2, c_3, c_4\}$ be a set of four distinct colors. As $\chi(H) = 4$ by assumption, it follows that there exist subsets V_1, V_2, V_3, V_4 of $V(H)$ such that $V(H) = \cup_{i=1}^{4} V_i$, where $V_i = \{v \in V(H) | v \text{ receives the color } c_i\}$ for each $i \in \{1, 2, 3, 4\}$.

Since the subgraph of H induced on $\{0 + I, x, x + y, x + y + z\}$ is a clique, we obtain that no two of $\{0 + I, x, x + y, x + y + z\}$ can be in the same V_i for any $i \in \{1, 2, 3, 4\}$. Without loss of generality, we can assume that $0+I \in V_1, x \in V_2, x+y \in V_3$, and $x+y+z \in V_4$. It is now clear that $V_1 = \{0+I\}$. Since z is adjacent to both $x+y$ and $x+y+z$ in H, z must be in V_2. As x and $x + z$ are adjacent in H, we obtain that $x+z$ cannot be in $V_1 \cup V_2$. Hence, either $x+z \in V_3$ or $x+z \in V_4$. Suppose that $x + z \in V_3$. As $y + z$ is adjacent to each member of $\{z, x + z, x + y + z\}$ in H, we get that $y + z$ cannot be in $\cup_{i=1}^{4} V_i$. This is a contradiction. Suppose that $x + z \in V_4$. Since y is adjacent to each member of $\{z, x + y, x + z\}$, it follows that y cannot be in $\cup_{i=1}^{4} V_i$. This is a contradiction. Therefore, $\chi(H) \geq 5$. We now verify that the vertices of H can be properly colored using a set of five distinct colors. Let $\{c_i | i \in \{1, 2, 3, 4, 5\}\}$ be a set consisting of five distinct colors. Let us assign the color c_1 to $0 + I$, the color c_2 to x, the color c_3 to $x + y$, the color c_4 to $x + y + z$, the color c_2 to z, the color c_4 to y, the color c_3 to $x + z$, and the color c_5 to $y + z$. The above assignment of colors is indeed a proper vertex coloring of H and so, $\chi(H) \leq 5$. Therefore, $\chi(H) = 5$. Since $|U(R)| = 16$ and the subgraph $G_1(R)$ of $G(R)$ induced on $U(R)$ is complete, it follows as is remarked in Remark 2.13 that $\omega(G(R)) = \omega(G_1(R)) + \omega(G_2(R)) = |U(R)| + \omega(H) = 16 + 4 = 20$ and $\chi(G(R)) = \chi(G_1(R)) + \chi(G_2(R)) = |U(R)| + \chi(H) = 16 + 5 = 21$. □

Let $(R, \mathfrak{m})$ be a finite local ring which is not reduced. In Proposition 2.15, we provide another sufficient condition in order that $\chi(G(R)) = \omega(G(R))$.

Proposition 2.15. *Let $(R, \mathfrak{m})$ be a finite SPIR which is not reduced. Then $\chi(G(R)) = \omega(G(R)) = |U(R)| + 2$.*

Proof. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Let $m \in \mathfrak{m}$ be such that $\mathfrak{m} = Rm$. Note that $\{Rm^i | i \in \{1, \ldots, n - 1\}\}$ is the set of all nonzero proper ideals of R. We claim that $G_2(R)$, the subgraph of $G(R)$ induced on $NU(R) = \mathfrak{m}$ is a star graph. It is clear that 0 is adjacent to all the nonzero elements of R in $G(R)$ and hence, 0 is adjacent to all the elements of $\mathfrak{m}\backslash\{0\}$ in $G_2(R)$. Let $x, y \in \mathfrak{m}\backslash\{0\}$ be such that $x \neq y$. Note that there exist $i, j \in \{1, \ldots, n - 1\}$ such that $Rx = \mathfrak{m}^i$ and $Ry = \mathfrak{m}^j$. Since the ideals of R are comparable under the inclusion relation, it follows that $Rx \cap Ry$ is either $\mathfrak{m}^i$ or $\mathfrak{m}^j$. However, $Rxy = \mathfrak{m}^{i+j} \notin \{\mathfrak{m}^i, \mathfrak{m}^j\}$. Hence, x and y are not adjacent in $G(R)$. This shows that $G_2(R)$ is a star graph. Therefore, $\chi(G_2(R)) = \omega(G_2(R)) = 2$. Now, it follows as is remarked in Remark 2.13 that $\chi(G(R)) = \chi(G_1(R)) + \chi(G_2(R))$, where $G_1(R)$ is the subgraph of

$G(R)$ induced on $U(R)$. Therefore, $\chi(G(R)) = |U(R)| + 2$. Note that $\omega(G(R)) = \omega(G_1(R)) + \omega(G_2(R)) = |U(R)| + 2$. This proves that $\chi(G(R)) = \omega(G(R)) = |U(R)| + 2$. □

Proposition 2.16. *Let $R = F \times S$, where F is a finite field and $(S, \mathfrak{m})$ is a finite SPIR which is not a field. Then $\chi(G(R)) = \omega(G(R)) = |U(R)| + |F\backslash\{0\}| + |U(S)| + 2$.*

Proof. We know from Remark 2.13 that $\chi(G(R)) = \chi(G_1(R)) + \chi(G_2(R))$ and $\omega(G(R)) = \omega(G_1(R)) + \omega(G_2(R))$, where $G_1(R)$ is the subgraph of $G(R)$ induced on $U(R)$ and $G_2(R)$ is the subgraph of $G(R)$ induced on $NU(R)$. Since $G_1(R)$ is a complete graph on $|U(R)|$ vertices, we get that $\chi(G_1(R)) = \omega(G_1(R)) = |U(R)|$. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Let us next determine $\chi(G_2(R))$ and $\omega(G_2(R))$. Note that R is a principal ideal ring. The set of all proper ideals of R equals $\{(0) \times (0), (0) \times \mathfrak{m}^i, (0) \times S, F \times (0), F \times \mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$. Let $x \in \mathfrak{m}$ be such that $\mathfrak{m} = Sx$. It is clear that the subgraph of $G_2(R)$ induced on $A = \{(0,0), (\alpha, 0), (0, u), (0, x) | \alpha \in F\backslash\{0\}, u \in U(S)\}$ is a clique. Therefore, $\omega(G_2(R)) \geq |F\backslash\{0\}| + |U(S)| + 2$. Let $F\backslash\{0\} = \{\alpha_i | i \in \{1, \ldots, l\}\}$ and let $U(S) = \{u_j | j \in \{1, \ldots, m\}\}$ so that $U(R) = \{(\alpha_i, u_j) | i \in \{1, \ldots, l\}, j \in \{1, \ldots, m\}\}$. We next verify that the vertices of $G_2(R)$ can be properly colored using a set of $l+m+2$ distinct colors. Let $\{d_0, d_i, d_{l+j}, d_{l+m+1} | i \in \{1, \ldots, l\}, j \in \{1, \ldots, m\}\}$ be a set of $l+m+2$ distinct colors. Let us assign the color d_0 to $(0,0)$, the color d_i to $(\alpha_i, 0)$ for each $i \in \{1, \ldots, l\}$, the color d_{l+j} to $(0, u_j)$ for each $j \in \{1, \ldots, m\}$, and the color d_{l+m+1} to $(0, x)$. Let $r \in R$ be such that $r \in NU(R)$ with $r \notin A$. Observe that $Rr \in \{(0) \times \mathfrak{m}^k, F \times \mathfrak{m}^k | k \in \{1, \ldots, n-1\}\}$. Either $Rr = (0) \times \mathfrak{m}^k$ or $Rr = F \times \mathfrak{m}^t$ for some $k, t \in \{1, \ldots, n-1\}$. Note that r and $(0, x)$ are not adjacent in $G_2(R)$. Let us assign the color d_{l+m+1} to r. Let $r_1, r_2 \in NU(R)\backslash A$ with $r_1 \neq r_2$. Then it is clear that $\{r_1, r_2, (0, x)\}$ is an independent set of $G_2(R)$. Hence, the above assignment of $l+m+2$ colors to the vertices of $G_2(R)$ is a proper vertex coloring of $G_2(R)$. Therefore, $\chi(G_2(R)) \leq l+m+2$. Hence, $l+m+2 \leq \omega(G_2(R)) \leq \chi(G_2(R)) \leq l+m+2$ and so, $\chi(G_2(R)) = \omega(G_2(R)) = l+m+2$. Therefore, we obtain that $\chi(G(R)) = \chi(G_1(R)) + \chi(G_2(R)) = \omega(G_1(R)) + \omega(G_2(R)) = \omega(G(R)) = lm + l + m + 2 = |U(R)| + |F\backslash\{0\}| + |U(S)| + 2$. □

3. On the planarity of $G(R)$

The aim of this section is to classify rings R such that $\Gamma(R)$ (respectively, $G(R)$) is planar. Let R be a ring. We know from Lemma 2.1 that $\Gamma(R)$ is a spanning subgraph of $G(R)$. Thus if $G(R)$ is planar, then $\Gamma(R)$ is planar. We first classify rings R such that $\Gamma(R)$ is planar.

Lemma 3.1. *Let R be a ring. If $\Gamma(R)$ satisfies (Ku_1), then the following hold.*

(i) *R is a finite ring and $|U(R)| \leq 3$.*

(ii) *$|Max(R)| \leq 3$ and if $|Max(R)| \geq 2$, then $|U(R)| \leq 2$.*

Proof. We are assuming that $\Gamma(R)$ satisfies (Ku_1). Hence, $\omega(\Gamma(R)) \leq 4$. Now, it follows from [14, Proposition 2.2] that R is finite. Suppose that $|U(R)| \geq 4$. Let $\{1, u, v, w\} \subseteq U(R)$. Observe that the subgraph of $\Gamma(R)$ induced on $\{0, 1, u, v, w\}$ is a clique on five vertices. This is impossible, since $\omega(\Gamma(R)) \leq 4$. Therefore, we get that $|U(R)| \leq 3$.

(ii) We know from [14, Theorem 2.3] that $\chi(\Gamma(R)) = \omega(\Gamma(R)) = t + l$, where t is the number of maximal ideals of R and l is the number of units of R. It follows from $\omega(\Gamma(R)) \leq 4$ and $1 \leq |U(R)|$ that $t + 1 \leq t + l \leq 4$. Therefore, $t = |Max(R)| \leq 3$. Suppose that $t \geq 2$. Then $2 + l \leq t + l \leq 4$. Hence, we obtain that $|U(R)| = l \leq 2$. □

Theorem 3.2. *Let R be a ring such that $|Max(R)| = 3$. The following statements are equivalent:*

(i) *$\Gamma(R)$ is planar.*

(ii) *$\Gamma(R)$ satisfies (Ku_1^*) and (Ku_2^*).*

(iii) *$\Gamma(R)$ satisfies (Ku_1).*

(iv) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.*

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [8, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ We know from Lemma 3.1 (i) that R is a finite ring. As $|Max(R)| = 3$, it follows that $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a finite local ring for each $i \in \{1, 2, 3\}$. Let us denote the ring $R_1 \times R_2 \times R_3$ by T. Since $R \cong T$ as rings, we obtain that $\Gamma(T)$ satisfies (Ku_1). We assert that $|U(R_i)| = 1$ for each $i \in \{1, 2, 3\}$. Suppose that $|U(R_i)| > 1$ for some $i \in \{1, 2, 3\}$. Without loss of generality, we can assume that $|U(R_1)| > 1$. Let $u \in U(R_1) \backslash \{1\}$. Observe that the subgraph of $\Gamma(T)$ induced on $\{(1, 1, 1), (u, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a clique on five vertices. This is in contradiction to the fact that $\omega(\Gamma(T)) \leq 4$. Therefore, $|U(R_i)| = 1$ for each $i \in \{1, 2, 3\}$. Let $i \in \{1, 2, 3\}$. Let $x \in \mathfrak{m}_i$. As $1 + x \in U(R_i)$, it follows that $1 + x = 1$ and so, $x = 0$. This proves that $R_i = \{0, 1\}$. Therefore, we obtain that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.

$(iv) \Rightarrow (i)$ Let us denote the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by T. Note that $|T| = 8$ and $T = \{v_1 = (1, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 1, 1), v_4 = (1, 1, 1), v_5 = (0, 0, 1), v_6 = (0, 1, 0), v_7 = (1, 0, 0), v_8 = (0, 0, 0)\}$. Observe that $\Gamma(T)$ is the union of the cycles Γ_i for $i \in \mathbb{N}$ such that $i \leq 7$ and the edge

$v_4 - v_8$, where the cycles Γ_i are given by $\Gamma_1 : v_1 - v_2 - v_3 - v_1$ $\Gamma_2 : v_1 - v_2 - v_4 - v_1$, $\Gamma_3 : v_2 - v_3 - v_4 - v_2$, $\Gamma_4 : v_1 - v_4 - v_3 - v_1$, $\Gamma_5 : v_1 - v_5 - v_4 - v_1$, $\Gamma_6 : v_2 - v_4 - v_6 - v_2$, and $\Gamma_7 : v_3 - v_4 - v_7 - v_3$. The cycle Γ_1 can be represented by means of a triangle Δ_1 whose vertices are v_1, v_2, and v_3. The vertex v_4 can be plotted inside Δ_1 representing Γ_1 and on joining the vertex v_i and v_4 by a line segment for each $i \in \{1, 2, 3\}$, we obtain triangles $\Delta_2, \Delta_3, \Delta_4$, where the vertices of Δ_2 are v_1, v_2, and v_4 and it represents Γ_2; the vertices of Δ_3 are v_2, v_3, and v_4 and it represents Γ_3; the vertices of Δ_4 are v_1, v_4, and v_3 and it represents Γ_4. The vertex v_5 can be plotted inside Δ_2 and on joining v_1 and v_5 (respectively, v_4 and v_5) by a line segment, we obtain a triangle Δ_5 whose vertices are v_1, v_5, and v_4 and it represents Γ_5. Similarly, the vertex v_6 can be plotted inside Δ_3 and on joining v_2 and v_6 (respectively, v_4 and v_6) by a line segment, we obtain a triangle Δ_6 whose vertices are v_2, v_4, and v_6 and it represents Γ_6. The vertex v_7 can be plotted inside Δ_3 and on joining v_7 and v_3 (respectively, v_4 and v_7) by a line segment, we obtain a triangle Δ_7 whose vertices are v_3, v_4, and v_7 and it represents Γ_7. The vertex v_8 can be plotted inside Δ_4 and the edge $v_4 - v_8$ can be drawn inside Δ_4 in such a way that there are no crossing over of the edges. This shows that $\Gamma(T)$ is planar. As $R \cong T$ as rings, we get that $\Gamma(R)$ is planar. □

Let R b a ring such that $|Max(R)| = 2$. We next try to classify such rings R in order that $\Gamma(R)$ satisfies (Ku_1). We denote the polynomial ring in one variable X over a ring T by $T[X]$.

Proposition 3.3. *Let R be a ring such that $|Max(R)| = 2$. The following statements are equivalent:*

(i) $\Gamma(R)$ satisfies (Ku_1).

(ii) R is isomorphic to one of the rings from the collection $\mathcal{C} = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$.

Proof. $(i) \Rightarrow (ii)$ We know from Lemma 3.1 (i) that R is finite and since $|Max(R)| = 2$, we obtain from Lemma 3.1 (ii) that $|U(R)| \leq 2$. Moreover, we obtain that $R \cong R_1 \times R_2$ as rings, where $(R_i, \mathfrak{m}_i)$ is a finite local ring for each $i \in \{1, 2\}$. Let us denote the ring $R_1 \times R_2$ by T. As $R \cong T$ as rings, we obtain that $|U(T)| \leq 2$. Note that $U(T) = U(R_1) \times U(R_2)$. We consider the following cases.

Case (A) $|U(T)| = 1$

As $|U(T)| = |U(R_1)||U(R_2)|$, we obtain that $|U(R_i)| = 1$ for each $i \in \{1, 2\}$. Since R_i is local for each $i \in \{1, 2\}$, it follows that $R_i = \{0, 1\}$. Hence, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case (B) $|U(T)| = 2$

As $2 = |U(T)| = |U(R_1)||U(R_2)|$, it follows that $|U(R_i)| = 1$ for exactly one $i \in \{1, 2\}$. Without loss of generality, we can assume that $|U(R_1)| = 1$. Then $R_1 = \{0, 1\}$ and $|U(R_2)| = 2$. Either R_2 is a field or R_2 is not a field. If R_2 is a field, then it follows from $R_2 = \{0\} \cup U(R_2)$ that $|R_2| = 3$ and so, $R_2 \cong \mathbb{Z}_3$. Therefore, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ as rings. Suppose that R_2 is not a field. Then $|\mathfrak{m}_2| \geq 2$ and it follows from $|U(R_2)| = 2$ that $|\mathfrak{m}_2| = 2$ and so, $|R_2| = 4$. Therefore, either $R_2 \cong \mathbb{Z}_4$ or $R_2 \cong \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$. Hence, either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$ as rings.

Thus if $\Gamma(R)$ satisfies (Ku_1), then R is isomorphic to one of the rings from the collection $\mathcal{C}$, where $\mathcal{C} = \{\mathbb{Z}_2\times\mathbb{Z}_2, \mathbb{Z}_2\times\mathbb{Z}_3, \mathbb{Z}_2\times\mathbb{Z}_4, \mathbb{Z}_2\times\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$.
$(ii) \Rightarrow (i)$ We are assuming that R is isomorphic to one of the rings from the collection $\mathcal{C}$. For any member $T \in \mathcal{C}$, $|Max(T)| = 2$ and $|U(T)| \in \{1, 2\}$. Therefore, by [14, Theorem 2.3], we get that $\omega(\Gamma(T)) = |Max(T)| + |U(T)| \in \{3, 4\}$. Hence, $\Gamma(T)$ satisfies (Ku_1). As R is isomorphic to one of the rings T from the collection $\mathcal{C}$, we obtain that $\Gamma(R)$ satisfies (Ku_1). □

Lemma 3.4. *Let R_1, R_2 be rings and let $R = R_1 \times R_2$. If R_2 is not a field and if $|U(R_2)| \geq 2$, then $\Gamma(R)$ does not satisfy (Ku_2).*

Proof. Let $x \in R_2\backslash\{0\}$ be such that x is not a unit in R_2. Let $u, v \in U(R_2)$ be such that $u \neq v$. Let $V_1 = \{(0, u), (0, v), (1, u)\}$ and let $V_2 = \{(1, 0), (1, x), (1, v)\}$. Observe that $V_1 \cup V_2 \subseteq V(\Gamma(R))$, $V_1 \cap V_2 = \emptyset$, and the subgraph of $\Gamma(R)$ induced on $V_1 \cup V_2$ contains $K_{3.3}$ as a subgraph. Therefore, we obtain that $\Gamma(R)$ does not satisfy (Ku_2). □

Corollary 3.5. *Let $R \in \mathcal{D} = \{\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Then $\Gamma(R)$ does not satisfy (Ku_2).*

Proof. Note that $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$ are not fields and $|U(\mathbb{Z}_4)| = |U(\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]})| = 2$. Therefore, we obtain from Lemma 3.4 that if $R \in \mathcal{D}$, then $\Gamma(R)$ does not satisfy (Ku_2). □

Lemma 3.6. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\Gamma(R)$ is planar.*

Proof. Since $\Gamma(R)$ is a simple graph on four vertices, it is clear that $\Gamma(R)$ is planar. Indeed, $\Gamma(R)$ is the union of the cycle $\Gamma : (1, 1) - (1, 0) - (0, 1) - (1, 1)$ and the edge $(0, 0) - (1, 1)$. □

Lemma 3.7. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then $\Gamma(R)$ is planar.*

Proof. Note that $V(\Gamma(R)) = \{v_1 = (1, 0), v_2 = (1, 1), v_3 = (0, 1), v_4 = (1, 2), v_5 = (0, 2), v_6 = (0, 0)\}$. It is not hard to show that $\Gamma(R)$ is the union of the cycles Γ_i where $i \in \mathbb{N}$ is such that $i \leq 7$ and the cycles Γ_i are given by $\Gamma_1 : v_1 - v_2 - v_3 - v_1$, $\Gamma_2 : v_1 - v_2 - v_4 - v_1$, $\Gamma_3 : v_2 - v_3 - v_4 - v_2$, $\Gamma_4 : v_1 - v_4 - v_3 - v_1$, $\Gamma_5 : v_1 - v_4 - v_5 - v_1$, $\Gamma_6 : v_2 - v_4 - v_5 - v_2$, and $\Gamma_7 : v_2 - v_4 - v_6 - v_2$. Note that Γ_1 can be represented by means of a triangle Δ_1, whose vertices are v_1, v_2, and v_3. The vertex v_4 can be plotted inside Δ_1 and on joining v_i to v_4 by a line segment for each $i \in \{1, 2, 3\}$, we obtain triangles Δ_2, Δ_3, and Δ_4, where the vertices of Δ_2 are v_1, v_2, and v_4 and it represents Γ_2; v_2, v_3, and v_4 are vertices of Δ_3 and it represents Γ_3; v_1, v_4, and v_3 are vertices of Δ_4 and it represents Γ_4. Now, v_5 can be plotted inside Δ_2 and on joining v_1 and v_5 (respectively, v_4 and v_5) by a line segment, we obtain triangle Δ_5 whose vertices are v_1, v_4, and v_5 and it represents Γ_5; now, on joining v_2 and v_5 by a line segment, we obtain triangle Δ_6 whose vertices are v_2, v_4, and v_5 and it represents Γ_6; v_6 can be plotted inside Δ_3 and on joining v_2 and v_6 (respectively, v_4 and v_6)by a line segment, we obtain triangle Δ_7 whose vertices are v_2, v_4, and v_6 and it represents Γ_7. From the above given arguments, it is clear that $\Gamma(R)$ can be drawn in a plane in such a way that there are no crossing over of the edges. This proves that $\Gamma(R)$ is planar. □

Theorem 3.8. *Let R be a ring such that $|Max(R)| = 2$. The following statements are equivalent:*

(i) $\Gamma(R)$ is planar.

(ii) $\Gamma(R)$ satisfies both (Ku_1^*) and (Ku_2^*).

(iii) $\Gamma(R)$ satisfies both (Ku_1) and (Ku_2).

(iv) R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3\}$.

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [8, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ Assume that $\Gamma(R)$ satisfies both (Ku_1) and (Ku_2). As the number of maximal ideals of R is exactly two and $\Gamma(R)$ satisfies (Ku_1), we obtain from $(i) \Rightarrow (ii)$ of Proposition 3.3 that R is isomorphic to one of the rings from the collection $\mathcal{C} = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Let $\mathcal{D} = \{\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. We know from Corollary 3.5 that if $T \in \mathcal{D}$, then $\Gamma(T)$ does not satisfy (Ku_2). Thus if $\Gamma(R)$ satisfies both (Ku_1) and (Ku_2), then R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3\}$.

$(iv) \Rightarrow (i)$ We know from Lemma 3.6 (respectively, from Lemma 3.7) that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ (respectively, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$) is planar. Thus if (iv) holds, then $\Gamma(R)$ is planar. □

Let $(R, \mathfrak{m})$ be a local ring. In Proposition 3.9, we try to classify such rings R in order that $\Gamma(R)$ satisfies (Ku_1).

Proposition 3.9. *Let $(R, \mathfrak{m})$ be a local ring. The following statements are equivalent:*

(i) $\Gamma(R)$ satisfies (Ku_1).

(ii) R is isomorphic to one of the rings from the collection $\mathcal{R}$, where $\mathcal{R} = \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\Gamma(R)$ satisfies (Ku_1). We know from Lemma 3.1 (i) that $|U(R)| \leq 3$. We consider the following cases.

Case(A) $|U(R)| = 1$

In such a case, it is already noted in the proof of $(i) \Rightarrow (ii)$ of Proposition 3.3 (See **Case**(A)) that $R = \{0, 1\}$ and so, $R \cong \mathbb{Z}_2$ as rings.

Case(B) $|U(R)| = 2$

It is already observed in the proof of $(i) \Rightarrow (ii)$ of Proposition **??** (See **Case**(B)) that R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$.

Case(C) $|U(R)| = 3$

In this case, we first verify that R is a field. Suppose that R is not a field. Since we are assuming that $|U(R)| = 3$, it follows that $2 \leq |\mathfrak{m}| \leq 3$. If $|\mathfrak{m}| = 2$, then $|R| = |\mathfrak{m}| + |U(R)| = 5$. Hence, $R \cong \mathbb{Z}_5$ as rings. This contradicts the assumption that R is not a field. If $|\mathfrak{m}| = 3$, then $|R| = |\mathfrak{m}| + |U(R)| = 6$. This is impossible, since the number of elements in any finite local ring is a power of a prime number. Therefore, R is a field. Hence, $|R| = |\mathfrak{m}| + |U(R)| = 4$ and so, $R \cong \mathbb{F}_4$ as rings.

Thus if $\Gamma(R)$ satisfies (Ku_1), then R is isomorphic to one of the rings from the collection $\mathcal{R} = \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$.

$(iv) \Rightarrow (i)$ We are assuming that R is isomorphic to one of the rings from the collection $\mathcal{R} = \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$. Note that if $T \in \mathcal{R}$, then $|T| \leq 4$ and so, $\omega(\Gamma(T)) \leq 4$. Therefore, $\Gamma(T)$ satisfies (Ku_1) and so, $\Gamma(R)$ satisfies (Ku_1). □

In Theorem 3.10, we classify local rings $(R, \mathfrak{m})$ such that $\Gamma(R)$ is planar.

Theorem 3.10. *Let $(R, \mathfrak{m})$ be a local ring. The following statements are equivalent:*

(i) $\Gamma(R)$ is planar.

(ii) $\Gamma(R)$ *satisfies* (Ku_1^*) *and* (Ku_2^*).
(iii) $\Gamma(R)$ *satisfies* (Ku_1).
(iv) *R is isomorphic to one of the rings from the collection* $\mathcal{R}$*, where* $\mathcal{R}$ *is as in statement* (ii) *of Proposition 3.9S.*

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [8, Theorem 5.9].
$(ii) \Rightarrow (iii)$ This is clear.
$(iii) \Rightarrow (iv)$ This follows from $(i) \Rightarrow (ii)$ of Proposition 3.9.
$(iv) \Rightarrow (i)$ If T is any member of $\mathcal{R}$, then $|T| \leq 4$. Since any simple graph on at most four vertices is planar, we obtain that $\Gamma(T)$ is planar. As R is isomorphic to one of the rings from the collection $\mathcal{R}$, we obtain that $\Gamma(R)$ is planar. □

In Theorem 3.11, we classify rings R such that $G(R)$ is planar.

Theorem 3.11. *Let R be a ring. The following statements are equivalent:*
(i) $G(R)$ *is planar.*
(ii) $G(R)$ *satisfies both* (Ku_1^*) *and* (Ku_2^*).
(iii) $G(R)$ *satisfies* (Ku_1).
(iv) *R is isomorphic to one of the rings from the collection* $\mathcal{E}$*, where* $\mathcal{E} = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$.

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [8, Theorem 5.9].
$(ii) \Rightarrow (iii)$ This is clear.
$(iii) \Rightarrow (iv)$ We are assuming that $G(R)$ satisfies (Ku_1). We know from Lemma 2.1 that $\Gamma(R)$ is a spanning subgraph of $G(R)$. Hence, $\Gamma(R)$ satisfies (Ku_1). Therefore, we obtain from Lemma **??** (ii) that $|Max(R)| \leq 3$. Suppose that $|Max(R)| = 3$. It follows from $(iii) \Rightarrow (iv)$ of Theorem 3.2 that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. Hence, we obtain that R is a finite Boolean ring. We know from $(ii) \Rightarrow (i)$ of Proposition 2.4 that $G(R)$ is complete. Therefore, $\omega(G(R)) = |R| = 8$. Thus if $G(R)$ satisfies (Ku_1), then $|Max(R)| \leq 2$. Suppose that $|Max(R)| = 2$. Since $\Gamma(R)$ satisfies (Ku_1), we obtain from $(i) \Rightarrow (ii)$ of Proposition 3.3 that R is isomorphic to one of the rings from the collection $\mathcal{C} = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. We know from the moreover part of Proposition 2.10 that $\omega(G(\mathbb{Z}_2 \times \mathbb{Z}_3)) = |\mathbb{Z}_2 \times \mathbb{Z}_3| = 6$, and so, $G(\mathbb{Z}_2 \times \mathbb{Z}_3)$ does not satisfy (Ku_1). Note that $|U(\mathbb{Z}_2)| = 1$, $|U(\mathbb{Z}_4)| = |U(\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]})| = 2$, and so, we obtain from Proposition 2.16 that $\omega(G(\mathbb{Z}_2 \times \mathbb{Z}_4)) = \omega(G(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]})) = 2 + 1 + 2 + 2 = 7$. Hence,

$G(\mathbb{Z}_2 \times \mathbb{Z}_4)$ (respectively, $G(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]})$) does not satisfy (Ku_1). Thus if $G(R)$ satisfies (Ku_1), then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. Suppose that $|Max(R)| = 1$. Since $\Gamma(R)$ satisfies (Ku_1), we obtain from $(i) \Rightarrow (ii)$ of Proposition 3.9 that R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$. From the above given arguments, it is clear that if $G(R)$ satisfies (Ku_1), then R is isomorphic to one of the rings from the collection $\mathcal{E} = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}, \mathbb{F}_4\}$.

$(iv) \Rightarrow (i)$ Let $\mathcal{E}$ be as in the statement (iv) of this theorem. If $T \in \mathcal{E}$, then $|T| \leq 4$. Since any simple graph on at most four vertices is planar, we obtain that $G(T)$ is planar. As R is isomorphic to one of the rings from the collection $\mathcal{E}$, we get that $G(R)$ is planar. □

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