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A NOTE ON THE EXTENDED TOTAL GRAPH OF COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring and H a nonempty proper subset of R. In this paper, the extended total graph, denoted by $ET_H(R)$ is presented, where H is a multiplicative-prime subset of R. It is the graph with all elements of R as vertices, and for distinct $p, q \in R$, the vertices p and q are adjacent if and only if $rp+sq \in H$ for some $r, s \in R \setminus H$. We also study the two (induced) subgraphs $ET_H(H)$ and $ET_H(R \setminus H)$, with vertices H and $R \setminus H$, respectively. Among other things, the diameter and the girth of $ET_H(R)$ are also studied.

1. Introduction

Throughout this paper R is a commutative ring with nonzero identity. Recently, there has been considerable attention in the work to associating graphs with algebraic structures (see [4],[5],[6], [8], [9] and [10]). Anderson and Badawi in [3] defined a nonempty proper subset H of R to be a *multiplicative-prime* subset of R if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $rs \in H$ for some $r, s \in R$, then either $r \in H$ or $s \in H$. They introduced the notion of the generalized total graph of a commutative ring $GT_H(R)$ with the vertices of this graph are all elements of R and two vertices $x, y \in R$ are adjacent if and only if $x + y \in H$ where H is a *multiplicative-prime* subset of R. In this paper, we introduce an extension of the graph $GT_H(R)$, denoted by $ET_H(R)$, such that its vertex set consist of all

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elements of R and for distinct $p, q \in R$, the vertices p and q are adjacent if and only if $rp + sq \in H$ for some $r, s \in R \setminus H$, where H is a *multiplicative-prime* subset of R. Let $ET_H(H)$ be the (induced) subgraph of $ET_H(R)$ with vertex set H, and let $ET_H(R \setminus H)$ be the (induced) subgraph $ET_H(R)$ with vertices consisting of $R \setminus H$.
Obviously, the total graph $GT_H(R)$ is a subgraph of $ET_H(R)$. It follows that each edge (path) of $GT_H(R)$ is an edge (path) of $ET_H(R)$. The study of $ET_H(R)$ breaks naturally into two cases depending on whether or not H is an ideal of R. In the second section, , we handle the case when H is an ideal of R; in the third section, we do the case when H is not an ideal of R. For every case, we characterize the girths and diameters of $ET_H(R)$, $ET_H(H)$ and $ET_H(R \setminus H)$.

We begin with some notation and definitions. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we mean the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two of it's distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$. We also define $d(a, a) = 0$. The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). For a graph Γ, the degree of a vertex v in Γ, denoted $deg(v)$, is the number of edges of Γ incident with v. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertices of Γ_1 is adjacent(in Γ) to some vertex of Γ_2.

2. The case when H is an ideal of R

In this section, we study the case when H is a an ideal of R. It is clear that if H an ideal of R, then H is a prime ideal of R. If $H = R$, then it is clear that $ET_H(R)$ is a complete graph and $ET_H(R)$ is disconnected when $H = 0$ and $|R| \geq 2$. So we may assume that $H \neq 0$ and $H \neq R$.
First we begin with the following example that shows $ET_H(R) \neq GT_H(R)$.

Example 2.1. Let $R = \mathbb{Z}_8$. Set $H = \{\bar{0}, \bar{4}\}$. It is clear that H is an ideal of R. Since $\bar{7} + \bar{3} = \bar{2} \notin H$, so $\bar{7} - \bar{3}$ is not an edge in $GT_H(R)$. But $\bar{1}(\bar{7}) + \bar{3}(\bar{3}) = \bar{0} \in H$ and $\bar{1}, \bar{3} \in R \setminus H$. Then $\bar{7} - \bar{3}$ is an edge in $ET_H(R)$. Hence $ET_H(R) \neq GT_H(R)$.

The main goal of this section is a general structure theorem (Theorem 2.4) for $ET_H(R \setminus H)$ when H is an ideal of R. But first, we record the trivial observation that H is an ideal of R, the $ET_H(H)$ is a complete subgraph of $ET_H(R)$ and is disjoint from $ET_H(R \setminus H)$. Thus we will concentrate on the subgraph $ET_H(R \setminus H)$ throughout this section.

Theorem 2.2. *Let R be a commutative ring and H be a prime ideal of R. Then $ET_H(H)$ is a complete subgraph of $ET_H(R)$ and is disjoint from $ET_H(R \setminus H)$. In particular, $ET_H(H)$ is connected and $ET_H(R)$ is disconnected.*

Proof. Let $p, q \in H$. Then it is clear that $p + q \in H$ since H is an ideal of R. If $x \in H$ is adjacent to $y \in R \setminus H$, then $rx + sy \in H$ for some $r, s \in R \setminus H$. This implies that $sy \in H$, so either $y \in H$ or $s \in H$ since H is a prime ideal which is a contradiction. The "in particular" state is clear. □

Theorem 2.3. *Let R be a commutative ring and H be a prime ideal of R. Then the following hold:*
(1) Suppose that G is an induced subgraph of $ET_H(R \setminus H)$ and let x and y be distinct vertices of G that are connected by a path in G. Then there exists a path in G of length 2 between x and y. In particular, if $ET_H(R \setminus H)$ is connected, then $diam(ET_H(R \setminus H)) \leq 2$.
(2) Let x and y be distinct elements of $ET_H(R \setminus H)$ that are connected by a path. If $x + y \notin H$ then $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $ET_H(R \setminus H)$.

Proof. (1) Let x_1, x_2, x_3 and x_4 are distinct vertices of G. It suffices to show that if there is a path $x_1 - x_2 - x_3 - x_4$ from x_1 to x_4, then x_1 and x_4 are adjacent. Now, $r_1x_1 + r_2x_2, r'_2x_2 + r'_3x_3, r_3x_3 + r_4x_4 \in H$ for some $r_1, r_2, r_3, r_4, r'_2, r'_3 \in R \setminus H$. Hence $(r_1r_3r'_2)x_1 + (r_2r'_3r_4)x_4 = r_3r'_2(r_1x_1 + r_2x_2) - r_2r_3(r'_2x_2 + r'_3x_3) + r_2r'_3(r_3x_3 + r_4x_4) \in H$. Since H is a prime ideal of R, so $r_1r_3r'_2, r_2r'_3r_4 \notin H$. Then x_1 and x_4 are adjacent. So if $ET_H(R \setminus H)$ is connected, then $diam(ET_H(R \setminus H)) \leq 2$.
(2) Since $x, y \in R \setminus H$ and $x + y \notin H$, there exists $z \in R \setminus H$ such that $x - z - y$ is a path of length 2 by part (1) above. So $rx + sz, s'z + r'y \in H$ for some $r, s, r', s' \in R \setminus H$. Therefore $rs', r's \notin H$ since H is a prime ideal of R. Then $rs'x - r'sy = s'(rx + sz) - s(s'z + r'y) \in H$ and x is adjacent to y. So $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $ET_H(R \setminus H)$. □

Now, we give the main theorem of this section. Since $ET_H(H)$ is a complete subgraph of $ET_H(R)$ by Theorem 2.2, the next theorem gives a complete description of $ET_H(R \setminus H)$. Let $|H| = \alpha$. We allow α to be infinite cardinal. Compare the next theorem with [3, Theorem 2.2].

Theorem 2.4. *Let R be a commutative ring and H be a prime ideal of R and let $|H| = \alpha$.*
(1) If $r + s \in H$ for some $r, s \in R \setminus H$, then $ET_H(R \setminus H)$ is the union of complete subgraphs.
(2) If $r + s \notin H$ for all $r, s \in R \setminus H$, then $ET_H(R \setminus H)$ is the union of totally disconnected subgraphs and some connected subgraphs.

Proof. (1) Suppose that $r + s \in H$ for some $r, s \in R \setminus H$. For $x, x' \in R \setminus H$, we write $x \sim x'$ if and only if $tx + t'x' \in H$ and $t + t' \in H$ for some $t, t' \in R \setminus H$. It is straightforward to check that $\sim$ is an equivalence relation on $R \setminus H$, since H is a prime ideal. For $x \in R \setminus H$, we denote the equivalence class which contains x by $[x]$. Now let $x \in R \setminus H$. If $[x] = \{x\}$, then $r(x + h_1) + s(x + h_2) = (r + s)x + rh_1 + sh_2 \in H$ for every $h_1, h_2 \in H$ since $r + s \in H$. Then $x + H$ is a complete subgraph of $ET_H(R \setminus H)$ with at most α vertices. Now let $|[x]| = \nu$ and $x' \in [x]$. Then $tx + t'x' \in H$ and $t + t' \in H$ for some $t, t' \in R \setminus H$. So $t(x+h_1)+t'(x'+h_2) = tx+t'x'+th_1+t'h_2 \in H$ for every $h_1, h_2 \in H$. Thus $x + H$ is a part of complete graph k^{μ} where $\mu \leq \alpha\nu$.
(2) Assume that $r + s \notin H$ for all $r, s \in R \setminus H$. Set

$$A_x = \{x' \in R \setminus H : rx + sx' \in H \ for\ some\ r, s \in R \setminus H\}$$

be the set of all adjacent vertices to x. If $A_x = \emptyset$, then $px + qx' \notin H$ for every $x' \in R \setminus H$ and every $p, q \in R \setminus H$. In this case, we show that $x + H$ is a totally disconnected subgraph of $ET_H(R \setminus H)$. If $r(x + x_1) + s(x + x_2) \in H$ for some $r, s \in R \setminus H$ and $x_1, x_2 \in H$, then $(r + s)x \in H$. Since H is a prime ideal of R and $x \notin H$, then $r + s \in H$ which is a contradiction. Therefore $x + H$ is a totally disconnected subgraph of $ET_H(R \setminus H)$. Now, we may assume that $A_x \neq \emptyset$. Then $rx+sx' \in H$ for some $r, s \in R \setminus H$ and $x' \in R \setminus H$. Hence $r(x + h_1) + s(x' + h_2) = rx + sx' + rh_1 + sh_2 \in H$ for every $h_1, h_2 \in H$; hence each element of $x + H$ is adjacent to each element of $x' + H$. If $| A_x |= \nu$, then we have a connected subgraph of $ET_H(R \setminus H)$ with at most $\alpha\nu$ vertices. So $ET_H(R \setminus H)$ is the union of totally disconnected subgraphs and some connected subgraphs. □

Now it is easy to compute the girth of $ET_H(R \setminus H)$ using Theorem 2.4.

Theorem 2.5. *Let R be a commutative ring and H be a prime ideal of R. Then $gr(ET_H(R \setminus H)) = 3, 4$ or ∞. In particular, $gr(ET_H(R \setminus H)) \leq 4$ if $ET_H(R \setminus H)$ contains a cycle.*

Proof. Let $ET_H(R \setminus H)$ contains a cycle. Then $ET_H(R \setminus H)$ is not a totally disconnected graph, so by the proof of Theorem 2.4, $ET_H(R\setminus H)$ has either a complete or a complete bipartite subgraph. Therefore it must contain either a 3-cycle or 4-cycle. Thus $gr(ET_H(R\setminus H)) \leq 4$. □

Theorem 2.6. *Let R be a commutative ring and H be a prime ideal of R.*
(1) $gr(ET_H(R \setminus H)) = 3$ if and only if $r + s \in H$ and $|y + H| \geq 3$ for some $r, s \in R \setminus H$ and $y \in R \setminus H$.
(2) $gr(ET_H(R \setminus H)) = 4$ if and only if $r + s \notin H$ for every $r, s \in R \setminus H$ and $px + qx' \in H$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$.
(3) Otherwise, $gr(ET_H(R \setminus H)) = \infty$.

Proof. (1) Assume that $gr(ET_H(R \setminus H)) = 3$. Then by Theorem 2.4, $ET_H(R \setminus H)$ is a complete graph K^λ where $\lambda \geq 3$. Then $r + s \in H$ for some $r, s \in R \setminus H$ and $|y + H| \geq 3$ for some $y \in R \setminus H$ by Theorem 2.4.
(2) If $gr(ET_H(R \setminus H)) = 4$, then $ET_H(R \setminus H)$ has a complete bipartite subgraph. So $r + s \notin H$ for every $r, s \in R \setminus H$ and $px + qx' \in H$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$ by Theorem 2.4.
The other implications of (1) and (2) follow directly from Theorem 2.4. □

We end this section with the following theorem.

Theorem 2.7. *Let R be a commutative ring and H be a prime ideal of R.*
(1) $gr(ET_H(R)) = 3$ if and only if $|H| \geq 3$.
(2) $gr(ET_H(R)) = 4$ if and only if $r + s \notin H$ for every $r, s \in R \setminus H$, $|H| < 3$ and $px + qx' \in U$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$.
(3) Otherwise, $gr(ET_H(R)) = \infty$.

Proof. (1) This follows from Theorem 2.2.
(2) Assume that $gr(ET_H(R)) = 4$. Since $gr(ET_H(H)) = 3$ or ∞, then $gr(ET_H(R \setminus H)) = 4$. Therefore $r + s \notin H$ for every $r, s \in R \setminus H$ and $px + qx' \in H$ for some $x, x' \in R \setminus H$ and $p, q \in R \setminus H$ by Theorem 2.6. On the other hand, $gr(ET_H(R)) \neq 3$; so $|H| < 3$. The other implication follows from Theorem 2.4. □

3. The case when H is not an ideal of R

In this section, we study $ET_H(R)$, when the multiplicative-prime subset H is not an ideal of R. Since H is always closed under multiplication by elements of R, this just means that $0 \in H$ and there are distinct $x, y \in H$ such that $x + y \in R \setminus H$.
First we begin with the following example that shows $ET_H(R) \neq GT_H(R)$.

Example 3.1. Let $R = \mathbb{Z}$. Set $H = 4\mathbb{Z} \cup 6\mathbb{Z}$. It is clear that H is not an ideal of R since $4, 6 \in H$, but $4 + 6 = 10 \notin H$. So $4 - 6$ is not an edge in $GT_H(R)$. But $2(4) + 2(6) = 20 \in H$ and $2 \in R \setminus H$. Then $4 - 6$ is an edge in $ET_H(R)$. Hence $ET_H(R) \neq GT_H(R)$.

Now, we have the following theorem that shows $ET_H(H)$ is always connected (but never complete), $ET_H(H)$ and $ET_H(R \setminus H)$ are never disjoint subgraphs of $ET_H(R)$ and $ET_H(R)$ is connected when $ET_H(R \setminus H)$ is connected.

Theorem 3.2. *Let R be a commutative ring such that H is a multiplicative-prime subset of R that is not an ideal of R. Then the following hold:*
(1) $ET_H(H)$ is connected with $diam(ET_H(H)) = 2$.
(2) Some vertex of $ET_H(H)$ is adjacent to a vertex of $ET_H(R \setminus H)$. In particular, the subgraphs $ET_H(H)$ and $ET_H(R \setminus H)$ are not disjoint.
(3) If $ET_H(R \setminus H)$ is connected, then $ET_H(R)$ is connected.

Proof. (1) Let $x \in H^* = H \setminus \{0\}$. Then x is adjacent to 0. Thus $x - 0 - x'$ is a path in $ET_H(H)$ of length two between any two distinct $x, x' \in H^*$. Moreover, there exist nonadjacent $x, x' \in H^*$ since H is not an ideal of R; thus $diam(ET_H(H)) = 2$.
(2) Since H is not an ideal of R, there exist distinct $x, y \in H^*$ such that $x + y \notin H$. Then $-x \in H$ and $x + y \in H$ are adjacent vertices in $ET_H(R)$. Finally, the "in particular" statement is clear.
(3) Since $ET_H(H)$ and $ET_H(R \setminus H)$ are connected and there is an edge between $ET_H(H)$ and $ET_H(R \setminus H)$, so $ET_H(R)$ is connected. □

We determine when $ET_H(R)$ is connected and compute $diam(ET_H(R))$ with the following theorem. Compare the next theorem with [3, Theorem 3.2].

Theorem 3.3. *Let R be a commutative ring such that H is a multiplicative-prime subset of R that is not an ideal of R. Then $ET_H(R)$ is connected if and only if for every $x \in R$ there exists $r \in R \setminus H$ such that $rx \in \langle H \rangle$.*

Proof. Suppose that $ET_H(R)$ is connected, and $x \in R$. Then there exists a path $0 - x_1 - x_2 - \ldots - x_n - x$ from 0 to x in $ET_H(R)$.

Thus $r_1x_1, r_2x_1+r_3x_2, ..., r_{2n-2}x_{n-1}+r_{2n-1}x_n, r_{2n}x_n+sx \in H$ for some $r_1, r_2, ..., r_{2n}, s \in R \setminus H$. Then

$$sr_1r_3r_5...r_{2n-1}x = (r_1r_3r_5...r_{2n-1})(sx + r_{2n}x_n) -$$
$$(r_1r_3r_5...r_{2n-3}r_{2n})(r_{2n-2}x_{n-1} + r_{2n-1}x_n) + ...$$
$$-(r_1r_3...r_{2n-2k-5}r_{2n-2k-3}r_{2n-2k}r_{2n-2k-2}...r_{2n})(r_{2n-2k-1}x_{n-k}+r_{2n-2k-2}x_{n-(k+1)})$$
$$+(r_1r_3...r_{2n-2k-5}r_{2n-2k-2}r_{2n-2k}r_{2n-2k-2}...r_{2n})(r_{2n-2k-3}x_{n-(k+1)}+r_{2n-2k-4}x_{n-(k+2)})$$
$$... - (r_2r_4r_6...r_{2n})(r_1x_1) \in \langle H\rangle$$

Since H is a *multiplicative-prime* subset of R, so $r = sr_1r_3r_5...r_{2n-1} \in R \setminus H$ and $rx \in \langle H\rangle$. Conversely, suppose that for every $x \in R$ there exists $r \in R\setminus H$ such that $rx \in \langle H\rangle$. We show that for each $0 \neq x \in R$, there exists a path in $ET_H(R)$ from 0 to x. By assumption, there are elements $h_1, h_2, ..., h_n \in H$ such that $rx = h_1 + h_2 + ... + h_n$. Set $y_0 = 0$ and $y_k = (-1)^{n+k}(h_1 + h_2 + ... + h_k)$ for each integer k with $1 \leq k \leq n$. Then $y_k + y_{k+1} = (-1)^{n+k+1}h_{k+1} \in H$ for each integer $1 \leq k \leq n-1$. Also, $y_{n-1}+rx = y_{n-1}+y_n = h_n \in H$. Thus $0-y_1-y_2-...-y_{n-1}-x$ is a path from 0 to x in $ET_H(R)$. Now, let $0 \neq x, y \in R$. Then by the preceding argument, there are paths from x to 0 and 0 to y in $ET_H(R)$. Hence there is a path from x to y in $ET_H(R)$. So $ET_H(R)$ is connected. □

Theorem 3.4. *Let R be a commutative ring such that H is a multiplicative-prime subset of R that is not an ideal of R, and let for every $x \in R$ there exists $r \in R \setminus H$ such that $rx \in \langle H\rangle$. Let $n \geq 2$ be the least integer such that $\langle H\rangle =< h_1, h_2, ..., h_n >$ for some $h_1, h_2, ..., h_n \in H$. Then $diam(ET_H(R)) \leq n$.*

Proof. Let x and x' be distinct elements in R. We show that there exists a path from x to x' in $ET_H(R)$ with length at most n. By hypothesis, $rx, r'x' \in \langle H\rangle$ for some $r, r' \in R \setminus H$, so we can write $rx = \sum_{i=1}^n r_ih_i$ and $r'x' = \sum_{i=1}^n s_ih_i$ for some $r_i, s_i \in R$. Define $x_0 = x$ and $x_k = (-1)^k(\sum_{i=k+1}^n r_ih_i + \sum_{i=1}^k s_ih_i)$, so $x_k + x_{k+1} = (-1)^k h_{k+1}(r_{k+1} - s_{k+1}) \in H$ for each integer k with $1 \leq k \leq n-1$. On the other hand, $rx + x_1 = (r_1 - s_1)h_1 \in H$ and $r'x' + (-1)^n x_{n-1} = (s_n - r_n)h_n \in H$. So $x - x_1 - x_2 - ... - x_{n-1} - x'$ is a path from x to x' in $ET_H(R)$ with length at most n since $1, (-1)^n \notin H$. □

We end the paper with the following theorem.

Theorem 3.5. *Let R be a commutative ring such that H is a multiplicative-prime subset of R that is not an ideal of R. Then the following hold:*
(1) Either $gr(ET_H(H)) = 3$ or $gr(ET_H(H)) = \infty$.
(2) If $gr(ET_H(R)) = 4$, then $gr(ET_H(H)) = \infty$.

Proof. (1) If $rx+sx' \in H$ for some distinct $x, x' \in H$ and $r, s \in R \setminus H$, then $0-x-x'-0$ is a cycle of length 3 in $ET_H(H)$, so $gr(ET_H(H)) = 3$. Otherwise $rx+sx' \in R \setminus H$ for all distinct $x, x' \in H$ and all elements $r, s \in R \setminus H$. Therefore in this case, each nonzero element $x \in H$ is adjacent to 0, and no two distinct $x, x' \in H$ are adjacent. Thus $gr(ET_H(H)) = \infty$.
(2) If $gr(ET_H(R)) = 4$, then it is clear $gr(ET_H(H)) \neq 3$. So $gr(ET_H(H)) = \infty$ by part (1) above. □

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