A NOTE ON THE EXTENDED TOTAL GRAPH OF COMMUTATIVE RINGS

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Abstract. Let $R$ be a commutative ring and $H$ a nonempty proper subset of $R$. In this paper, the extended total graph, denoted by $ET_H(R)$, is presented, where $H$ is a multiplicative-prime subset of $R$. It is the graph with all elements of $R$ as vertices, and for distinct $p, q \in R$, the vertices $p$ and $q$ are adjacent if and only if $rp + sq \in H$ for some $r, s \in R \setminus H$. We also study the two (induced) subgraphs $ET_H(H)$ and $ET_H(R \setminus H)$, with vertices $H$ and $R \setminus H$, respectively. Among other things, the diameter and the girth of $ET_H(R)$ are also studied.

1. Introduction

Throughout this paper $R$ is a commutative ring with nonzero identity. Recently, there has been considerable attention in the work to associating graphs with algebraic structures (see [4],[5],[6], [8], [9] and [10]). Anderson and Badawi in [3] defined a nonempty proper subset $H$ of $R$ to be a multiplicative-prime subset of $R$ if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $rs \in H$ for some $r, s \in R$, then either $r \in H$ or $s \in H$. They introduced the notion of the generalized total graph of a commutative ring $GT_H(R)$ with the vertices of this graph are all elements of $R$ and two vertices $x, y \in R$ are adjacent if and only if $x + y \in H$ where $H$ is a multiplicative-prime subset of $R$. In this paper, we introduce an extension of the graph $GT_H(R)$, denoted by $ET_H(R)$, such that its vertex set consist of all

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elements of \( R \) and for distinct \( p, q \in R \), the vertices \( p \) and \( q \) are adjacent if and only if \( rp + sq \in H \) for some \( r, s \in R \setminus H \), where \( H \) is a multiplicative-prime subset of \( R \). Let \( ET_H(H) \) be the (induced) subgraph of \( ET_H(R) \) with vertex set \( H \), and let \( ET_H(R \setminus H) \) be the (induced) subgraph \( ET_H(R) \) with vertices consisting of \( R \setminus H \). Obviously, the total graph \( GT_H(R) \) is a subgraph of \( ET_H(R) \). It follows that each edge (path) of \( GT_H(R) \) is an edge (path) of \( ET_H(R) \).

The study of \( ET_H(R) \) breaks naturally into two cases depending on whether or not \( H \) is an ideal of \( R \). In the second section, we handle the case when \( H \) is an ideal of \( R \); in the third section, we do the case when \( H \) is not an ideal of \( R \). For every case, we characterize the girths and diameters of \( ET_H(R) \), \( ET_H(H) \) and \( ET_H(R \setminus H) \).

We begin with some notation and definitions. For a graph \( \Gamma \), by \( E(\Gamma) \) and \( V(\Gamma) \), we mean the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two of its distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices \( a \) and \( b \), denoted by \( d(a, b) \), is the length of a shortest path connecting them (if such a path does not exist, then \( d(a, b) = \infty \)). We also define \( d(a, a) = 0 \). The diameter of a graph \( \Gamma \), denoted by \( \text{diam}(\Gamma) \), is equal to \( \sup \{ d(a, b) : a, b \in V(\Gamma) \} \). A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph \( \Gamma \), denoted \( \text{gr}(\Gamma) \), is the length of a shortest cycle in \( \Gamma \), provided \( \Gamma \) contains a cycle; otherwise; \( \text{gr}(\Gamma) = \infty \). We denote the complete graph on \( n \) vertices by \( K^n \) and the complete bipartite graph on \( m \) and \( n \) vertices by \( K^{m,n} \) (we allow \( m \) and \( n \) to be infinite cardinals). For a graph \( \Gamma \), the degree of a vertex \( v \) in \( \Gamma \), denoted \( \text{deg}(v) \), is the number of edges of \( \Gamma \) incident with \( v \). We say that two (induced) subgraphs \( \Gamma_1 \) and \( \Gamma_2 \) of \( \Gamma \) are disjoint if \( \Gamma_1 \) and \( \Gamma_2 \) have no common vertices and no vertices of \( \Gamma_1 \) is adjacent (in \( \Gamma \)) to some vertex of \( \Gamma_2 \).

2. The case when \( H \) is an ideal of \( R \)

In this section, we study the case when \( H \) is an ideal of \( R \). It is clear that if \( H \) an ideal of \( R \), then \( H \) is a prime ideal of \( R \). If \( H = R \), then it is clear that \( ET_H(R) \) is a complete graph and \( ET_H(R) \) is disconnected when \( H = 0 \) and \( |R| \geq 2 \). So we may assume that \( H \neq 0 \) and \( H \neq R \).

First we begin with the following example that shows \( ET_H(R) \neq GT_H(R) \).
Example 2.1. Let $R = \mathbb{Z}_8$. Set $H = \{0, 4\}$. It is clear that $H$ is an ideal of $R$. Since $7 + 3 = 2 \notin H$, so $7 - 3$ is not an edge in $G_{T_H(R)}$. But $\overline{1}(7) + 3(3) = 0 \in H$ and $\overline{1}, 3 \in R \setminus H$. Then $7 - 3$ is an edge in $E_{T_H(R)}$. Hence $E_{T_H(R)} \neq G_{T_H(R)}$.

The main goal of this section is a general structure theorem (Theorem 2.4) for $E_{T_H(R \setminus H)}$ when $H$ is an ideal of $R$. But first, we record the trivial observation that $H$ is an ideal of $R$, the $E_{T_H(H)}$ is a complete subgraph of $E_{T_H(R)}$ and is disjoint from $E_{T_H(R \setminus H)}$. Thus we will concentrate on the subgraph $E_{T_H(R \setminus H)}$ throughout this section.

Theorem 2.2. Let $R$ be a commutative ring and $H$ be a prime ideal of $R$. Then $E_{T_H(H)}$ is a complete subgraph of $E_{T_H(R)}$ and is disjoint from $E_{T_H(R \setminus H)}$. In particular, $E_{T_H(H)}$ is connected and $E_{T_H(R)}$ is disconnected.

Proof. Let $p, q \in H$. Then it is clear that $p + q \in H$ since $H$ is an ideal of $R$. If $x \in H$ is adjacent to $y \in R \setminus H$, then $rx + sy \in H$ for some $r, s \in R \setminus H$. This implies that $sy \in H$, so either $y \in H$ or $s \in H$ since $H$ is a prime ideal which is a contradiction. The ”in particular” state is clear.

Theorem 2.3. Let $R$ be a commutative ring and $H$ be a prime ideal of $R$. Then the following hold:

(1) Suppose that $G$ is an induced subgraph of $E_{T_H(R \setminus H)}$ and let $x$ and $y$ be distinct vertices of $G$ that are connected by a path in $G$. Then there exists a path in $G$ of length 2 between $x$ and $y$. In particular, if $E_{T_H(R \setminus H)}$ is connected, then $diam(E_{T_H(R \setminus H)}) \leq 2$.

(2) Let $x$ and $y$ be distinct elements of $E_{T_H(R \setminus H)}$ that are connected by a path. If $x + y \notin H$ then $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between $x$ and $y$ in $E_{T_H(R \setminus H)}$.

Proof. (1) Let $x_1, x_2, x_3$ and $x_4$ are distinct vertices of $G$. It suffices to show that if there is a path $x_1 - x_2 - x_3 - x_4$ from $x_1$ to $x_4$, then $x_1$ and $x_4$ are adjacent. Now, $r_1x_1 + r_2x_2 + r_4^3x_3 + r_3x_3 + r_4x_4 \in H$ for some $r_1, r_2, r_3, r_4, r_4' \in R \setminus H$. Hence $(r_1r_3r_2^3)\{x_1 + (r_2r_3r_4)x_4 \in r_3r_2^3(r_1x_1 + r_2x_2) - r_2r_3^3(r_2x_2 + r_3x_3) + r_2r_3^3(r_3x_3 + r_4x_4) \in H$. Since $H$ is a prime ideal of $R$, so $r_1r_3r_2, r_2r_3^3x_4 \notin H$. Then $x_1$ and $x_4$ are adjacent. So if $E_{T_H(R \setminus H)}$ is connected, then $diam(E_{T_H(R \setminus H)}) \leq 2$.

(2) Since $x, y \in R \setminus H$ and $x + y \notin H$, there exists $z \in R \setminus H$ such that $x - z - y$ is a path of length 2 by part (1) above. So $rx + sz, s'z + r'y \in H$ for some $r, s, r', s' \in R \setminus H$. Therefore $rs', r's \notin H$ since $H$ is a prime ideal of $R$. Then $rs'x - r'y = s'(rx + sz) - s(s'z + r'y) \in H$ and $x$ is adjacent to $y$. So $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between $x$ and $y$ in $E_{T_H(R \setminus H)}$. 

\[\square\]
Now, we give the main theorem of this section. Since $ET_H(H)$ is a complete subgraph of $ET_H(R)$ by Theorem 2.2, the next theorem gives a complete description of $ET_H(R \setminus H)$. Let $|H| = \alpha$. We allow $\alpha$ to be infinite cardinal. Compare the next theorem with [3, Theorem 2.2].

**Theorem 2.4.** Let $R$ be a commutative ring and $H$ be a prime ideal of $R$ and let $|H| = \alpha$.

1. If $r + s \in H$ for some $r, s \in R \setminus H$, then $ET_H(R \setminus H)$ is the union of complete subgraphs.

2. If $r + s \notin H$ for all $r, s \in R \setminus H$, then $ET_H(R \setminus H)$ is the union of totally disconnected subgraphs and some connected subgraphs.

**Proof.** (1) Suppose that $r + s \in H$ for some $r, s \in R \setminus H$. For $x, x' \in R \setminus H$, we write $x \sim x'$ if and only if $tx + t'x' \in H$ and $t + t' \in H$ for some $t, t' \in R \setminus H$. It is straightforward to check that $\sim$ is an equivalence relation on $R \setminus H$, since $H$ is a prime ideal. For $x \in R \setminus H$, we denote the equivalence class which contains $x$ by $[x]$. Now let $x \in R \setminus H$. If $[x] = \{x\}$, then $r(x + h_1) + s(x + h_2) = (r + s)x + rh_1 + sh_2 \in H$ for every $h_1, h_2 \in H$ since $r + s \in H$. Then $x + H$ is a complete subgraph of $ET_H(R \setminus H)$ with at most $\alpha$ vertices. Now let $|\{x\}| = \nu$ and $x' \in [x]$. Then $tx + t'x' \in H$ and $t + t' \in H$ for some $t, t' \in R \setminus H$. So $t(x + h_1) + t'(x' + h_2) = tx + t'x' + th_1 + t'h_2 \in H$ for every $h_1, h_2 \in H$. Thus $x + H$ is a part of complete graph $k^\mu$ where $\mu \leq \alpha \nu$.

(2) Assume that $r + s \notin H$ for all $r, s \in R \setminus H$. Set

$$A_x = \{x' \in R \setminus H : rx + sx' \in H \text{ for some } r, s \in R \setminus H\}$$

be the set of all adjacent vertices to $x$. If $A_x = \emptyset$, then $px + qx' \notin H$ for every $x' \in R \setminus H$ and every $p, q \in R \setminus H$. In this case, we show that $x + H$ is a totally disconnected subgraph of $ET_H(R \setminus H)$. If $r(x + x_1) + s(x + x_2) \in H$ for some $r, s \in R \setminus H$ and $x_1, x_2 \in H$, then $(r + s)x \in H$. Since $H$ is a prime ideal of $R$ and $x \notin H$, then $r + s \in H$ which is a contradiction. Therefore $x + H$ is a totally disconnected subgraph of $ET_H(R \setminus H)$. Now, we may assume that $A_x \neq \emptyset$. Then $rx + sx' \in H$ for some $r, s \in R \setminus H$ and $x' \in R \setminus H$. Hence $r(x + h_1) + s(x' + h_2) = rx + sx' + rh_1 + sh_2 \in H$ for every $h_1, h_2 \in H$; hence each element of $x + H$ is adjacent to each element of $x' + H$. If $|A_x| = \nu$, then we have a connected subgraph of $ET_H(R \setminus H)$ with at most $\alpha \nu$ vertices. So $ET_H(R \setminus H)$ is the union of totally disconnected subgraphs and some connected subgraphs. \qed

Now it is easy to compute the girth of $ET_H(R \setminus H)$ using Theorem 2.4.
Theorem 2.5. Let $R$ be a commutative ring and $H$ be a prime ideal of $R$. Then $gr(ET_H(R \setminus H)) = 3, 4$ or $\infty$. In particular, $gr(ET_H(R \setminus H)) \leq 4$ if $ET_H(R \setminus H)$ contains a cycle.

Proof. Let $ET_H(R \setminus H)$ contains a cycle. Then $ET_H(R \setminus H)$ is not a totally disconnected graph, so by the proof of Theorem 2.4, $ET_H(R \setminus H)$ has either a complete or a complete bipartite subgraph. Therefore it must contain either a 3-cycle or 4-cycle. Thus $gr(ET_H(R \setminus H)) \leq 4$. \qed

Theorem 2.6. Let $R$ be a commutative ring and $H$ be a prime ideal of $R$.

1. $gr(ET_H(R \setminus H)) = 3$ if and only if $r + s \in H$ and $|y + H| \geq 3$ for some $r, s \in R \setminus H$ and $y \in R \setminus H$.
2. $gr(ET_H(R \setminus H)) = 4$ if and only if $r + s \notin H$ for every $r, s \in R \setminus H$ and $px + qx' \in H$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$.
3. Otherwise, $gr(ET_H(R \setminus H)) = \infty$.

Proof. (1) Assume that $gr(ET_H(R \setminus H)) = 3$. Then by Theorem 2.4, $ET_H(R \setminus H)$ is a complete graph $K^\lambda$ where $\lambda \geq 3$. Then $r + s \in H$ for some $r, s \in R \setminus H$ and $|y + H| \geq 3$ for some $y \in R \setminus H$ by Theorem 2.4.

2. If $gr(ET_H(R \setminus H)) = 4$, then $ET_H(R \setminus H)$ has a complete bipartite subgraph. So $r + s \notin H$ for every $r, s \in R \setminus H$ and $px + qx' \in H$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$ by Theorem 2.4.

The other implications of (1) and (2) follow directly from Theorem 2.4. \qed

We end this section with the following theorem.

Theorem 2.7. Let $R$ be a commutative ring and $H$ be a prime ideal of $R$.

1. $gr(ET_H(R)) = 3$ if and only if $|H| \geq 3$.
2. $gr(ET_H(R)) = 4$ if and only if $r + s \notin H$ for every $r, s \in R \setminus H$, $|H| < 3$ and $px + qx' \in U$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$.
3. Otherwise, $gr(ET_H(R)) = \infty$.

Proof. (1) This follows from Theorem 2.2.

2. Assume that $gr(ET_H(R)) = 4$. Since $gr(ET_H(H)) = 3$ or $\infty$, then $gr(ET_H(R \setminus H)) = 4$. Therefore $r + s \notin H$ for every $r, s \in R \setminus H$ and $px + qx' \in H$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$ by Theorem 2.6. On the other hand, $gr(ET_H(R)) \neq 3$; so $|H| < 3$. The other implication follows from Theorem 2.4. \qed
3. The Case When \( H \) is Not an Ideal of \( R \)

In this section, we study \( ET_H(R) \), when the multiplicative-prime subset \( H \) is not an ideal of \( R \). Since \( H \) is always closed under multiplication by elements of \( R \), this just means that \( 0 \in H \) and there are distinct \( x, y \in H \) such that \( x + y \in R \setminus H \).

First we begin with the following example that shows \( ET_H(R) \neq GT_H(R) \).

**Example 3.1.** Let \( R = \mathbb{Z} \). Set \( H = 4\mathbb{Z} \cup 6\mathbb{Z} \). It is clear that \( H \) is not an ideal of \( R \) since \( 4, 6 \in H \), but \( 4 + 6 = 10 \notin H \). So \( 4 - 6 \) is not an edge in \( GT_H(R) \). But \( 2(4) + 2(6) = 20 \in H \) and \( 2 \in R \setminus H \). Then \( 4 - 6 \) is an edge in \( ET_H(R) \). Hence \( ET_H(R) \neq GT_H(R) \).

Now, we have the following theorem that shows \( ET_H(H) \) is always connected (but never complete), \( ET_H(H) \) and \( ET_H(R \setminus H) \) are never disjoint subgraphs of \( ET_H(R) \) and \( ET_H(R) \) is connected when \( ET_H(R \setminus H) \) is connected.

**Theorem 3.2.** Let \( R \) be a commutative ring such that \( H \) is a multiplicative-prime subset of \( R \) that is not an ideal of \( R \). Then the following hold:

1. \( ET_H(H) \) is connected with \( diam(ET_H(H)) = 2 \).
2. Some vertex of \( ET_H(H) \) is adjacent to a vertex of \( ET_H(R \setminus H) \). In particular, the subgraphs \( ET_H(H) \) and \( ET_H(R \setminus H) \) are not disjoint.
3. If \( ET_H(R \setminus H) \) is connected, then \( ET_H(R) \) is connected.

**Proof.**
1. Let \( x \in H^* = H \setminus \{0\} \). Then \( x \) is adjacent to 0. Thus \( x - 0 - x' \) is a path in \( ET_H(H) \) of length two between any two distinct \( x, x' \in H^* \). Moreover, there exist nonadjacent \( x, x' \in H^* \) since \( H \) is not an ideal of \( R \); thus \( diam(ET_H(H)) = 2 \).
2. Since \( H \) is not an ideal of \( R \), there exist distinct \( x, y \in H^* \) such that \( x + y \notin H \). Then \( -x \in H \) and \( x + y \in H \) are adjacent vertices in \( ET_H(R) \). Finally, the ”in particular” statement is clear.
3. Since \( ET_H(H) \) and \( ET_H(R \setminus H) \) are connected and there is an edge between \( ET_H(H) \) and \( ET_H(R \setminus H) \), so \( ET_H(R) \) is connected. \( \Box \)

We determine when \( ET_H(R) \) is connected and compute \( diam(ET_H(R)) \) with the following theorem. Compare the next theorem with [3, Theorem 3.2].

**Theorem 3.3.** Let \( R \) be a commutative ring such that \( H \) is a multiplicative-prime subset of \( R \) that is not an ideal of \( R \). Then \( ET_H(R) \) is connected if and only if for every \( x \in R \) there exists \( r \in R \setminus H \) such that \( rx \in \langle H \rangle \).

**Proof.** Suppose that \( ET_H(R) \) is connected, and \( x \in R \). Then there exists a path \( 0 - x_1 - x_2 - \ldots - x_n - x \) from 0 to \( x \) in \( ET_H(R) \).
Thus \( r_1x_1, r_2x_1 + r_3x_2, \ldots, r_{2n-2}x_{n-1} + r_{2n-1}x_n, r_{2n}x_n + sx \in H \) for some \( r_1, r_2, \ldots, r_{2n}, s \in R \setminus H \). Then

\[
\begin{align*}
\text{sr}_1r_3r_5^\ldots r_{2n-1}x &= (r_1r_3r_5^\ldots r_{2n-1})(sx + r_{2n}x_n) - \\
&= (r_1r_3r_5^\ldots r_{2n-3}r_{2n})(r_{2n-2}x_{n-1} + r_{2n-1}x_n) + \\
&\quad - (r_1r_3^\ldots r_{2n-2}r_{2n-2k-5}r_{2n-2k-3}r_{2n-2k-2}r_{2n-2k-2}r_{2n})(r_{2n-2k-1}x_n + r_{2n-2k-2}x_n - (k+1)) \\
&\quad + (r_1r_3^\ldots r_{2n-2k-5}r_{2n-2k-3}r_{2n-2k-2}r_{2n-2k-2}r_{2n})(r_{2n-2k-3}x_n - (k+1) + r_{2n-2k-4}x_n - (k+2)) \\
&\quad - (r_2r_4^\ldots r_{2n})(r_1x_1) \in \langle H \rangle.
\end{align*}
\]

Since \( H \) is a multiplicative-prime subset of \( R \), so \( r = sr_1r_3r_5^\ldots r_{2n-1} \in R \setminus H \) and \( rx \in \langle H \rangle \). Conversely, suppose that for every \( x \in R \) there exists \( r \in R \setminus H \) such that \( rx \in \langle H \rangle \). We show that for each \( 0 \neq x \in R \), there exists a path in \( ET_H(R) \) from 0 to \( x \). By assumption, there are elements \( h_1, h_2, \ldots, h_n \in H \) such that \( rx = h_1 + h_2 + \ldots + h_n \). Set \( y_0 = 0 \) and \( y_k = (-1)^{n+k}(h_1 + h_2 + \ldots + h_k) \) for each integer \( k \) with \( 1 \leq k \leq n \). Then \( y_k + y_{k+1} = (-1)^{n+k+1}h_{k+1} \in H \) for each integer \( 1 \leq k \leq n - 1 \). Also, \( y_{n-1} + rx = y_{n-1} + y_n = h_n \in H \). Thus \( 0 - y_1 - y_2 - \ldots - y_{n-1} - x \) is a path from 0 to \( x \) in \( ET_H(R) \). Now, let \( 0 \neq x, y \in R \). Then by the preceding argument, there are paths from \( x \) to 0 and 0 to \( y \) in \( ET_H(R) \). Hence there is a path from \( x \) to \( y \) in \( ET_H(R) \). So \( ET_H(R) \) is connected.

\[\Box\]

**Theorem 3.4.** Let \( R \) be a commutative ring such that \( H \) is a multiplicative-prime subset of \( R \) that is not an ideal of \( R \), and let for every \( x \in R \) there exists \( r \in R \setminus H \) such that \( rx \in \langle H \rangle \). Let \( n \geq 2 \) be the least integer such that \( \langle H \rangle = \langle h_1, h_2, \ldots, h_n \rangle \) for some \( h_1, h_2, \ldots, h_n \in H \). Then \( \text{diam}(ET_H(R)) \leq n \).

**Proof.** Let \( x \) and \( x' \) be distinct elements in \( R \). We show that there exists a path from \( x \) to \( x' \) in \( ET_H(R) \) with length at most \( n \). By hypothesis, \( rx, r'x' \in \langle H \rangle \) for some \( r, r' \in R \setminus H \), so we can write \( rx = \sum_{i=1}^n r_ih_i \) and \( r'x' = \sum_{i=1}^n s_ih_i \) for some \( r_i, s_i \in R \). Define \( x_0 = x \) and \( x_k = (-1)^k(\sum_{i=k+1}^n r_ih_i + \sum_{i=1}^k s_ih_i) \), so \( x_k + x_{k+1} = (-1)^k h_{k+1}(r_{k+1} - s_{k+1}) \in H \) for each integer \( k \) with \( 1 \leq k \leq n - 1 \). On the other hand, \( rx + x_1 = (r_1 - s_1)h_1 \in H \) and \( r'x' + (-1)^nx_{n-1} = (s_n - r_n)h_n \in H \). So \( x - x_1 - x_2 - \ldots - x_{n-1} - x' \) is a path from \( x \) to \( x' \) in \( ET_H(R) \) with length at most \( n \) since \( 1, (-1)^n \notin H \). \[\Box\]

We end the paper with the following theorem.

**Theorem 3.5.** Let \( R \) be a commutative ring such that \( H \) is a multiplicative-prime subset of \( R \) that is not an ideal of \( R \). Then the following hold:

1. Either \( \text{gr}(ET_H(H)) = 3 \) or \( \text{gr}(ET_H(H)) = \infty \).
2. If \( \text{gr}(ET_H(R)) = 4 \), then \( \text{gr}(ET_H(H)) = \infty \).
Proof. (1) If \(rx + sx' \in H\) for some distinct \(x, x' \in H\) and \(r, s \in R \setminus H\), then \(-x - x' - 0\) is a cycle of length 3 in \(ET_H(H)\), so \(gr(ET_H(H)) = 3\). Otherwise, \(rx + sx' \in R \setminus H\) for all distinct \(x, x' \in H\) and all elements \(r, s \in R \setminus H\). Therefore in this case, each nonzero element \(x \in H\) is adjacent to 0, and no two distinct \(x, x' \in H\) are adjacent. Thus \(gr(ET_H(H)) = \infty\).

(2) If \(gr(ET_H(R)) = 4\), then it is clear \(gr(ET_H(H)) \neq 3\). So \(gr(ET_H(H)) = \infty\) by part (1) above.

\[\square\]

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