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CLASSICAL ZARISKI TOPOLOGY ON PRIME SPECTRUM OF LATTICE MODULES

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ABSTRACT. Let M be a lattice module over a C-lattice L. Let $Spec^p(M)$ be the collection of all prime elements of M. In this article, we consider a topology on $Spec^p(M)$, called the classical Zariski topology and investigate the topological properties of $Spec^p(M)$ and the algebraic properties of M. We investigate this topological space from the point of view of spectral spaces. By Hochster's characterization of a spectral space, we show that for each lattice module M with finite spectrum, $Spec^p(M)$ is a spectral space. Also we introduce finer patch topology on $Spec^p(M)$ and we show that $Spec^p(M)$ with finer patch topology is a compact space and every irreducible closed subset of $Spec^p(M)$ (with classical Zariski topology) has a generic point and $Spec^p(M)$ is a spectral space, for a lattice module M which has ascending chain condition on prime radical elements.

1. INTRODUCTION

A lattice L is said to be *complete*, if for any subset S of L, we have $\vee S, \wedge S \in L$. A complete lattice L is said to be a *multiplicative lattice*, if there is defined a binary operation "." called multiplication on L satisfying the following conditions:
(1) $a.b = b.a$, for all $a, b \in L$;
(2) $a.(b.c) = (a.b).c$, for all $a, b, c \in L$;
(3) $a.(\vee_\alpha b_\alpha) = \vee_\alpha (a.b_\alpha)$, for all $a, b_\alpha \in L$;

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(4) $a.1 = a$, for all $a \in L$.
Henceforth, $a.b$ will be simply denoted by ab.

For $a, b \in L$, we write $(a : b) = \vee\{x \in L | bx \leq a\}$. An element a in L is called compact if $a \leq \vee_{\alpha \in I} b_\alpha$ (I is an indexed set) implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \cdots \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I. By a *C-lattice*, we mean a multiplicative lattice L, with least element 0_L and greatest element 1_L which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset C of compact elements of L.

An element $a \in L$ is said to be *proper*, if $a < 1$. A proper element p of a multiplicative lattice L is said to be *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for $a, b \in L$. The collection of all prime elements of L is denoted by $Spec(L)$.

The Zariski topology on the set $Spec(L)$ of all prime elements in multiplicative lattices is being studied in [20] by Thakare, Manjarekar and Maeda and in [21], by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.
A proper element m of a multiplicative lattice L is said to be *maximal* if for every $x \in L$ with $m < x \leq 1_L$ implies $x = 1_L$.

A complete lattice M is said to be a *lattice module* over the multiplicative lattice L, or L-module, if there is a multiplication between elements of M and L, denoted by $aN \in M$, for $a \in L$ and $N \in M$, which satisfies the following properties:

(1) $(ab)N = a(bN)$;
(2) $(\vee_\alpha a_\alpha)(\vee_\beta N_\beta) = (\vee_{\alpha\beta} a_\alpha N_\beta)$;
(3) $1_L N = N$;
(4) $0_L N = 0_M$; for all $a, b, a_\alpha \in L$, and for all $N, N_\beta \in M$.

Let M be a lattice module over a C-lattice L. The greatest element of M will be denoted by 1_M and the smallest element will be denoted by 0_M. For $N \in M, b \in L$, denote $(N : b) = \vee\{K \in M | bK \leq N\}$ and for $A, B \in M, (A : B) = \vee\{x \in L | Bx \leq A\}$. An element $N \in M$ is said to be compact if $N \leq \vee_{\alpha \in I} A_\alpha$ (I is an indexed set) implies $N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I.

An element $N \in M$ is said to be *proper* if $N < 1_M$. A proper element N of a lattice module M is said to be *prime* if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e., $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$. The prime spectrum of a lattice module M is the set of all prime elements of M and it is denoted by $Spec^p(M)$. In [6], Sachin Ballal and Vilas Kharat studied the Zariski topology over $Spec^p(M)$ as a generalization of the results carried out in [[20], [21]]. Also in [11], Fethi Callialp et. al. studied the Zariski topology on $Spec^p(M)$ over multiplicative lattice L.

A non-zero element $N \in M$ is said to be *second*, if for $a \in L$, either $aN = N$ or $aN = 0_M$. The Zariski topology on the second spectrum of lattice modules is studied by Narayan Phadatare et. al. in [19] . An element $N < 1_M$ of M is said to be *maximal* if $N \leq B$ implies either $N = B$ or $B = 1_M, B \in M$. A non-zero element $K \neq 1_M$ of M is said to be *minimal* if $0_M \leq N < K$ implies $N = 0_M, N \in M$. If 1_M is compact, then M has a maximal element by [18] and every maximal element is a prime element by [2].

Let M be a lattice module over a C-lattice L and $Spec^p(M)$ be the prime spectrum of M. For any element N of M, $D(N) = \{P \in Spec^p(M) | N \leq P\}$. Note that $D(0_M) = Spec^p(M)$ and $D(1_M)$ is an empty set. It is easy to see that for any family of elements K_i $(i \in I)$ of M, $\cap_{i \in I} D(K_i) = D(\vee_{i \in I} K_i)$ and $D(N) \cup D(K) \subseteq D(N \wedge K)$. Thus if $\tau(M)$ denotes the collection of all subsets $D(N)$ of $Spec^p(M)$, then $\tau(M)$ contains the empty set and $Spec^p(M)$ and $\tau(M)$ is closed under arbitrary intersections. In general $\tau(M)$ is not closed under finite unions. A lattice module M is called a top lattice module, if $\tau(M)$ is closed under finite unions. In this case, $\tau(M)$ is called the quasi Zariski topology [11].

M. Behboodi and M. R. Haddadi introduced and studied the classical Zariski topology on the set of all prime submodules of modules as a generalization of the Zariski topology of rings in [7] and [8]. H. Ansari-Toroghy et. al. studied various topological properties of set of all prime submodules of a module over a commutative ring in [3] and the second classical Zariski topology on the second spectrum of modules over a commutative ring is introduced and studied by H. Ansari-Toroghy et. al. in [4]. In this paper, we generalize the concepts of submodules studied in [7] and [8] to the lattice modules.

Let M be a lattice module over a C-lattice L. For each element N of M, we define $E(N) = Spec^p(M) - D(N)$ and $\mathcal{E}(M) = \{E(N) | N \in M\}$, then we define topology $\psi(M)$ on $Spec^p(M)$ by the subbasis $\mathcal{E}(M)$ and call it the Classical Zariski topology of M. In fact $\psi(M)$ to be the collection U of all unions of finite intersections of elements of $\mathcal{E}(M)$(see [16]).

Further all these concepts and for more information on multiplicative lattices, lattice modules and topology, the reader may refer ([1],[2],[9],[14]).

2. Classical Zariski topology

Let M be a lattice module over a C-lattice L. An element P of M is called maximal prime if P is a prime element of M and there is no prime element Q of M such that $P \leq Q$.

Proposition 2.1. *Let M be a lattice module over a C-lattice L. Then the following statements are equivalent:*

(1) *For any elements $N_1, N_2 \in M$, $D(N_1) = D(N_2)$ implies that $N_1 = N_2$.*
(2) *Every proper element of M is a meet of prime elements.*

Proof. 1) $\Longrightarrow$ 2) Suppose that N_1 is a proper element of M. Then $D(N_1) \neq \phi$, because if $D(N_1) = \phi = D(1_M)$ and therefore $N_1 = 1_M$ by part (1), a contradiction. Now let $N_2 = \wedge_{P \in D(N_1)} P$. Clearly, by definition $D(N_1) = D(N_2)$ and therefore by part (1), $N_1 = N_2$. Hence $N_1 = N_2 = \wedge_{P \in D(N_1)} P$ is a meet of prime elements.
2) $\Longrightarrow$ 1) Assume that for $N_1, N_2 \in M$, $D(N_1) = D(N_2)$. By (2), $N_1 = \wedge_{P \in D(N_1)} P$ and $N_2 = \wedge_{P \in D(N_2)} P$. Since $D(N_1) = D(N_2)$, $N_1 = \wedge_{P \in D(N_1)} P = \wedge_{P \in D(N_2)} P = N_2$, as required. □

Let X be a topological space and x and y be points in X. We say that x and y can be separated if each lies in an open set which does not contain the other point. X is a T_1-space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if all points of X are closed in X(i.e. given any x in X, the singleton set $\{x\}$ is a closed set). Also X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods. This is why Hausdorff spaces are also called T_2-spaces or separated spaces.
For a lattice module M, $dim^p(M)$ denote the supremum of the length of chains of prime elements of M. Note that, if $Spec^p(M) = \phi$, then $dim^p(M) = -1$.
We obtain a characterization of $Spec^p(M)$ to be T_1-space in the following result.

Theorem 2.2. *Let M be a lattice module over a C-lattice L. Then $Spec^p(M)$ is a T_1-space if and only if $dim^p(M) \leq 0$.*

Proof. Suppose that $Spec^p(M)$ is a T_1-space. If $Spec^p(M) = \phi$, then $dim^p(M) = -1$. If $Spec^p(M) \neq \phi$, then $\{P\}$ is a closed set in $Spec^p(M)$ for $P \in Spec^p(M)$. Now, assume that $P \leq Q$, for P, Q in $Spec^p(M)$. Since $\{P\}$ is closed set, $\{P\} = \cap_{k \in J}(\cup_{l=1}^{n_k} D(N_{kl}))$, $N_{kl} \in M$ and J is an index set, $n_k \in \mathbb{N}$. Therefore, for each $k \in J$, $P \in \cup_{l=1}^{n_k} D(N_{kl})$ and hence there exists $1 \leq s \leq n_k$ such that $P \in D(N_{ks})$ and so $N_{ks} \leq P$. Now $P \leq Q$ and $N_{ks} \leq P$ implies that $N_{ks} \leq Q$, therefore $Q \in D(N_{ks})$

for all $k \in J$ and $1 \leq s \leq n_k$. It follows that, $Q \in \cup_{l=1}^{n_k} D(N_{kl})$ for each $k \in J$. Thus $Q \in \cap_{k\in J}(\cup_{l=1}^{n_k} D(N_{kl})) = \{P\}$. This implies that every prime element of M is maximal. Consequently, $dim^p(M) \leq 0$.
Conversely, assume that $dim^p(M) \leq 0$. If $dim^p(M) = -1$, then $Spec^p(M) = \phi$, *i.e.* $Spec^p(M)$ is a trivial space and hence it is T_1-space. If $dim^p(M) = 0$, then $Spec^p(M) \neq \phi$ and every prime element is maximal. Thus for each $P \in Spec^p(M)$, we have, $D(P) = \{P\}$ and so $\{P\}$ is a closed set in $Spec^p(M)$. Consequently, $Spec^p(M)$ is a T_1-space. □

Proposition 2.3. *Let M be a lattice module over a C-lattice L, then the following statements are equivalent:*

1. *Every proper element of M is a meet of maximal elements and $dim^p(M) = 0$.*
2. *$Spec^p(M)$ is a T_1-space and $D(N_1) = D(N_2)$ implies that $N_1 = N_2$ for any $N_1, N_2 \in M$.*

Proof. 1) $\Longrightarrow$ 2) Since every proper element of M is a meet of maximal elements of M and every maximal element is prime, therefore by Proposition 2.1, $D(N_1) = D(N_2)$ implies that $N_1 = N_2$ for any $N_1, N_2 \in M$. Also, since $dim^p(M) = 0$, by Theorem 2.2, $Spec^p(M)$ is a T_1-space.
2) $\Longrightarrow$ 1) Assume that $Spec^p(M)$ is a T_1-space and $D(N_1) = D(N_2)$ implies that $N_1 = N_2$ for any $N_1, N_2 \in M$. Therefore every proper element is a meet of prime elements, by Proposition 2.1 and every prime element is maximal, because $Spec^p(M)$ is a T_1-space. Hence every proper element is meet of maximal elements and $dim^p(M) = 0$. □

The cofinite topology(or finite complement topology) is a topology which can be defined on every set X. It has precisely the empty set and all cofinite subsets of X as open sets. As a consequence, in the cofinite topology, the only closed subsets are finite sets or the whole of X [5].
Now, we have characterization of $Spec^p(M)$ to be the cofinite topology.

Theorem 2.4. *Let M be a lattice module over a C-lattice L. Then the following statements are equivalent:*

1. *$Spec^p(M)$ is the cofinite topology.*
2. *$dim^p(M) \leq 0$ and for each element N of M either $D(N) = Spec^p(M)$ or $D(N)$ is finite.*

Proof. 1) $\Longrightarrow$ 2) Suppose that $Spec^p(M)$ is the cofinite topology. Since every cofinite topology satisfies the T_1-axiom, by Theorem 2.2, we have, $dim^p(M) \leq 0$. Suppose that there exists an element N of M such that it is contained in infinite number of prime elements of M, *i.e.*, $|D(N)| = \infty$ and $D(N) \neq Spec^p(M)$. Then $E(N) =$

$Spec^p(M) - D(N)$ is an open set in $Spec^p(M)$ with infinite complement, a contradiction.
2) $\Longrightarrow$ 1) Suppose that $dim^p(M) \leq 0$ and for each element N of M either $D(N) = Spec^p(M)$ or $D(N)$ is finite. Then the complement of every open set in $Spec^p(M)$ is of the form $\cap_{k\in J}(\cup_{l=1}^{n_k} D(N_{kl}))$, where $N_{kl} \in M$. This implies that every closed set in $Spec^p(M)$ is either finite or $Spec^p(M)$. Consequently, $Spec^p(M)$ is the cofinite topology. □

Theorem 2.5. *Let M be a lattice module over a C-lattice L with $|Spec^p(M)| \geq 2$. If $Spec^p(M)$ is a Hausdorff space, then $dim^p(M) = 0$ and there exists elements $N_1, N_2 \cdots N_k$ of M such that $D(N_i) \neq Spec^p(M)$, for all i and $D(N_1) \cup D(N_2) \cup \cdots \cup D(N_k) = Spec^p(M)$.*

Proof. Suppose that $Spec^p(M)$ is a Hausdorff space and $|Spec^p(M)| \geq 2$. Let $P, Q \in Spec^p(M)$, such that $P \neq Q$. Then there exist open sets $\cup_{k\in J}(\cap_{l=1}^{n_k} E(N_{kl}))$, $\cup_{p\in J'}(\cap_{q=1}^{n_p} E(N_{pq}))$, $N_{kl}, N_{pq} \in M$, $n_k, n_p \in \mathbb{N}$, J, J' are an index set such that $P \in \cup_{k\in J}(\cap_{l=1}^{n_k} E(N_{kl}))$, and $Q \in \cup_{p\in J'}(\cap_{q=1}^{n_p} E(N_{pq}))$ and $[\cup_{k\in J}(\cap_{l=1}^{n_k} E(N_{kl}))] \cap [\cup_{p\in J'}(\cap_{q=1}^{n_p} E(N_{pq}))] = \phi$. Therefore there exists $s \in J, t \in J'$ such that $P \in \cap_{l=1}^{n_s} E(N_{sl})$, and $Q \in \cap_{q=1}^{n_t} E(N_{tq})$ and $[\cap_{l=1}^{n_s} E(N_{sl})] \cap [\cap_{q=1}^{n_t} E(N_{tq})] = \phi$. This implies that $P \nleq Q, Q \nleq P$ and $[\cup_{l=1}^{n_s} D(N_{sl})] \cup [\cup_{q=1}^{n_t} D(N_{tq})] = Spec^p(M)$. Consequently, $dim^p(M) = 0$ and $Spec^p(M) = \cup_{i=1}^{k} D(N_i)$. □

3. Classical Zariski Topology and Spectral Spaces

Let M be a lattice module over a C-lattice L and let $Spec^p(M)$ be equipped with the classical Zariski topology. Let $Y \subseteq Spec^p(M)$, then $Cl(Y)$ denotes the closure of Y in $Spec^p(M)$ and meet of all elements of Y denoted by $\Upsilon(Y)$. Note that if $Y = \phi$, then $\Upsilon(Y) = 1_M$.

A topological space X is called irreducible if $X \neq \phi$ and every finite intersection of non-empty open sets of X is non-empty. A non-empty subset Y of a topological space X is called an irreducible set if the subspace Y of X is irreducible, i.e., if $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$, where Y_1 and Y_2 are closed subsets of X.

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = Cl(\{y\})$. Note that, a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is T_0-space.

Lemma 3.1. *Let M be a lattice module over a C-lattice L and let Y be a finite non-empty subset of $Spec^p(M)$. Then $Cl(Y) = \cup_{P\in Y} D(P)$.*

Proof. Suppose that $Y \subseteq Spec^p(M)$. Clearly $Y \subseteq \cup_{P\in Y} D(P)$. Now, let B be any closed subset of $Spec^p(M)$ such that $Y \subseteq B$. Thus $B = \cap_{k\in J}(\cup_{l=1}^{n_k} D(N_{kl}))$, for some $N_{kl} \in M, k \in J, n_k \in \mathbb{N}$. Let $Q \in$

$\cup_{P\in Y} D(P)$. Then there exists $P^{'} \in Y$ such that $Q \in D(P^{'})$ and so $P^{'} \leq Q$. Now, $P^{'} \in Y \subseteq B$, therefore $P^{'} \in B$. But $B = \cap_{k\in J}(\cup_{l=1}^{n_k} D(N_{kl}))$, therefore for each $k \in J$, there exist $l \in \{1, 2, \cdots, n_k\}$ such that $P^{'} \in D(N_{kl})$ and therefore $N_{kl} \leq P^{'} \leq Q$. It follows that $Q \in \cap_{k\in J}(\cup_{l=1}^{n_k} D(N_{kl})) = B$. Hence $\cup_{P\in Y} D(P) \subseteq B$. Thus $\cup_{P\in Y} D(P)$ is the smallest closed set in $Spec^p(M)$ containing Y. Consequently, $Cl(Y) = \cup_{P\in Y} D(P)$. □

Corollary 3.2. *Let M be a lattice module over a C-lattice L. Then*

1. *$Cl(\{P\}) = D(P)$, for all $P \in Spec^p(M)$.*
2. *$Q \in Cl(\{P\})$ if and only if $P \leq Q$ if and only if $D(Q) \subseteq D(P)$, for $Q \in Spec^p(M)$.*
3. *The set $\{P\}$ is closed in $Spec^p(M)$ if and only if P is a maximal prime element of M.*

Proof. 1) By Lemma 3.1, for $Y \subseteq Spec^p(M)$, $Cl(Y) = \cup_{P\in Y} D(P)$. Let $Y = \{P\}$, then $\cup_{P\in Y} D(P) = D(P)$, hence $Cl(\{P\}) = D(P)$.
2) Suppose that $Q \in Cl(\{P\})$. Then by part (1), $Q \in Cl(\{P\}) = D(P)$, therefore $P \leq Q$. It implies that $D(Q) \subseteq D(P)$. Conversely, suppose that, $D(Q) \subseteq D(P)$. Since $Q \in D(Q) \subseteq D(P)$, we have $P \leq Q$ and $Q \in D(P) = Cl(\{P\})$ by part (1).
3) Suppose that P is a maximal prime element of M. Let $Q \in Cl(\{P\})$, then by part (1), $Q \in Cl(\{P\}) = D(P)$, implies $P \leq Q$. But P is maximal, therefore $P = Q$ and hence $Cl(\{P\}) = \{P\}$. Consequently, $\{P\}$ is closed in $Spec^p(M)$.
Conversely, suppose that $\{P\}$ is closed in $Spec^p(M)$ and P is not maximal, then there exists Q such that $P \leq Q$, which implies that $Q \in Cl(\{P\})$ by part (2). Since $\{P\}$ is closed, $Q \in Cl(\{P\}) = \{P\}$, hence $P = Q$. Consequently, P is a maximal prime element of M. □

Lemma 3.3. *Let M be a lattice module over a C-lattice L and Y be a finite non-empty closed subset of $Spec^p(M)$, then $Y = \cup_{P\in Y} D(P)$.*

Proof. Suppose that Y is a non-empty closed subset of $Spec^p(M)$. It is clear that $Y \subseteq \cup_{P\in Y} D(P)$. By Corollary 3.2(1), for each $P \in Y$, we have $D(P) = Cl(\{P\}) \subseteq Cl(Y)$ and $Cl(Y) = Y$. Therefore $\cup_{P\in Y} D(P) \subseteq Y$. Consequently, $Y = \cup_{P\in Y} D(P)$. □

Lemma 3.4. *Let M be a lattice module over a C-lattice L. Then for each $P \in Spec^p(M)$, $D(P)$ is irreducible.*

Proof. Suppose that $D(P) \subseteq X_1 \cup X_2$, where X_1 and X_2 are closed sets of $Spec^p(M)$. Since $P \in D(P)$ and $D(P) \subseteq X_1 \cup X_2$, therefore $P \in X_1 \cup X_2$, which implies that either $P \in X_1$ or $P \in X_2$. Suppose that $P \in X_1$. Since X_1 is closed in $Spec^p(M)$, we have $X_1 = \cap_{k\in J}(\cup_{l=1}^{n_k} D(N_{kl}))$, for

$N_{kl} \in M$ and $k \in J, n_k \in \mathbb{N}$. Thus $P \in \cup_{l=1}^{n_k} D(N_{kl})$ for each $k \in J$. It follows that $D(P) \subseteq \cup_{l=1}^{n_k} D(N_{kl})$ for each $k \in J$. Hence $D(P) \subseteq \cap_{k\in J}(\cup_{l=1}^{n_k} D(N_{kl})) = X_1$. Consequently, $D(P)$ is irreducible. □

Theorem 3.5. *Let M be a lattice module over a C-lattice L and $Y \subseteq Spec^p(M)$. Then*

(1) *If Y is irreducible, then $\Upsilon(Y)$ is a prime element.*
(2) *If $\Upsilon(Y)$ is a prime element and $\Upsilon(Y) \in Cl(Y)$, then Y is irreducible.*

Proof. 1) Suppose that Y is an irreducible subset of $Spec^p(M)$. Clearly, $\Upsilon(Y) = \wedge_{P\in Y} P < 1_M$ and $Y \subseteq D(\Upsilon(Y))$. Let $aX \leq \Upsilon(Y)$, for $a \in L, X \in M$. Now for $P \in Y \subseteq D(\Upsilon(Y))$, $\Upsilon(Y) \leq P$ and $aX \leq \Upsilon(Y) \leq P$. Since P is prime, $X \leq P$ or $a1_M \leq P$, which implies that $P \in D(X)$ or $P \in D(a1_M)$. Hence $Y \subseteq D(X) \cup D(a1_M)$. Since Y is irreducible, either $Y \subseteq D(X)$ or $Y \subseteq D(a1_M)$. If $Y \subseteq D(X)$, then $X \leq P$, for all $P \in Y$. Therefore $X \leq \Upsilon(Y)$. If $Y \subseteq D(a1_M)$, then $a1_M \leq P$, for all $P \in Y$. Therefore $a1_M \leq \Upsilon(Y)$. Consequently, $\Upsilon(Y)$ is a prime element of M.
2) Suppose that $\Upsilon(Y)$ is a prime element of M and $\Upsilon(Y) \in Cl(Y)$. Since $\Upsilon(Y) \leq P$, for each $P \in Y$, we have $D(P) \subseteq D(\Upsilon(Y))$ for each $P \in Y$, by Corollary 3.2(2). Therefore $\cup_{P\in Y} D(P) \subseteq D(\Upsilon(Y))$ and so by Lemma 3.1, $Cl(Y) \subseteq D(\Upsilon(Y))$. Since $\Upsilon(Y)$ is a prime element of M and $\Upsilon(Y) \in Cl(Y)$, we have, $D(\Upsilon(Y)) \subseteq Cl(Y)$. Hence $D(\Upsilon(Y)) = Cl(Y)$. Now, let $Y \subseteq X_1 \cup X_2$, where X_1 and X_2 are closed sets of $Spec^p M$. Then $Cl(Y) \subseteq X_1 \cup X_2$. Since $D(\Upsilon(Y)) = Cl(Y) \subseteq X_1 \cup X_2$ and $D(\Upsilon(Y))$ is irreducible, by Lemma 3.4, we have either $D(\Upsilon(Y)) \subseteq X_1$ or $D(\Upsilon(Y)) \subseteq X_2$. It follows that either $Y \subseteq X_1$ or $Y \subseteq X_2$. Consequently, Y is irreducible. □

Definition 3.6. [15] Let M be a lattice module over a C-lattice L and N be an element of M. Then the prime radical of N is defined to be the meet of all prime elements containing N, that is $\sqrt[p]{N} = \wedge\{P \in Spec^p(M) | N \leq P\}$.
Note that, $\sqrt[p]{N} = 1_M$, if there is no prime element containing N. If $N = \sqrt[p]{N}$, then N is called as prime radical element of M.

Corollary 3.7. *Let M be a lattice module over a C-lattice L and N be an element of M. Then the subset $D(N)$ of $Spec^p(M)$ is irreducible if and only if $\sqrt[p]{N}$ is a prime element.*

Proof. Suppose that the subset $D(N)$ of $Spec^p(M)$ is irreducible. Then by Theorem 3.5, $\Upsilon(D(N))$ is a prime element. Now, we have $\Upsilon(D(N)) = \wedge\{P \in D(N)\} = \wedge\{P \in Spec^p(M) | N \leq P\} = \sqrt[p]{N}$. Hence $\sqrt[p]{N}$ is a

prime element.
Conversely, suppose that $\sqrt[p]{N}$ is a prime element. Clearly for each element N of M, $D(N) = D(\sqrt[p]{N})$. Since $\sqrt[p]{N}$ is a prime element, $D(\sqrt[p]{N})$ is irreducible by Lemma 3.4. Hence $D(N)$ is irreducible. □

The following Lemma shows that for any lattice module M over a $C-$lattice L, $Spec^p(M)$ is always a T_0-space and every finite irreducible closed subset of $Spec^p(M)$ has a generic point.

Lemma 3.8. *Let M be a lattice module over a C-lattice L. Then*

1. *$Spec^p(M)$ is always a T_0-space.*
2. *Every $P \in Spec^p(M)$ is a generic point of the irreducible closed subset $D(P)$.*
3. *Every finite irreducible closed subset of $Spec^p(M)$ has a generic point.*

Proof. 1) Let $P, Q \in Spec^p(M)$. Then by Corollary 3.2(1), $Cl(\{P\}) = D(P)$, $Cl(\{Q\}) = D(Q)$ and therefore $Cl(\{P\}) = Cl(\{Q\})$ if and only if $D(P) = D(Q)$ if and only if $P = Q$, by Corollary 3.2(2). Now, by the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct, we conclude that, $Spec^p(M)$ is a T_0-space.
2) By Corollary 3.2(1), it is clear that, for $P \in Spec^p(M)$, $D(P) = Cl(\{P\})$. Hence P is a generic point of the irreducible closed subset $D(P)$.
3) Let Y be an irreducible closed subset of $Spec^p(M)$ and $Y = \{P_1, P_2, \cdots, P_k\}$, where $P_i \in Spec^p(M)$, $k \in \mathbb{N}$. By Lemma 3.1, $Y = Cl(Y) = D(P_1) \cup D(P_2) \cup \cdots \cup D(P_k)$. Since Y is irreducible, $Y = D(P_i)$, for some $i(1 \leq i \leq k)$. Now by (2), P_i is a generic point of $D(P_i) = Y$. □

Definition 3.9. [13] A topological space X is a spectral space if X satisfy the following conditions(Hochster's characterization):

1. X is a T_0-space.
2. X is a quasi-compact.
3. The quasi-compact open subsets of X are closed under finite intersection and form an open basis.
4. Each irreducible closed subset of X has a generic point.

Theorem 3.10. *Let M be a lattice module over a C-lattice L with finite prime spectrum. Then $Spec^p(M)$ is a spectral space(with Classical Zariski topology).*

Proof. Assume that $Spec^p(M)$ is finite. Then by Lemma 3.8, $Spec^p(M)$ is a T_0-space and every irreducible closed subset of $Spec^p(M)$ has a generic point. Also, since $Spec^p(M)$ is finite, the quasi-compact open subsets of $Spec^p(M)$ are closed under finite intersection and form an open basis(This basis is $\mathbb{B} = \{E(N_1) \cap E(N_2) \cap \cdots \cap E(N_k) | N_i \in M, 1 \leq i \leq k$, for some $k \in \mathbb{N}\})$[10]. Now by Definition 3.9, we conclude that $Spec^p(M)$ is a spectral space. □

4. Finer Patch Topology and Spectral Spaces

Let X be a topological space. By the patch topology on X, we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology(see [12],[17]).

Definition 4.1. Let M be a lattice module over a C-lattice L and let $U(M)$ be the family of all elements of $Spec^p(M)$ of the form $D(N) \cap E(K)$, where $N, K \in M$. Clearly both $Spec^p(M) = D(0_M) \cap E(1_M)$ and the empty set $\phi = D(0_M) \cap E(0_M)$ are members of $U(M)$. Let $T(M)$ to be the collection of all unions of finite intersections of elements of $U(M)$. Then $T(M)$ is a topology on $Spec^p(M)$ and is called the finer patch topology. In fact $U(M)$ is a sub-basis for the finer patch topology of M.

Note that, finer patch topology on $Spec^p(M)$ is finer than classical Zariski topology on $Spec^p(M)$.

Theorem 4.2. *Let M be a lattice module over a C-lattice L. Then $Spec^p(M)$ with the finer patch topology is Hausdorff. Moreover, $Spec^p(M)$ with this topology is disconnected if and only if $|Spec^p(M)| > 1$.*

Proof. Suppose that $P, Q \in Spec^p(M)$ and $P \neq Q$. Since $P \neq Q$, either $P \nleq Q$ or $Q \nleq P$. By Definition 4.1, $U_1 = E(1_M) \cap D(P)$ is a finer patch-neighborhood of P and $U_2 = E(P) \cap D(Q)$ is a finer patch-neighborhood of Q. It is clear that $E(P) \cap D(P) = \phi$ and hence $U_1 \cap U_2 = \phi$. Thus $Spec^p(M)$ is a Hausdorff space. Now, for each element $N \in M$, $E(N)$ and $D(N)$ are open in finer patch topology, because $D(N) = E(1_M) \cap D(N)$ and $E(N) = E(1_M) \cap D(0_M)$. Since $E(N)$ and $D(N)$ are complements of each other, these sets are closed. Therefore $Spec^p(M)$ with finer patch topology is disconnected if and only if $|Spec^p(M)| > 1$. □

Theorem 4.3. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on prime radical elements. Then $Spec^p(M)$ with the finer patch topology is a compact space.*

Proof. Suppose that M is a lattice module over a $C-$lattice L such that M has ascending chain condition on prime radical elements. Let A be a family of finer patch-open sets which covers $Spec^p(M)$ and suppose that no finite subfamily of A covers $Spec^p(M)$. Since $D(\sqrt[p]{0_M}) = D(0_M) = Spec^p(M)$, we may use the ascending chain condition on prime radical

elements to choose an element N maximal with respect to the property that no finite subfamily of A covers $D(N)$(we may assume $N = \sqrt[p]{N}$ because $D(N) = D(\sqrt[p]{N})$).
Suppose that N is not prime element of M. Then there exists $X \in M$ and $a \in L$, such that $aX \leq N$ and $X \nleq N$, $a1_M \nleq N$. Thus $N < N \vee X \leq \sqrt[p]{N \vee X}$ and $N < N \vee a1_M \leq \sqrt[p]{N \vee a1_M}$. Hence without loss of generality, there must be a finite subfamily A' of A that covers both $D(N \vee X)$ and $D(N \vee a1_M)$. Let $P \in D(N)$, then $N \leq P$ and so $aX \leq N \leq P$. Since P is prime, $X \leq P$ or $a1_M \leq P$ and hence $N \vee X \leq P$ and $N \vee a1_M \leq P$. Thus either $P \in D(N \vee X)$ or $P \in D(N \vee a1_M)$, therefore $D(N) \subseteq D(N \vee a1_M) \cup D(N \vee X)$. Thus $D(N)$ is covered with the finite subfamily A', which is contradiction. Hence N is prime element of M.
Now choose $U \in A$ such that $N \in U$. Thus N must have a patch-neighborhood $\cap_{i=1}^{n}[E(K_i) \cap D(N_i)]$, for some $K_i, N_i \in M, n \in \mathbb{N}$, such that $\cap_{i=1}^{n}[E(K_i) \cap D(N_i)] \subseteq U$. Suppose for each $i(1 \leq i \leq n)$, $P \in E(K_i \vee N) \cap D(N)$. Then $P \in E(K_i \vee N)$, $P \in D(N)$ and so that $K_i \vee N \nleq P$ and $N \leq P$. Thus $K_i \nleq P, i.e., P \in E(K_i)$. On the other hand $N \in D(N_i)$, $i.e., N_i \leq N$, therefore $N_i \leq P$ and $P \in D(N_i)$. Consequently, $N \in [E(K_i \vee N) \cap D(N)] \subseteq [E(K_i) \cap D(N_i)]$ and hence $N \in \cap_{i=1}^{n}[E(K_i \vee N) \cap D(N)] \subseteq \cap_{i=1}^{n}[E(K_i) \cap D(N_i)] \subseteq U$. Thus $[\cap_{i=1}^{n} E(K_i \vee N)] \cap D(N)$, where $N < K_i \vee N$, is a neighborhood of N, with $[\cap_{i=1}^{n} E(K_i \vee N)] \cap D(N) \subseteq U$.
Since for each $i(1 \leq i \leq n)$, $N < K_i \vee N$, $D(K_i \vee N)$ is covered by some finite subfamily A_i' of A. But $D(N) - [\cup_{i=1}^{n} D(K_i \vee N)] = D(N) - [\cap_{i=1}^{n} E(K_i \vee N)]^c = [\cap_{i=1}^{n} E(K_i \vee N)] \cap D(N) \subseteq U$ and so $D(N)$ can be covered by $A_1' \cup A_2' \cup \cdots \cup A_n' \cup \{U\}$, which is contradiction to our choice of N. Thus there must exist a finite subfamily of A which covers $Spec^p(M)$. Therefore $Spec^p(M)$ is compact in the finer patch topology of M. $\square$

Let τ_1 and τ_2 be two topologies on X such that $\tau_1 \subseteq \tau_2$. If X is quasi-compact (i.e. any open cover of it has a finite subcover) in τ_2, then X is also quasi-compact in τ_1 (see [16]).

Theorem 4.4. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on prime radical elements. Then for each $n \in \mathbb{N}$ and elements $N_i (1 \leq i \leq n)$ of M, $E(N_1) \cap E(N_2) \cap \cdots \cap E(N_n)$ is a quasi-compact subset of $Spec^p(M)$ with the classical Zariski topology. Consequently, $Spec^p(M)$ is quasi-compact and has a basis of quasi-compact open subsets.*

Proof. By Definition 4.1, we have, for each element $N \in M$, $D(N) = D(N) \cap E(1_M)$ as an open subset of $Spec^p(M)$ with finer patch topology and $E(N)$ is complement of $D(N)$, therefore $E(N)$, for each $N \in M$, is a closed subset of $Spec^p(M)$. Thus for each $n \in \mathbb{N}$ and $N_i \in M (1 \leq i \leq n)$, $E(N_1) \cap E(N_2) \cap \cdots \cap E(N_n)$ is also a closed subset in $Spec^p(M)$ with finer patch topology. By Theorem 4.3, $Spec^p(M)$ is a compact space with finer patch topology and since every closed subset of a compact space is compact, $E(N_1) \cap E(N_2) \cap \cdots \cap E(N_n)$ is compact in $Spec^p(M)$ with finer patch topology and therefore, it is quasi-compact in $Spec^p(M)$ with the classical Zariski topology.
Since $Spec^p(M) = E(1_M)$ and $\mathbb{B} = \{E(N_1) \cap E(N_2) \cap \cdots \cap E(N_n) | N_i \in M, n \in \mathbb{N}\}$ is a basis for the classical Zariski topology of M, $Spec^p(M)$ is quasi-compact and has a basis of quasi-compact open subsets. □

Lemma 4.5. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on prime radical elements. Then every irreducible closed subset of $Spec^p(M)$(with the classical Zariski topology) has a generic point.*

Proof. Suppose that Y is an irreducible closed subset of $Spec^p(M)$. Note that $D(N)$ and $E(N)$ are both open and closed in finer patch topology. Hence for each $P \in Y, D(P)$ is an open subset of $Spec^p(M)$. Now, since Y is closed subset of $Spec^p(M)$ with classical Zariski topology, the complement of Y is open by this topology, hence complement of Y is open subset with finer patch topology and Y is closed subset of $Spec^p(M)$ with finer patch topology. By Theorem 4.3, $Spec^p(M)$ is compact and Y is a closed subset of $Spec^p(M)$, Y is also compact. We have, by Lemma 3.3, $Y = \cup_{P \in Y} D(P)$. Since Y is compact and each $D(P)$ is finer patch-open, there exists a finite subset Y_1 of Y such that $Y = \cup_{P \in Y_1} D(P)$. Since Y is irreducible, $Y = D(P)$ for some $P \in Y$. Consequently, P is a generic point for Y. □

Corollary 4.6. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on prime radical elements. Then quasi-compact open sets of $Spec^p(M)$ (with classical Zariski topology) are closed under finite intersections.*

Proof. Let U_1 and U_2 be two quasi-compact open sets of $Spec^p(M)$ and let $U = U_1 \cap U_2$. Each of U_1 and U_2 is a finite union of members of $\mathbb{B} = \{E(N_1) \cap E(N_2) \cap \cdots \cap E(N_n) | N_i \in M, n \in \mathbb{N}\}$. Hence $U = \cup_{i=1}^{m} (\cap_{j=1}^{n_i} E(N_j))$. Let Π be any open cover of U. Then Π also covers each $\cap_{j=1}^{n_i} E(N_j)$ which is quasi-compact by Theorem 4.4. Hence each $\cap_{j=1}^{n_i} E(N_j)$ has a finite subcover of Π and hence U has a finite subcover of Π. Thus U is quasi-compact, as required. □

Theorem 4.7. *Let M be a lattice module over a C-lattice L such that M has ascending chain condition on prime radical elements. Then $Spec^p(M)$(with the classical Zariski topology) is a spectral space.*

Proof. By Lemma 3.8, $Spec^p(M)$ is a T_0-space. Since M has ascending chain condition on prime radical elements, $\mathbb{B} = \{E(N_1) \cap E(N_2) \cap \cdots \cap E(N_n) | N_i \in M, n \in \mathbb{N}\}$ is a basis for $Spec^p(M)$ with the property that each basis element, in particular $E(1_M) = Spec^p(M)$ is quasi-compact by Theorem 4.4. By Corollary 4.6, the quasi-compact open sets are closed under finite intersections. And finally, by Lemma 4.5, each irreducible closed set has a generic point. Therefore, by Definition 3.9, $Spec^p(M)$ is a spectral space. □

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