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ON THE ADDITIVE MAPS SATISFYING SKEW-ENGEL CONDITIONS

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Abstract. Let R be a prime ring, I be any nonzero ideal of R and $f : I \to R$ be an additive map. Then skew-Engel condition $\langle ...\langle\langle\ f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0$ implies that $f(x) = 0\ \forall\, x \in I$ provided $2 \neq$ char $(R) > n_1 + n_2 + ... + n_k$, where $n_1, n_2, ..., n_k$ are natural numbers. This extends some existing results. In the end, we also generalise this result in the setting of MA-semirings.

1. Introduction

Except in the last section of this paper, R will be an associative ring with center $Z(R)$, Martindale right ring of quotients Q and extended centroid C. The Lie and Jordan product of x and y in R will be denoted by $[x, y] = xy - yx$ and $\langle x, y\rangle = xy + yx$ respectively. An Engel condition is a polynomial equation $[...[[y, x^{n_1}], x^{n_2}], ..., x^{n_k}] = 0$ in noncommutative indeterminate x and y, where $k, n_1, n_2, ...n_k$ are natural numbers. We call polynomial equation $\langle ...\langle\langle\ y, x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0$ as skew-Engel condition. A polynomial equation in which both Lie and Jordan products are involved is called mixed Engel condition such as $\langle ...\langle[\langle ...\langle[y, x\underbrace{], x\rangle, ..., x\rangle}_{n_1}, ..., x\underbrace{], x\rangle, ..., x\rangle}_{n_k} = 0$. A ring R is said to be prime if $aRb = 0\ \forall a, b \in R$ implies either $a = 0$ or $b = 0$ and semiprime if $aRa = 0$ implies $a = 0$. An additive mapping $d : R \to R$ is called derivation on R if $d(xy) = d(x)y + xd(y)\ \forall x, y \in R$. A mapping $f : R \to R$ is said to be commuting (centralizing) on R if

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$[f(x), x] = 0$ $([f(x), x] \in Z(R))\, \forall x \in R$. These mappings were initiated by E. C. Posner [19]. He proved that if R is a non-commutative prime ring and d is a centralizing derivation of R, then $d = 0$. J. Vukman [21] proved that if prime ring R of characteristic not 2 satisfies the identity $[[d(x), x], x] = 0$ $\forall x \in R$, then $d = 0$. C. Lanski [17] extended these results by studying the Engel condition $[d(x), x]_n = [...[[d(x), x], x], ..., x] = 0$ $\forall x \in R$, where n is any positive integer. M. Brešar and B. Hvava [8] proved that if R is a prime ring of characteristic not 2, I is nonzero ideal of R and $f : I \to R$ is an additive map, then $[f(x), x^2] = 0$ implies $[f(x), x] = 0$ $\forall x \in I$ and f is of the form $\lambda x + \xi(x)$, where $\lambda \in C$ and $\xi : R \to C$ is an additive map. This result has been proved in semiprime ring setup by P. Ara and M. Mathieu [1]. A. Fošner and N. Rehman [11] proved that if R is a $n!$-torsion free semiprime ring and an additive mapping $f : R \to R$ satisfies the relation $[f(x), x^n] = 0$, then $[f(x), x] = 0$ $\forall x$ R. M. Brešar [5] proved that if there is an additive mapping f of a prime rings of characteristic not 2 with $[[f(x), x], x] = 0$ $\forall x \in R$, then $[f(x), x] = 0$ $\forall x$ R. He [7] further proved that if f satisfies $[f(x), x]_n = [...[[f(x), x], x], ..., x] = 0$ $\forall x \in R$, then $[f(x), x] = 0$ $\forall x \in R$ provided $\text{char}(R) = 0$ or char $> n$. M. Fošner et al. [14] generalized this result in semiprime ring. Beidar et al. [3] proved that if R is a prime ring with nonzero right ideal I and $f : I \to R$ is a additive mapping satisfying $[...[[f(x), x^{n_1}], x^{n_2}, ...x^{n_k}] = 0$ $\forall x \in I$, then $[f(x), x] = 0$ $\forall x \in I$ provided that R is of characteristic greater than $n_1 + n_2 + ...n_k$, where $k, n_1, n_2, ...n_k$ are natural numbers. By using a well known method, we generalise this result for semiprime ring. We also prove a Jordan version of this result which extends some existing results as we discuss later.

An additive mapping $f : R \to R$ is said to be skew-commuting on R if $\langle f(x), x\rangle = 0\, \forall x \in R$. An additive mapping $f : R \to R$ is said to be n-skew-commuting on R if $\langle f(x), x^n\rangle = 0$. M. Brešar [6] proved that if R is a prime ring of characteristic not 2 and map $f : R \to R$ is skew-commuting, then $f = 0$. He [9] also proved that if R is a prime ring with $\text{char}(R) \neq 2$ and mapping $f : R \to R$ is 2-skew-commuting, then $f = 0$. This result has been proved in semiprime ring setup by M. Fošner [12]. M. Fošner et al. raised a conjecture in [11] about n-skew-commuting and solved it in [13] as if R is a prime ring with $2 \neq \text{char}(R) > n$ and an additive mapping $f : R \to R$ satisfies the relation $\langle f(x), x^n\rangle = 0\, \forall x \in R$, then $f = 0$. They also proved semiprime version of this result. We extend these results by proving that skew-Engel condition$\langle ...\langle\langle\ f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0$ implies $f(x) = 0$ for all $x \in I$, where I is any nonzero ideal of R and $f : I \to R$ is an additive map and $2 \neq$ char $(R) > n_1 + n_2 + ... + n_k$. In the last section of this paper, we study these identities in MA-semiring setup. An algebraic system $(R, +, \cdot)$ is called a semiring if $(R, \cdot)$ is a semigroup; $(R, +)$ is a commutative semigroup with 0 and distributive laws of multiplication over addition hold; furthermore, $0s = s0 = 0\ \forall s \in R$. An inverse semiring is an algebraic system $(R, +, \cdot, ')$ such that $(R, +, \cdot)$ is such a semiring in which for each element a, there exits an element $a' \in S$ such that $a + a + a' = a$ and $a' = a' + a + a'$. The element a' is called pseudo-inverse of a. It is easy to check that a' is unique for each a. In [16] Anjum and Aslam introduced a special class of inverse semirings called MA-semiring. A MA-semiring is an inverse semiring $(R, +, \cdot, ')$ with $y(x + x') = (x + x')y$ for all x, y in R. A subset S of $(R, +, \cdot, ')$ is said to be MA-subsemiring, if $a \in S$ implies $a' \in S$. A semiring ideal I of $(R, +, \cdot, ')$ is called MA-semiring ideal if it is MA-subsemiring. The Lie and Jordan commutator of x and y in MA-semiring R can be defined as $[x, y] = xy + (yx)' = xy + y'x = xy + yx'$ and $\langle x, y\rangle = xy + yx$ respectively. A ring $(R, +, \cdot)$ is a natural example of MA-semiring with $-x = x'\ \forall x \in R$. Similarly commutative inverse semirings and distributive lattices are MA-semirings. For more examples (noncommutative) we refer reader to [2]. When there is no ambiguity we write R for $(R, +, \cdot, ')$. For more details about MA-semirings, see [2, 15, 20]

2. Preliminary results

Here we state some useful results which we will use in our main theorems.

Fact 2.1. [6] *Let R be any nonzero prime ring and I be any ideal of R. Set $I_n = \{x^n | x \in I\}$, where n is any positive number. If $I_n a = 0$ (or $aI_n = 0$) for $a \in R$, then $a = 0$.*

Fact 2.2. [3] *Let R be a prime ring with nonzero right ideal I and $q \in Q$. Suppose that $[...[[q, x^{n_1}], x^{n_2}], ..., x^{n_k}] = 0\ \forall x \in I$, then $q \in C$.*

Fact 2.3. [3] *Let R be a prime ring, $f : I \to R$ an additive mapping and $k, n_1, n_2, ...n_k$ be natural numbers. Suppose $[...[[f(x), x^{n_1}], x^{n_2}], ..., x^{n_k}] = 0\ \forall x \in I$, then $[f(x), x] = 0$ provided that char(R) $= 0$ or char(R) $> n_1 + n_2 + ... + n_k$.*

Fact 2.4. [4] *Let n be any positive fixed integer and R be a prime ring. If an additive mapping $f : R \to R$ satisfies the identity $f(x)x^n = 0$ for all $x \in R$, then $f = 0$.*

Fact 2.5. [10] *Let R be an $n-$torsion free ring. Suppose $y_1, ..., y_n \in R$ satisfy $\alpha y_1 + ... + \alpha^n y_n = 0$ for $\alpha = 1, 2, ..., n$, then $y_i = 0$ for all i.*

As there is no convenient reference known to us, we include simple proofs of the following lemmas for the sake of completeness.

Lemma 2.6. *Let R be a semiprime ring. If an additive mapping $f : R \to R$ satisfies the identity $f(x)x^n = 0$ for all $x \in R$, where n is any positive integer, then $f = 0$.*

Proof. Let $f(x)x^n = 0\,\forall x \in R$. As R is semiprime, there exits a collection τ of prime ideals such that $\cap\tau = 0$. Take a prime ideal $P \in \tau$, so R/P becomes a prime ring. Let $p \in P$ and replace x by $x + p$, we get $(f(x) + f(p))(x^n + p^n + S) = 0$, where S is sum of terms involving x and p. This gives $f(x)p^n + f(x)S + f(p)x^n + f(p)S = 0$. Which, in turn, implies $f(p)x^n \in P$. As R/P is a prime ring, so by result 2.1, we have $f(p) \in P$. Hence f induces a mapping $g(x + P) = f(x) + P$ on R/P. Since g is an additive mapping and $g(x + P)(x + P)^n \subseteq P$. So by result 2.4, we obtain $g(x + P) = P\ \ \forall(x + P) \in R/P$. This gives $f(x) + P = P$, which implies $f(x) \in P\ \forall x \in R$. Hence we conclude $f(x) \in \cap\tau = 0$. This ends the proof. □

Example 2.7. Let R be an arbitrary ring, then $R_2(R) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| \text{a,b} \in R \right\}$ becomes a ring. Define an additive mapping $f : R_2(R) \to R_2(R)$ by $f(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, then $f(X)X^n = 0\,\forall X \in R_2(R)$, where n is any positive integer. This shows that the above lemma does not hold for arbitrary ring in general.

Lemma 2.8. *Let R be a semiprime ring, $f : R \to R$ be a nonzero additive mapping and $k, n_1, n_2, ...n_k$ be natural numbers. Suppose that $[...[[f(x), x^{n_1}], x^{n_2}], ..., x^{n_k}] = 0\ \ \forall x \in R$, then $[f(x), x] = 0$ provided that R is $(n_1 + n_2 + ... + n_k)!$-torsion free.*

Proof. Let

$$[...[[f(x), x^{n_1}], x^{n_2}], ..., x^{n_k}] = 0\,\forall x \in R. \tag{2.1}$$

As R is a semiprime ring, there exits a collection of prime ideals $\{P_\alpha : \alpha \in I\}$ such that $\cap_{\alpha\in I} P_\alpha = 0$. Take a prime ideal P_α, $\alpha \in I$ so that R/P_α becomes a prime ring. Let C_α be extended centroid of R/P_α and A be central closure of R/P_α. As a vector space over the field C_α, let A has a subspace B. Since C_α is a subspace of A, one can written as $A = B + C_\alpha$. Let π be canonical projection of A onto B. Replacing x by $x + p$ in (1), where $p \in P$, we get $[...[[f(x + p), (x + p)^{n_1}], (x + p)^{n_2}], ..., (x + p)^{n_k}] = 0\,\forall x \in R, \forall p \in P$. This can be written as $[...[[f(x) + f(p), x^{n_1} + p_{n_1}(x, p) + p^{n_1}], x^{n_2} + p_{n_2}(x, p) + p^{n_2}], ..., x^{n_k} + p_{n_k}(x, p) + p^{n_k}] = 0$, where $p_{n_i}(x, p)$ is sum of all terms in the expansion of $(x+p)^{n_i}$ except x^{n_i} and p^{n_i}. By expanding, one can get the following relation

$$[...[[f(x), x^{n_1}], x^{n_2}], ..., x^{n_k}] + [...[[f(p), x^{n_1}], x^{n_2}], ..., x^{n_k}] + [...[[f(x), p_{n_1}(x,p)], x^{n_2}], ..., x^{n_k}] + ...[...[[f(x), p^{n_1}], p^{n_2}], ..., p^{n_k}] + [...[[f(p), p^{n_1}], p^{n_2}], ..., p^{n_k}] = 0 \, \forall x \in R, \forall p \in P.$$

Using (2.1), the last relation reduces to

$$[...[[f(p), x^{n_1}], x^{n_2}], ..., x^{n_k}] + [...[[f(x), p_{n_1}(x,p)], x^{n_2}], ..., x^{n_k}] + ...[...[[f(x), p^{n_1}], p^{n_2}], ..., p^{n_k}] = 0 \, \forall x \in R, \forall p \in P.$$

This implies $[...[[f(p), x^{n_1}], x^{n_2}], ..., x^{n_k}] + P_\alpha = P_\alpha$. Hence, one can get the result $[...[[f(p), x^{n_1}], x^{n_2}], ..., x^{n_k}] \in P_\alpha$. This gives $[...[[\overline{f(p)}, \overline{x}^{n_1}], \overline{x}^{n_2}], ..., \overline{x}^{n_k}] = 0$. By using fact 2.2, we get $[\overline{f(p)}, \overline{x}] = 0$. This implies $\overline{f(p)}$ lies in the center of R/P_α. This gives $\pi \overline{f(p)} = 0$. So $\overline{f} : R/P_\alpha \to A$ with $\overline{f}(\overline{x}) = \pi \overline{f(x)}$ is well defined. It can be observed that $\overline{f}$ is additive map and satisfies the identity $[...[[\overline{f}(\overline{x}), \overline{x}^{n_1}], \overline{x}^{n_2}], ..., \overline{x}^{n_k}] = 0$. By using result 2.3, we get $[\overline{f}(\overline{x}), \overline{x}] = 0$. This implies $[f(x), x] \in P_\alpha$. Hence $[f(x), x] = 0 \;\; \forall x \in R$. This ends the proof. □

Remark 2.9. Let R be an arbitrary ring and $T_n(R) = \left\{ \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\}$ be ring of strictly upper triangular matrices over R. Define an additive mapping $f : T_n(R) \to T_n(R)$ by $f(\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}) = \begin{bmatrix} 0 & a_{12} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Although f satisfies identity 2.1 but $[f(x), x)] \neq 0 \; \forall x \in T_n(R)$. So primeness of a ring is necessary in last lemma.

The last lemma generalised the following recent result under the mild condition of torsion.

Theorem 1.1 [18]

Let R be a semiprime ring with an anti-automorphism f of finite order. If $[...[[f(x), x^{n_1}], x^{n_2}], ..., x^{n_k}] = 0 \, \forall x \in R$, then f is a commuting map.

3. Skew-Engel conditions in rings.

We first study skew-Engel condition in the algebras of matrices.

Theorem 3.1. *Let $m > 1$ be an integer and $M_m(K)$ be the algebra of all $m \times m$ matrices over an arbitrary field K. Suppose that the mapping $F : M_m(K) \to M_m(K)$ is a K-linear map such that the identity $\langle ...\langle\langle\ F(X), X^{n_1}\rangle, X^{n_2}\rangle, ..., X^{n_k}\rangle = 0$, then $F(X) = 0\ \ \forall X \in M_m(K)$.*

Proof. Let we have

$$\langle ...\langle\langle F(X), X^{n_1}\rangle, X^{n_2}\rangle, ..., X^{n_k}\rangle = 0. \tag{3.1}$$

Now replace X by an idempotent matrix E to get

$$\langle ...\langle\langle F(E), E\rangle, E\rangle, ..., E\rangle = 0. \tag{3.2}$$

This becomes

$$[...[[F(E), E], E]..., E] = 0.$$

This reduces to $[F(E), E] = 0$ in view of [3, theorem 2.4(iii)] . From (3.2), we have $F(E)E = EF(E) = 0$. Now replace X by $X + tY$ in (3.1), where t is any positive integer, and using result fact 2.5 to have

$$\begin{gathered}\langle ...\langle\langle F(Y), X^{n_1}\rangle, X^{n_2}\rangle, ..., X^{n_k}\rangle + \\ \langle ...\langle\langle F(X), \textstyle\sum_{i=0}^{n_1-1} X^iYX^{n_1-1-i}\rangle, X^{n_2}\rangle ..., X^{n_k}\rangle + \\ \langle ...\langle\langle F(X), X^{n_1}\rangle, \textstyle\sum_{i=0}^{n_2-1} X^iYX^{n_2-1-i}\rangle, X^{n_3}\rangle ..., X^{n_k}\rangle + ... + \\ \langle ...\langle\langle F(X), X^{n_1}\rangle, X^{n_2}\rangle, ..., X^{n_{k-1}}\rangle, \textstyle\sum_{i=0}^{n_k-1} X^iYX^{n_k-1-i}\rangle = 0.\end{gathered}$$

Replace X by E in last relation such that $YE = YE = 0$, we have

$$\langle ...\langle\langle F(Y), E\rangle, E\rangle, ..., E\rangle = 0.$$

This becomes

$$[...[[F(X), E], E]..., E] = 0.$$

Again by using [3, theorem 2.4(iii)], we have $[F(Y), E] = 0$. This gives $F(Y)E = EF(Y) = 0$.

Let $E_{ij} \in M_m(K)$ be the matrix with (i,j)-entry equal to one and all the others equal to zero. Furthermore, suppose $F(E_{ii}) = \sum_{s,t=1}^{m} X_{st}E_{st}$. From $F(E_{ii})E_{jj} = 0$, where $i \neq j$, we have $X_{sj} = 0$ for $1 \leq s \leq m$, so $F(E_{ii}) = \sum_{s=1}^{m} X_{si}$. The relation $F(E_{ii})E_{ii} = 0$ gives $X_{si} = 0$ for $1 \leq s \leq m$. Hence $F(E_{ii}) = 0$ for $1 \leq i \leq m$. Now suppose $F(E_{ij}) =$

$\sum_{u,v=1}^{m} X_{uv}E_{uv}$ for $i \neq j$, then $F(E_{ij})E_{pp} = 0$ for $p \neq i, j$ implies $X_{up} = 0$ where $1 \leq u \leq m$ and $E_{pp}F(E_{ij}) = 0$ implies $X_{pv} = 0$ where $1 \leq v \leq m$. From $F(E)E = 0$, we have $F(E_{ij} + E_{jj})(E_{ij} + E_{jj}) = F(E_{ij})(E_{ij} + E_{jj}) = 0$. This gives $X_{ii} + X_{ij} = 0$ and $X_{ji} + X_{jj} = 0$. Also $(E_{ij}+E_{jj})F(E_{ij}+E_{jj}) = (E_{ij}+E_{jj})F(E_{ij}) = 0$ gives $X_{jj} = 0$ and $X_{ji} = 0$. The relation $F(E_{ij} + E_{ii})(E_{ij} + E_{ii}) = F(E_{ij})(E_{ij} + E_{ii}) = 0$ gives $X_{ij} = 0$ and so $X_{ii} = 0$. Hence $F(E_{ij}) = 0$ for $1 \leq i,j \leq m$. So $F(X) = F(\sum_{i,j=1}^{m} X_{ij}E_{ij}) = X_{ij}F(\sum_{i,j=1}^{m} E_{ij}) = 0 \ \forall X \in M_m(K)$. □

The following theorem is an extension of the [6, theorem 1], [8, theorem 2] and [9, lemma 4].

Theorem 3.2. *Let R be a prime ring, I be any nonzero ideal of R and $f : I \to R$ be an additive map. Next let $k, n_1, n_2, ...n_k$ be natural numbers, then skew-Engel condition $\langle ...\langle\langle\ f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0$ (or $\langle ...\langle\langle\ f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle \in Z(R)$) implies that $f(x) = 0\ \forall x \in I$ provided $2 \neq$ char $(R) > n_1 + n_2 + ... + n_k$.*

Proof. Let we have

$$\langle ...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0\, \forall x \in I. \tag{3.3}$$

Multiplying left and right separately by x^{n_k}, we get

$$x^{2n_k}\langle ...\langle\langle y, x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-1}}\rangle + x^{n_k}\langle ...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-1}}\rangle x^{n_k} = 0.$$

$$x^{n_k}\langle ...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-1}}\rangle x^{n_k} + \langle ...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-1}}\rangle x^{2n_k} = 0.$$

Comparing the last two relations, we get

$$[...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-1}}\rangle, x^{2n_k}] = 0.$$

On expansion, we have

$[x^{n_{k-1}}\langle ...\langle\langle f(x), x^{n_2}\rangle, x^{n_{k-1}}\rangle, ..., x^{n_{k-2}}\rangle + \langle ...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-2}}\rangle x^{n_{k-1}}, x^{2n_k}] = 0$. This gives

$$\begin{aligned} x^{n_{k-1}}[\langle ...\langle\langle f(x), x^{n_2}\rangle, x^{n_{k-1}}\rangle, ..., x^{n_{k-2}}\rangle, x^{2n_k}] \\ + [\langle ...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-2}}\rangle x^{n_{k-1}}, x^{2n_k}]x^{n_{k-1}} = 0. \end{aligned}$$

Now first multiplying left and right separately by x^{n_k-1}, then comparing the resultant equations, we have

$$[...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_{k-2}}\rangle, x^{2n_{k-1}}], x^{2n_k}] = 0.$$

Continuing this process, we obtain

$$[...[[f(x), x^{2n_1}], x^{2n_2}], ..., x^{2n_k}] = 0.$$

Applying result 2.3 to last relation, we get $[f(x), x] = 0$. So 3.3

reduces to

$$2^k f(x)x^n = 0.$$

where $n = n_1 + n_2 + ... + n_k$, this yields

$$f(x)x^n = 0\,\forall x \in I. \tag{3.4}$$

If $I = R$, then $f = 0$ by lemma 2.6, otherwise replace x by $x + ty$ in the last identity, where t is any positive number and $x, y \in I$, to obtain

$$t(f(x)\sum_{i=0}^{n-1} x^i y x^{n-1-i} + f(y)x^n) + ... + t^n(f(y)\sum_{i=0}^{n-1} y^i x y^{n-1-i} + f(x)y^n) = 0.$$

Using result 2.5, the last equation reduces to

$$f(y)\sum_{i=0}^{n-1} y^i x y^{n-1-i} + f(x)y^n = 0.$$

Left multiplying the last relation by y^n and using (3.4), we have

$$y^n f(x)y^n = 0. \tag{3.5}$$

Now linearize $[f(x), x] = 0$ to get

$$[f(x), y] = [x, f(y)]. \tag{3.6}$$

Replacing y by y^n in above equation, we obtain

$$[f(x), y^n] = [x, f(y^n)]. \tag{3.7}$$

Right multiplying by y^n and using (3.4), we have identity

$$[x, f(y^n)]y^n = f(x)y^{2n}. \tag{3.8}$$

Replacing x by x^n in last relation and using (3.5) to get $x^{3n}f(y^n)y^n = 0$. By using fact 2.1, we have $f(y^n)y^n = 0$. So the equation (3.8) reduces to equation $f(y^n)xy^n + f(x)y^{2n} = 0$. Now first replace x by xy^n in it and multiply it from right by y^n, then compare the resultant equations to obtain

$$f(xy^n)y^{2n} = f(x)y^{3n}. \tag{3.9}$$

Left multiplying (3.7) by y^n and using (3.4), we get equation $y^{2n}f(x) + y^n x f(y^n) = 0$. Replacing x by xy^n and using (3.4), we get

$$y^{2n} f(xy^n) = 0. \tag{3.10}$$

From (3.6), we have $[f(x), zy^n] = [x, f(zy^n)]$, where $z \in R$. First multiplying this by y^{2n} from left and right simultaneously then using (3.7) and (3.8), we get $y^{2n}f(x)zy^{3n} = y^{2n}xf(z)y^{3n}$. Left multiplying this by y^n, we get $y^{3n}f(x)zy^{3n} = y^{3n}xf(z)y^{3n}$. By replacing z by $zy^{3n}f(u)$ in it and using (3.4), we have $y^{3n}xf(zy^{3n}f(u))y^{3n} = 0$. As R be prime ring, so we have

$$f(zy^{3n}f(u))y^{3n} = 0. \tag{3.11}$$

From (3.6), we have identity $[f(x), zy^{3n}f(u)] = [x, f(zy^{3n}f(u))]$. Multiplying by y^{3n} from right and using (3.11), we get equation

$f(zy^{3n}f(u))xy^{3n} = zy^{3n}f(u)f(x)y^{3n}$. Now first replace z by tz in it and left multiply it by t separately, then compare the resultant equations to obtain $f(tzy^{3n}f(u))xy^{3n} = tf(zy^{3n}f(u))xy^{3n}$. Again left multiply by w^n and replace x by w^nx in the resultant identity, we obtain

$w^ntf(zy^{3n}f(u))w^nxy^{3n} = 0$ in view of (3.5). As R is a prime ring, so we have $f(zy^{3n}f(u))w^n = 0$. Using fact 2.1, we obtain

$$f(zy^{3n}f(u)) = 0. \tag{3.12}$$

Suppose $f(u) \neq 0$ for some u, otherwise the theorem is proved. Hence by result 2.1, we have $a = y^{3n}f(u) \neq 0$, then $L = Ra$ is a non-zero left ideal. From (3.12), we conclude $f(L) = 0$. By (3.6), we get $[f(x), l] = 0$, where $l \in L$. Replace l by rl, where $r \in R$, to get $[f(x), rl] = 0$. Substitute r by x^nr to have $x^nrlf(x) = 0$. This reduces to $lf(x) = 0$ due to primeness of R, this implies that $f(x)l = 0$. Now replace l by rl to obtain $f(x)rl = 0$. As R is a prime ring, so we get $f(x) = 0 \, \forall x \in I$. This completes the proof. □

Remark 3.3. Let R be a ring of $n_i \times n_i$ strictly upper triangular matrices, where i is any number such that $1 \leq i \leq k$, then any non-additive mapping on R satisfies identity (3.3). This implies that the above theorem does not hold for arbitrary rings in general.

With the help of lemmas 2.6 and 2.8, we can prove the semiprime version of above theorem which is an extension of [6, theorem 2], [11, theorem 4] and [12, theorem 1].

Corollary 3.4. *Let R be a prime ring and I be nonzero ideal such that $f : I \to R$ be an additive mapping. Next let $k, n_1, n_2, ...n_k$ be natural numbers such that char $(R) > n_1 + n_2 + ... + n_k$. If f satisfies any one of following mixed Engel conditions, then $[f(x), x] = 0 \, \forall x \in I$. In this case there exits $\lambda \in C$ and additive map $\xi : I \to C$ such that $f(x) = x + \xi(x)$ provided $[I, I]I \neq 0$.*

(i) $[...[[\langle...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle, x^{n_k}], x^{n_{k-1}}], ..., x^{n_1}] = 0.$
(ii) $\langle...\langle\langle[...[[f(x), x^{n_1}], x^{n_2}], ..., x^{n_k}], x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0.$
(iii) $[\langle...[\langle[\langle f(x), x^{n_1}\rangle, x^{n_1}], x^{n_2}\rangle, x^{n_2}], ..., x^{n_k}\rangle, x^{n_k}] = 0.$
(iv) $\langle[...\langle[\langle[f(x), x^{n_1}], x^{n_1}\rangle, x^{n_2}], x^{n_2}\rangle..., x^{n_k}], x^{n_k}\rangle = 0.$
(v) $\langle...\langle[\langle...\langle[f(x), x\underbrace{], x\rangle, ..., x\rangle}_{n_1}, ..., x\underbrace{], x\rangle, ..., x\rangle}_{n_k} = 0.$

Corollary 3.5. *Let R be a prime ring and $k, n_1, n_2, ...n_k$ be natural numbers. Suppose that $a \in R$ such that $\langle...\langle\langle\ a, x\rangle, x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0\,\forall x \in R$, then $a = 0$ provided $2 \neq char(R) > n_1 + n_2 + ... + n_k$.*

Proof. Let we have $\langle...\langle\langle a, x\rangle, x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0\,\forall x \in R$. By using theorem 3.2, we get $\langle a, x\rangle = 0$. This implies $[\langle a, x\rangle, x] = 0$. So we have $[a, x^2] = 0\,\forall x \in R$, which can be written as $\langle f_a(x), x\rangle = 0$, where $f_a(x) = [a, x]$ is an additive mapping. Now by using [6, theorem 1], we get $f_a(x) = [a, x] = 0\,\forall x \in R$. So we have $2ax = 0\,\forall x \in R$. This can be written as $ar(2x) = 0\,\forall x, r \in R$. As $\text{char}(R) \neq 2$ and R is a prime ring, so we concludes that $a = 0$. □

4. Skew-Engel conditions in MA-semirings.

A MA-semiring R is an inverse semiring with $y(x+x') = (x+x')y$ for all x, y in R. Every ring is a MA-semiring but the converse is not true in general. Commutative inverse semirings and distributive lattices are examples of MA-semirings. For more examples (non- commutative) we refer reader to [2]. Here we prove that our main theorem still holds in MA-semirings.

Lemma 4.1. *Let R be any MA-semiring and $x, y \in R$, then the following hold:*

(i) $xy = yx + [x, y]$.
(ii) $[x, y + z] = [x, y] + [x, y]$.
(iii) $[x, y]' = [y, x] = [x, y'] = [x', y]$.
(iv) $[x, yz] = [x, y]z + y[x, z]$.
(v) $[x, y] + [y, x] = x(y + y') = y(x + x')$.
(vi) $\langle...\langle\langle y, x^{n_1}\rangle, x^{n_2}\rangle...x^{n_k}\rangle = x^{n_1+n_2+...n_k}(y+y')+\langle...\langle\langle y, x^{n_1}\rangle, x^{n_2}\rangle...x^{n_k}\rangle$.

Lemma 4.2. *Let R be a prime MA-semiring and S be any nonzero subset of R. Let $(x + x')^n = 0$, then $x + x' = 0\,\forall x \in S$.*

Theorem 4.3. *Let R be a prime MA-semiring, I be an MA-semiring ideal and $f : I \to R$ be an additive mapping. Next let $n_1, n_2, ...n_k$ be natural numbers such that $2 \neq char\ (R) > n_1 + n_2 + ... + n_k$, then identity $\langle...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0$ implies $f(x) = 0\,\forall x \in I$.*

Proof. Let we have

$$\langle ...\langle\langle f(x), x^{n_1}\rangle, x^{n_2}\rangle, ..., x^{n_k}\rangle = 0\,\forall x \in I. \tag{4.1}$$

Replacing y by $f(x)$ in 4.1 (vi), we obtain $x^m(f(x)+f(x)') = 0$, where $m = n_1 + n_2 + ... + n_k$. This can be written as $(x(f(x)+f(x)'))^m = 0$, which gives $(xf(x) + (xf(x))')^m = 0$. By using lemma 4.2, we obtain $xf(x) + (xf(x))' = 0$. This implies $(x + x')f(x) = 0$. By definition, we have $(x + x')rf(x) = 0\ \ \forall r \in R$. Since R is a prime MA-semiring, so we have either $f(x) = 0$ or $x + x' = 0\ \ \forall x \in I$. Replace x by xy to get $(y+y')x = 0$, where $y \in R$. This implies $y+y' = 0$. So R becomes ring and hence by theorem 3.2, we again have $f(x) = 0\ \ \forall x \in I$. This ends the proof. □

Example 4.4. Let R be a MA-ring, then
$M_2(R) = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \Big| \text{a,b} \in R \right\}$ also becomes MA-semiring ring and $C_2(R) = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \Big| \text{a,b} \in R \right\}$ is a MA-semiring ideal of $M_2(R)$. Define an additive mapping $f : C_2(R) \to M_2(R)$ by $f\Big(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \Big) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, then f satisfies identity (4.1). This shows that the last lemma does not hold for arbitrary ring in general.

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