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ESSENTIAL SUBHYPERMODULES AND THEIR PROPERTIES

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ABSTRACT. Let R be a hyperring (in the sense of [8]) and M be a hypermodule on R. In this paper we will introduce and study a class of subhypermodules of M. We will study the intersection of this kind of subhypermodules and give some suitable results about them. We will proceed to give some interesting results about the complements, direct sums and independency of this kind of subhypermodules.

1. INTRODUCTION

The concepts of *hypergroups*, *hyperrings* and *hypermodules* and some generalizations of them were introduced by some authors, see for examples [1, 3, 4, 5, 6, 7, 8, 11, 12].

A *hyperstructure* is a nonvoid set H together with a function $. : H \times H \longrightarrow P^*(H)$, where . is called a hyperoperation and $P^*(H)$ is the set of all nonempty subsets of H.

For $A, B \subseteq H$ and $x \in H$ we define

$$A.B = \bigcup_{a\in A, b\in B} a.b, \quad x.B = \{x\}.B, \quad A.x = A.\{x\}.$$

Definition 1.1. A hyperstructure H with a hyperoperation "+" is called a *canonical hypergroup* if the following hold for H;

(i) $(x + y) + z = x + (y + z)$ for all $x, y, z \in H$;

(ii) $x + y = y + x$ for all $x, y \in H$;

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(iii) There exists an element 0 such that $0+x=\{x\}$, where $\{x\}$ is the single ton set with element x, for every $x \in H$;

(iv) For each $x \in H$ there exists a unique element $x' \in H$, such that $0 \in x+x'$. (we denote x' by $-x$ and it is called the opposite of x). Also we write $x-y$ instead of $x+(-y)$;

(v) If $z \in x+y$, then $y \in z-x$ for all x, y, z in H.

Note that 0 is unique and for every $x \in H$ we have $x+0=0+x=\{x\}$. We identify a singleton set $\{x\}$ by x.
A nonempty subset G of a canonical hypergroup H is called a *canonical subhypergroup* of H if G is a canonical hypergroup.
Canonical hypergroups were studied by J. Mittas in [11].

A non-void set R with a hyperoperation "+" and with a binary operation "." is called a *hyperring* if

(R_1) $(R,+)$ is a canonical hypergroup;

(R_2) $(R,.)$ is a multiplicative semigroup having 0, suchthat $x.0=0.x=0$ for all $x \in R$;
(R_3) $z.(x+y)=z.x+z.y$ and $(x+y).z=x.z+y.z$ for all $x, y, z \in R$.
If the operation "." in (R_2) is commutative, then R is called a *commutative hyperring*.

For more details about the theory of hyperrings see [5, 6].

2. Hypermodules

Throughout this paper R is a commutative hyperring and all related hypermodules are R–hypermodules.

Definition 2.1. ([7]) A left *hypermodule* over a unitary hyperring R is a canonical hypergroup $(M,+)$ together with an external composition $. : R \times M \longrightarrow M$, denoted by $(r,m) \mapsto rm \in M$, such that for all $(x,y) \in M^2$ and all $(r,s) \in R^2$, the following hold;

(M_1) $r.(x+y)=r.x+r.y$;
(M_2) $(r+s).x=r.x+s.x$;
(M_3) $(rs).x=r.(s.x)$;
(M_4) $1.m=m$ and $0.m=0$, for each $m \in M$.

A nonempty subset N of a hypermodule M is called a *subhypermodule* of M if $(N,+)$ is a canonical subhypergroup of $(M,+)$ and N is a hypermodule over R, under external composition "." from $R\times N$ to N. By $N\leq M$, we mean N is a subhypermodule of M.

Lemma 2.2. *Let M be a hypermodule and N be a nonvoid subset of M. Then N is a subhypermodule of M if and only if for every $x,y\in N$ and $r\in R$ we have $rx+y\subseteq N$.*

Proof. Suppose that N is a subhypermodule of M. Let $x,y\in N$ and $r\in R$. Since N is a hypermodule over R, so $rx\in N$. Also since $(N,+)$ is a subhypergroup of $(M,+)$, $rx+y\subseteq N$.
For converse it suffices to show that $(N,+)$ is a subhypergroup of $(M,+)$. This follows from the fact that $rx+y\subseteq N$ for every $x,y\in N$ and every $r\in R$. □

We refer to [2] for more information about hypermodules and subhypermodules.

Let M,N be two R–hypermodules. A function $f:M\longrightarrow N$ is called a *homomorphism* if for every $x,y\in M$ and every $r\in R$ the following hold

(1) $f(x+y)=f(x)+f(y)$;
(2) $f(rx)=rf(x)$,

and f is called a *weak homomorphism* if

(1) $f(x+y)\subseteq f(x)+f(y)$;
(2) $f(rx)=rf(x)$.

A homomorphism f is called a *monomorphism (monic)*, if f is one to one. f is called an *epimorphism (epic)*, if f is onto and f is called *isomorphism* if it is epic and monic.

For two hypermodules M,N and a homomorphism $f:M\longrightarrow N$, it is easy to see that $f(0)=0$.

Proposition 2.3. *Let M,N,K be R–hypermodules, $f:M\longrightarrow N$ a homomorphism and $g:M\longrightarrow K$ an epimorphism. Moreover suppose that $Ker(g)\subseteq Ker(f)$. Then there exists a homomorphism $h:K\longrightarrow N$ such that $f=hog$, and the following hold*

(1) *If f is epic, then h is epic.*
(2) *If $Ker(f)=Ker(g)$ then h is monic.*
(3) *If f is epic and $Ker(f)=Ker(g)$, then h is an isomorphism.*

Proof. See [9, Theorem 3.1]. □

Let M be a hypermodule over a hyperring R and $N \leq M$. Consider $M/N = \{m+N \mid m \in M\}$, then M/N becomes a hypermodule over R under hyperoperations defined by $+ : M/N \times M/N \longrightarrow P^*(M/N)$ and $. : R \times M/N \longrightarrow M/N$ such that $m+N+m'+N = \{x+N \mid x \in m+m'\}$ and $r.(m+N) = rm+N$ for $m, m' \in M$ and $r \in R$. Note that $m+N = N$ if and only if $m \in N$.

For a hypermodule M and a subhypermodule N of M there exists a canonical epimorphism $\pi : M \longrightarrow M/N$ defined by $\pi(m) = m+N$ with $Ker(\pi) = N$.

By $R - hmod$, we mean the category of all hypermodules over hyperring R. The following result is an immediate consequence of Proposition 2.3.

Corollary 2.4. *Let M, N be R–hypermodules. If $f : M \longrightarrow N$ is a homomorphism and $K \leq M$. Then*

1. *If $K \subseteq Ker(f)$, then there exists a unique homomorphism $\bar{f} : M/K \longrightarrow N$ such that $\bar{f}(m+K) = f(m)$ for every $m \in M$.*
2. *If f is epic, then $\bar{f}$ is epic.*
3. *If $K = Ker(f)$, then $\bar{f}$ is monic.*
4. *If f is epic and $K = Ker(f)$, then $\bar{f}$ is an isomorphism.*

Proposition 2.5. *Let M, N be R–hypermodules and $f : M \longrightarrow N$ be a homomorphism. Then the following are equivalent:*

1. *f is monic;*
2. *$Ker(f) = 0$;*
3. *For every hypermodule K and for homomorphisms $g, h : K \longrightarrow M$, if $fog = foh$, then $g = h$.*

Proof. See [9, Theorem 3.3]. □

Proposition 2.6. *For two hypermodules M, N and homomorphism $f : M \longrightarrow N$, the following are equivalent:*

1. *f is epic;*
2. *For every hypermodule K and for homomorphisms $g, h : N \longrightarrow K$, if $gof = hof$, then $g = h$.*

Proof. See [9, Theorem 3.4]. □

By Proposition 2.5 and Proposition 2.6, we conclude that monomorphisms (resp. epimorphisms) in R–hmod are homomorphisms which are one to one (resp. onto).

Lemma 2.7. *Let M be an R–hypermodule. If $\{M_\alpha\}_{\alpha \in A}$ is an indexed set of subhypermodules of M and $S \subseteq M$. Then*

(1) $\sum_A M_\alpha = \{t \mid t \in \sum_{F\subseteq A} m_\alpha; for\ some\ finite\ subset\ F \subseteq A\ and\ m_\alpha \in M_\alpha\}$ *is a subhypermodule of* M.
(2) $< S >= \{t \mid t \in \sum_{i=1}^{n} r_i s_i$ *for some* $r_i \in R,\ s_i \in S$ *and* $n \in \mathbb{N}\} = \bigcap\{N \mid N \leq M, N \supseteq S\}$ *is a subhypermodule of* M.
(3) $\sum_A M_\alpha =< \bigcup_A M_\alpha >$.

Proof. 1. Since $0 \in \sum_A M_\alpha$, so $\sum_A M_\alpha \neq \emptyset$. Let $x, y \in \sum_A M_\alpha$ and $r \in R$. So there exist finite subset F, G of A such that $x \in \sum_{F\subseteq A} m_\alpha$ and $y \in \sum_{G\subseteq A} m_\beta$, for some $m_\alpha \in M_\alpha$ and $m_\beta \in M_\beta$. Therefore

$$rx + y \subseteq \sum_{F\subseteq A} rm_\alpha + \sum_{G\subseteq A} m_\beta \subseteq \sum_A M_\alpha$$

Now by Lemma 2.2, $\sum_A M_\alpha$ is a subhypermodule of M.
(2) follows from (1) and (3) follows from (1) and (2). □

3. Essential Subhypermodules

Definition 3.1. Let M be a hypermodule and $N \leq M$, then N is called an *essential subhypermodule* of M (denoted by $N \trianglelefteq M$) if $N \cap K \neq 0$ for all nonzero subhypermodule K of M; or equivalently $N \cap K = 0$ implies that $K = 0$ for every $K \leq M$.

For two hypermodules M, N, a monomorphism $f : M \longrightarrow N$ is called an *essential monomorphism* if $Im(f) \trianglelefteq N$.

Proposition 3.2. *Let* M *be a hypermodule over hyperring* R *and* $K \leq M$. *Then the following are equivalent:*

(1) $K \trianglelefteq M$;
(2) *The inclusion map* $i_K : K \longrightarrow M$ *is an essential monomorphism.*
(3) *For every hypermodule* N *and for every homomorphism* $h : M \longrightarrow N$, $Ker(h) \cap K = 0$ *implies that* $Ker(h) = 0$.

Proof. Straightforward. □

Proposition 3.3. *A monomorphism* $f : L \longrightarrow M$ *in* R*–hmod is essential if and only if for all homomorphism* h *in* R*–hmod, if* hof *is monic, then* h *is monic.*

Proof. Suppose that $f : L \longrightarrow M$ is an essential monomorphism and hof is monic. Then $Im(f) \cap Ker(h) = 0$ and hence $Ker(h) = 0$. So h is monic.

For converse suppose that $Im(f) \cap K = 0$ for $K \leq M$. Now consider the canonical epimorphism $\pi_K : M \longrightarrow M/K$, then clearly $\pi_K of : L \longrightarrow M/K$ is monic. Hence, π_K is monic; i.e. $K = 0$ and so $Im(f) \trianglelefteq M$. □

Proposition 3.4. *Let M be a hypermodule and $K \leq N \leq M$. Then*

(1) $K \trianglelefteq M$ if and only if $K \trianglelefteq N$ and $N \trianglelefteq M$.

(2) $H \cap K \trianglelefteq M$ if and only if $H \trianglelefteq M$ and $K \trianglelefteq M$.

Proof. (1) $\Longrightarrow$: Suppose that $K \cap L = 0$ for subhypermodule L of N, then $L \leq M$ and so $L = 0$. Also, $K \trianglelefteq M$ implies $N \trianglelefteq M$, since $K \leq N$.

$\Longleftarrow$: Suppose that $K \trianglelefteq N$ and $N \trianglelefteq M$. Let L be a subhypermodule of M such that $K \cap L = 0$. Then $K \cap (L \cap N) = 0$, thus $L \cap N = 0$ and so $L = 0$.

(2) $\Longrightarrow$ is clear.

For converse suppose that $(H \cap K) \cap L = 0$. Since $K \trianglelefteq M$, we have $K \cap L = 0$ and similarly $L = 0$. Hence $H \cap K \trianglelefteq M$, as required. □

Proposition 3.5. *Let M, N be R–hypermodules, $X \leq N$ and $f : M \longrightarrow N$ a monomorphism. If $X \trianglelefteq N$, then $f^{-1}(X) \trianglelefteq M$.*

Proof. Suppose that $f^{-1}(X) \cap K = 0$ for subhypermodule K of M. We will show that $X \cap f(K) = 0$. Let $x \in X \cap f(K)$, then $x = f(y)$ for some $y \in K$ and so $y \in f^{-1}(X) \cap K = 0$. Thus $x = f(0) = 0$, that is $X \cap f(K) = 0$. Now $X \trianglelefteq N$, so we have $f(K) = 0$. Since f is monic, this implies $K = 0$. □

Let M be a hypermodule over the hyperring R and $x \in M$. Then it is not difficult to see that $Rx = \{rx \mid r \in R\}$ is a subhypermodule of M.

Lemma 3.6. *Let M be a hypermodule over hyperring R. Then a subhypermodule K of M is essential in M if and only if for every $0 \neq x \in M$, there exists $r \in R$, such that $0 \neq rx \in K$.*

Proof. Obvious. □

Proposition 3.7. *Let M, N be two hypermodules over hyperring R and $g : M \longrightarrow N$ be a monomorphism. Then for each homomorphism $f : K \longrightarrow M$, gof is an essential monomorphism if and only if g and f are essential monomorphisms.*

Proof. Suppose that g is a monomorphism and $gof : K \longrightarrow N$ is an essential monomorphism. Let L be a subhypermodule of M such that $Im(f) \cap L = 0$. Then $g(Im(f) \cap L) = 0$. Now we show that $g(Im(f) \cap L) = g(Im(f)) \cap g(L)$. Clearly $g(Im(f) \cap L) \subseteq g(Im(f)) \cap g(L)$. For converse let $x \in g(Im(f)) \cap g(L)$, then $x = g(f(y)) = g(z)$ for some $y \in K$ and $z \in L$, so $0 \in g(f(y)) - g(z) = g(f(y) - z)$. Thus there exists $a \in f(y) - z$ such that $g(a) = 0$, i.e $a \in Ker(g) = \{0\}$. Hence $0 \in f(y) - z$ and so $f(y) = z$. Therefore, $0 = g(Im(f) \cap L) = g(Im(f)) \cap g(L)$ and hence $Im(gof) \cap g(L) = 0$. Since

gof is an essential monomorphism, this implies $g(L) = 0$ and consequently $L = 0$ (since g is monic). These implies f is an essential monomorphism (since gf is monic clearly f is monic). Finally, since $Im(gof) \subseteq Im(g)$, g must be an essential monomorphism.

For converse suppose that $h : N \longrightarrow P$ is a homomorphism, such that $hogof$ is monic. It suffices show that h is monic. Suppose that $Ker(h) \neq 0$. WE have $Im(g) \cap Ker(h) \neq 0$ and hence $Ker(hog) \neq 0$. Now $Im(f) \cap Ker(hog) \neq 0$, since f is an essential monomorphism. Let $0 \neq x = f(k)$ such that $hog(f(k)) = 0$ for some $k \in K$. Since $hogof$ is monic, we have $k = 0$, a contradiction by $f(k) \neq 0$. Hence $Ker(h) = 0$. $\square$

A sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of hypermodules is called *exact* if f is one to one, g is onto and $Im(f) = Ker(g)$.

Proposition 3.8. *Assume that both rows in the following commutative diagram are exact and moreover γ is monic.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

If f' is an essential monomorphism, then so is f.

Proof. Let $0 \neq K \leq B$. We consider two cases:
Case 1: $\beta(K) \neq 0$; Since $Im(f') \trianglelefteq B'$, in this case we have $\beta(K) \cap Im(f') \neq 0$. So there exists $0 \neq x \in Im(f') = Ker(g')$ such that $x = \beta(k)$ for some $0 \neq k \in K$. Now $0 = g'(x) = g'(\beta(k)) = \gamma(g(k))$. Since γ is monic, $g(k) = 0$. Hence $0 \neq k \in Ker(g) = Im(f)$.
Case 2: $\beta(K) = 0$; therefore in this case we obtain $0 = g'(\beta(K)) = \gamma(g(K))$, so $g(K) = 0$. Thus $K \subseteq Ker(g) = Im(f)$.

In both above cases we obtain f is an essential monomorphism. $\square$

Definition 3.9. Let M be a hypermodule and N a subhypermodule of M. Then the subhypermodule N' of M is called a *complement* of N in M if N' is maximal with the property $N \cap N' = 0$.

By Zorn's Lemma, every subhypermodule of an arbitrary hypermodule has a complement.

We say two subhypermodule K, N of M are *independent*, if $K \cap N = 0$. If N, K are independent then $N + K$ is denoted by $N \oplus K$.

Also a subhypermodule N of M is called a *direct summand* of M if $M = N \oplus N'$ for some $N' \leq M$.

A hypermodule M is called *indecomposable* if whenever $M = M_1 \oplus M_2$, then $M_1 = 0$ or $M_2 = 0$.

Lemma 3.10. *(Modularity Law) Suppose that M is a hypermodule and A, B, C are subhypermodules of M such that $B \leq A$. Then $A \cap (B + C) = B + (A \cap C)$.*

Proof. Clearly $B + (A \cap C) \subseteq A \cap (B + C)$.

For converse let $x \in A \cap (B + C)$. Then $x = a \in b + c$ for some $a \in A, b \in B$ *and* $c \in C$. So we have $c \in a - b \subseteq A$, and hence $c \in A \cap C$. But $x \in b + c \subseteq B + A \cap C$. Thus $A \cap (B + C) \subseteq B + (A \cap C)$. $\square$

The condition $B \subseteq A$ in Lemma 3.10 is necessary. That is in general we have not $A \cap (B + C) = (A \cap B) + (A \cap C)$. For example consider the hypermodule $M = \{(x, y) \mid x, y \in \mathbb{Z}\}$ as $\mathbb{Z}$-hypermodule with trivial hyperoperations and let $A = \{(x, x) \mid x \in \mathbb{Z}\}$, $B = \{(x, 0) \mid x \in \mathbb{Z}\}$ and $C = \{(0, x) \mid x \in \mathbb{Z}\}$. Then $A \cap (B + C) = A$, but $(A \cap B) + (A \cap C) =< (0, 0) >\neq A$.

Proposition 3.11. *Let M be a hypermodule and $N \leq M$. Moreover let N' be a complement of N in M. Then*

(i) $N \oplus N' \trianglelefteq M$.

(ii) $(N \oplus N')/N' \trianglelefteq M/N'$.

Proof. (i) Suppose that $K \leq M$ and $(N \oplus N') \cap K = 0$. We show that $N \cap (N' \oplus K) = 0$ and then we conclude $K = 0$, by maximality of N'. To see $N \cap (N' \oplus K) = 0$, let $x \in N \cap (N' \oplus K)$. Then there exist $n \in N, n' \in N'$ *and* $k \in K$ such that $x = n \in n' + k$. Thus $k \in n - n' \subseteq N \oplus N'$ and so $k \in K \cap (N \oplus N') = 0$. Hence $x = n \in n' + 0 = \{n'\}$; i.e, $x = n = n' \in N \cap N' = 0$. Therefore $N \cap (N' \oplus K) = 0$.

(ii) Suppose that $N' \leq L$ and $(N \oplus N')/N' \cap L/N' = N'$. Then $(N \oplus N') \cap L \leq N'$ and by Lemma 3.10, $N' \oplus (N \cap L) \leq N'$. Hence $N \cap L \leq N'$ and so $L \cap N = 0$. Thus $L = N'$, by maximality of N'. That is $(N \oplus N')/N' \trianglelefteq M/N'$.

$\square$

Lemma 3.12. *Suppose that $M = M_1 \oplus M_2$ is a hypermodule where M_1, M_2 are subhypermodules of M. Then for each $m \in M$ there exist a unique element $m_1 \in M_1$ and a unique element $m_2 \in M_2$ such that $m \in m_1 + m_2$.*

Proof. Obviously, for each $m \in M$ there exist $m_1 \in M_1$ and $m_2 \in M_2$, such that $m \in m_1 + m_2$. Now suppose that $m \in m_1 + m_2$ and $m \in n_1 + n_2$ for some $m_1, n_1 \in M_1$ and $m_2, n_2 \in M_2$. Thus we have $0 \in m - m \subseteq (m_1 + m_2) - (n_1 + n_2) = (m_1 - n_1) + (m_2 - n_2)$ and so there exist $x \in m_1 - n_1 \subseteq M_1$ and

$y \in m_2 - n_2 \subseteq M_2$ such that $0 \in x + y$. Hence $x = -y \in M_1 \cap M_2 =< 0 >$; i.e, $0 \in m_1 - n_1$ and $0 \in m_2 - n_2$ that shows $m_1 = n_1$ and $m_2 = n_2$. $\square$

Proposition 3.13. *Let $M = M_1 \oplus M_2$ be a hypermodule. Then for each $i =$ 1,2, there exists a weak epimorphism $\pi_i : M \longrightarrow M_i$ such that $Im(\pi_i) = M_i$ and $Ker(\pi_i) = M_j$, $j \neq i$.*

Proof. Let $m \in M$. Then there exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m \in m_1 + m_2$. Now define $\pi_1 : M \longrightarrow M_1$ by $\pi_1(m) = m_1$.

By Lemma 3.12, π_1 is well defined.

To see that π_1 is a weak homomorphism, let $m \in M$ so that $m \in m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Let $r \in R$. Then $rm \in r(m_1 + n_1) = rm_1 + rn_1$ and hence $\pi_1(rm) = rm_1 = r\pi_1(m)$.

Now let $n \in M$ so that $n \in n_1 + n_2$ for some $n_1 \in M_1$ and $n_2 \in M_2$. We show that $\pi_1(m + n) \subseteq \pi_1(m) + \pi_1(n)$. Let $x \in \pi_1(m + n)$, so $x = \pi_1(y)$ for some $y \in m + n \subseteq (m_1 + m_2) + (n_1 + n_2) = (m_1 + n_1) + (m_2 + n_2)$. Thus there exist $k_1 \in m_1 + n_1 \subseteq M_1$ and $k_2 \in m_2 + n_2 \subseteq M_2$ such that $y \in k_1 + k_2$. Then we have $x = \pi_1(y) = k_1 \in m_1 + n_1 = \pi_1(m) + \pi_1(n)$; that is $\pi_1(m + n) \subseteq \pi_1(m) + \pi_1(n)$.

Obviously, f is epic.

Finally, let $m \in M$ so that $m \in m_1 + m_2$ $(m_1 \in M_1, m_2 \in M_2)$. By Lemma 3.12, we have $\pi_1(m) = 0$ iff $m_1 = 0$ iff $m \in 0 + m_2 = \{m_2\}$, iff $m = m_2 \in M_2$; i.e, $Ker(\pi_1) = M_2$.

$\square$

Proposition 3.14. *Suppose that M is a hypermodule and $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ such that $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$ if and only if $K_1 \trianglelefteq M_1$ and $K_2 \trianglelefteq M_2$.*

Proof. Suppose that $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$ but K_1 is not essential in M_1. So there exists a nonzero subhypermodule L_1 of M_1 such that $K_1 \cap L_1 = 0$. This implies $(K_1 + K_2) \cap L_1 = 0$ that is a contradiction. To see $(K_1 + K_2) \cap L_1 = 0$, let $x \in L_1 \cap (K_1 + K_2)$. Then there exist $l_1 \in L_1, k_1 \in K_1$ and $k_2 \in K_2$ such that $x = l_1 \in k_1 + k_2$. Thus $k_2 \in l_1 - k_1 \subseteq M_1$; i.e, $k_2 \in M_1 \cap M_2 = 0$ and so $x = l_1 \in k_1 + 0 = \{k_1\}$ that implies $x \in L_1 \cap K_1 = 0$. Similarly $K_2 \trianglelefteq M_2$.

Conversely suppose that $K_1 \trianglelefteq M_1$ and $K_2 \trianglelefteq M_2$. Let $0 \neq x \in M_1 \oplus M_2$, then $x \in m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Since $x \neq 0$, at least one of m_1 and m_1 must be nonzero. We consider two cases :

Case 1. $m_2 = 0$ or $m_1 = 0$: Suppose that $m_2 = 0$, then $m_1 \neq 0$ and by Lemma 3.6, there exists $r_1 \in R$ such that $0 \neq r_1 m_1 \in K_1$. Therefore

$r_1x \in r_1(m_1+0) = \{rm_1\} \neq 0$ so that $0 \neq r_1x \in K_1 \subseteq K_1 \oplus K_2$. Hence $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$ by Lemma 3.6. Similarly we have for $m_2 = 0$.

Case 2. Both m_1 and m_2 are nonzero: By using Lemma 3.6, there exists $0 \neq r_1 \in R$ such that $0 \neq r_1m_1 \in K_1$. If $r_1m_2 \in K_2$, then $r_1x \in r_1(m_1+m_2) = r_1m_1 + r_2m_2 \subseteq K_1 \oplus K_2$ and by independency $r_1x \neq 0$. If $r_1m_2 \notin K_2$, again by Lemma 3.6, there exists $r_2 \in R$ such that $0 \neq r_2r_1m_2 \in K_2$ and hence $0 \neq r_2r_1x \in r_2r_1m_1 + r_2r_1m_2 \subseteq K_1 \oplus K_2$. Thus $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$. □

Proposition 3.15. *Let M be a hypermodule and K an essential subhypermodule of M such that $K = (K \cap M_1) \oplus (K \cap M_2)$, whenever $M = M_1 \oplus M_2$. If K is indecomposable, then so is M.*

Proof. Suppose that $M = M_1 \oplus M_2$ is a decomposition of M. By hypothesis $K = (K \cap M_1) \oplus (K \cap M_2)$. Since K is indecomposable, so we have $K \cap M_1 = 0$ or $K \cap M_2 = 0$ and hence $M_1 = 0$ or $M_2 = 0$, as $K \trianglelefteq M$. That is M is indecomposable. □

Let M be any hypermodule and $\{M_\alpha\}_{\alpha \in A}$ an indexed set of subhypermodules of M. Then $\{M_\alpha\}_{\alpha \in A}$ is called *independent* if $M_\alpha \cap (\sum_{\beta \in A \setminus \{\alpha\}} M_\beta) =< 0 >$ for every $\alpha \in A$. If $\{M_\alpha\}_{\alpha \in A}$ is independent then the sum $\sum_{\alpha \in A} M_\alpha$ is denoted by $\bigoplus_{\alpha \in A} M_\alpha$.

Lemma 3.16. *Let M be any hypermodule and $\{M_\alpha\}_{\alpha \in A}$ an indexed set of subhypermodule of M. Then the following statements are equivalent*
1) $\{M_\alpha\}_{\alpha \in A}$ is independent;
2) $\{M_\alpha\}_{\alpha \in F}$ is independent for every finite subset F of A;
3) $(\{M_\beta\}_{\beta \in B}) \cap (\{M_\gamma\}_{\gamma \in C}) =< 0 >$, for every pair $B, C \subseteq A$ with $B \cap C = 0$.

Proof. Straightforward. □

Proposition 3.17. *Suppose that M is a hypermodule and $\{L_\alpha\}_{\alpha \in A}$ is a set of independent subhypermodules of M. Let $\{M_\alpha\}_{\alpha \in A}$ be a set of subhypermodules of M such that $L_\alpha \trianglelefteq M_\alpha$; for each $\alpha \in A$. Then the following hold*
1) $\{M_\alpha\}_{\alpha \in A}$ is independent.
2) $\bigoplus_{\alpha \in A} L_\alpha \trianglelefteq \bigoplus_{\alpha \in A} M_\alpha$.

Proof. First we show that (1), (2) hold for every finite subset F of A. Let L_1, L_2 be independent and $L_1 \trianglelefteq M_1$, $L_2 \trianglelefteq M_2$. So we have $(L_1 \cap M_2) \cap L_2 = L_1 \cap L_2 = 0$, and hence $L_1 \cap M_2 = 0$ as $L_1 \trianglelefteq M_1$. Moreover, $(M_1 \cap M_2) \cap L_1 \leq L_1 \cap M_2 = 0$ that implies $M_1 \cap M_2 = 0$ as $L_1 \trianglelefteq M_1$. Thus M_1, M_2 are independent. By Proposition 3.14, $L_1 \oplus L_2 \trianglelefteq M_1 \oplus M_2$. By using induction, we see that (1),

(2) hold for every finite subset $F \subseteq A$. Now by Lemma 3.16, $\{M_\alpha\}_{\alpha\in A}$ is independent. To obtaining (2), let $0 \neq x \in \bigoplus_A M_\alpha$. Then $x \in m_{\alpha_1} + m_{\alpha_2} + ... + m_{\alpha_n}$ for some $m_{\alpha i} \in M_{\alpha_i}$ and $n \in \mathbb{N}$, where $\{\alpha_1, \alpha_2, ..., \alpha_n\} \subseteq A$. That is $x \in \bigoplus_{i=1}^n M_{\alpha_i}$. Since $\bigoplus_{i=1}^n L_{\alpha_i} \trianglelefteq \bigoplus_{i=1}^n M_{\alpha_i}$, by Lemma 3.6 there is an $r \in R$ such that $0 \neq rx \in \bigoplus_{i=1}^n L_{\alpha_i} \leq \bigoplus_{\alpha\in A} L_\alpha$. Hence $\bigoplus_{\alpha\in A} L_\alpha \trianglelefteq \bigoplus_{\alpha\in A} M_\alpha$, that completes the proof. □

4. Examples

Example 4.1. Consider $\mathbb{Z}$–hypermodule $\mathbb{Z}$ with trivial hyperoperations. Then every subhypermodule of $\mathbb{Z}$ is essential in $\mathbb{Z}$ and the corresponding inclusion map is an essential monomorphism.

Example 4.2. Let M denote the $\mathbb{Z}$–hypermodule $\mathbb{Z}/6\mathbb{Z}$ with trivial hyperoperations. Then $\mathbb{Z}/6\mathbb{Z} = 2\mathbb{Z}/6\mathbb{Z} \oplus 3\mathbb{Z}/6\mathbb{Z}$ and hence neither $2\mathbb{Z}/6\mathbb{Z}$ nor $3\mathbb{Z}/6\mathbb{Z}$ is essential in $\mathbb{Z}/6Z$.

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