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ON THE RANKS OF CERTAIN SEMIGROUPS OF ORDER-PRESERVING PARTIAL ISOMETRIES OF A FINITE CHAIN

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Abstract. Let $X_n = \{1, 2, \ldots, n\}$ be a finite chain, $\mathcal{ODP}_n$ be the semigroup of order-preserving partial isometries on X_n and N be the set of all nilpotents in $\mathcal{ODP}_n$. In this work, we study the nilpotents in $\mathcal{ODP}_n$ and investigate the ranks of two subsemigroups of $\mathcal{ODP}_n$; the nilpotent generated subsemigroup $\langle N \rangle$ and the subsemigroup $L(n, r) = \{\alpha \in \mathcal{ODP}_n : |im\ \alpha| \le r\}$.

1. Introduction

Let $X_n = \{1, 2, \ldots, n\}$ be a finite chain, $\mathcal{P}_n$ be the Partial transformation semigroup on X_n and $\mathcal{I}_n$ be the set of all injective transformation on X_n. The set $\mathcal{I}_n$ is known to be an inverse subsemigroup of $\mathcal{P}_n$ (in the sense that $\forall \alpha \in \mathcal{I}_n$ there exist a unique $\alpha' \in \mathcal{I}_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$). This semigroup is more commonly known as the *symmetric inverse semigroup*.

For a transformation $\alpha \in \mathcal{P}_n$, we denote the domain set and image set of α as *dom* α and *im* α respectively, while the height of α is denoted and defined as h(α)=$|im\ \alpha|$.

A map $\alpha \in \mathcal{I}_n$ is said to be *order increasing* (resp., *order decreasing*) if, for all $x \in dom\ \alpha$, $x \le x\alpha$ (resp. $x\alpha \le x$); *order preserving* if (for all $x, y \in dom\ \alpha$) $x \le y \implies x\alpha \le y\alpha$; an *isometry* (i. e., *distance preserving*) if (for all $x, y \in dom\ \alpha$) $|x\alpha - y\alpha| = |x - y|$ and

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an *order preserving partial isometry* if it is both order-preserving and isometry. Adopting the symbols used in [1], we shall denote by $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ the semigroup of partial isometries and the semigroup of order-preserving partial isometries respectively. The semigroup $\mathcal{I}_n$ and some of its subsemigroups have been well studied and its algebraic and combinatorial properties were investigated over the years. Some of these interesting results can be found in [7, 10, 11, 14, 15]. The study of semigroup of partial isometries on a chain was only initiated recently by Alkharousi et al. [1, 2], where some of the combinatorial, algebraic and rank properties of the two semigroups $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ were investigated.

Let α be in $\mathcal{P}_n$, then α is said to be a nilpotent if there exists $m \in \mathbb{N}$ such that $\alpha^m = 0$. The study of nilpotents in some subsemigroups of $\mathcal{I}_n$ have been considered by many authors and some delightful results were obtained. See for example [3, 6, 8].

The main aim of this paper is to present the study of nilpotents in $\mathcal{ODP}_n$.

It was shown among other things in Alkharousi et al. [1] that, the semigroup $\mathcal{ODP}_n$ (as an inverse semigroup) has rank n. As part of what we want to achieve in this paper is to generalize this result and find the rank of the subsemigroup $L(n,r) = \{\alpha \in \mathcal{ODP}_n : |im\alpha| \leq r\}$ $(1 < r \leq n)$ of $\mathcal{ODP}_n$.

In this section we introduce some of the basic terminologies and also quote some results from related literature which we will later need in proving some of the results in this paper.

In section 2 we define nilpotents in $\mathcal{ODP}_n$ and describe the nilpotent generated subsemigroup $\langle N \rangle$ of $\mathcal{ODP}_n$.

In section 3 we define an equivalence relation on the subsets of X_n and computed the order of each equivalence class. This equivalence relations lead us to finding the minimal generating set for the inverse subsemigroup

$$M(n,r) = \{\alpha \in \langle N \rangle \,:\, |im\ \alpha| \leq r\}.$$

Consequently the rank $M(n,r)$ and $\langle N \rangle$ were obtained.

The results obtained in section 3 is extended to investigate the rank of the ideal of $\mathcal{ODP}_n$ which was obtained in section 4.

For standard concept in semigroup we refer to Howie [9] and Higgins [12].

Let

$$\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : (\text{for all } \ x, y \in dom\ \alpha)\ |x\alpha - y\alpha| = |x - y|\} \quad (1.1)$$

and

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : (\text{for all } \ x, y \in dom\ \alpha)\ x \leq y \implies x\alpha \leq y\alpha\} \tag{1.2}$$

be the semigroup of partial isometries and the semigroup of order-preserving partial isometries respectively. Then we have the following lemma:

Lemma 1.1. [1, Lemma 1.1] *$\mathcal{DP}_n$ and $\mathcal{ODP}_n$ are inverse subsemigroups of $\mathcal{I}_n$.*

For a transformation α, if $x\alpha = x$ for some $x \in dom\alpha$ then x is called a fix point of α. The set of all fix points of α is denoted by $F(\alpha)$, that is $F(\alpha) = \{x \in dom\alpha : x\alpha = x\}$, while $f(\alpha)$ is defined as the cardinality of the set $F(\alpha)$, that is $f(\alpha) = |F(\alpha)|$. The following results are found useful in this paper:

Lemma 1.2. [1, Lemma 1.7] *Let $\alpha \in \mathcal{ODP}_n$ and $f(\alpha) \geq 1$. Then α is partial identity.*

Lemma 1.3. *Let $\alpha \in \mathcal{ODP}_n$. If $\{1,\ n\} \subseteq dom\ \alpha$ then α is a partial identity.*

Theorem 1.4. [1, Theorem 2.6] *Let $\mathcal{ODP}_n$ be the semigroup of order preserving partial isometries. Then $|\mathcal{ODP}_n| = 3.2^n - 2(n+1)$.*

2. Nilpotents in $\mathcal{ODP}_n$

Let S be a semigroup with 0(zero) element. A non zero element $a \in S$ is said to be a nilpotent if there exist $m \in \mathbb{N}$ such that $a^m = 0$. Since the semigroup $\mathcal{ODP}_n$ is finite and contains a zero element (which is the empty map) therefore, it is natural to ask about its nilpotent elements and its nilpotent generated subsemigroup.

Let S be a semigroup, an element e in S is said to be an idempotent if $e^2 = e$. We write $E(S)$ to denote the set of all idempotents in a semigroup S.

Next, we have the following lemmas.

Lemma 2.1. [4, Lemma 2.1] *Let α be in $\mathcal{IO}_n$ such that $h(\alpha) < n$, then α is a nilpotent if and only if $x\alpha \neq x$ for every $x \in dom\ \alpha$.*

Since the semigroup $\mathcal{ODP}_n$ is a subsemigroup of $\mathcal{IO}_n$, every nilpotent/idempotent in $\mathcal{ODP}_n$ is also a nilpotent/idempotent in $\mathcal{IO}_n$. Therefore, the above Lemma holds for $\mathcal{ODP}_n$ as well.

Lemma 2.2. [8, corollary 2.7.3] *The element $\alpha \in \mathcal{I}_n$ is an idempotent if and only if α is the identity transformation of some $A \subseteq X_n$. In particular, $\mathcal{I}_n$ contains exactly 2^n idempotents.*

The next lemma follows directly from Lemma (1.2), Lemma (2.1) and Lemma (2.2).

Lemma 2.3. *Let α be in $\mathcal{ODP}_n$ such that $h(\alpha) \geq 1$, then α is either a nilpotent or an idempotent.*

Remark 2.4. It is clear that if $h(\alpha) = 0$, then α is both nilpotent and idempotent.

Let N be the set of all nilpotent elements in $\mathcal{ODP}_n$, Alkharousi [1, Proposition 2.8] obtained the cardinality of the set N from combinatorial point of view as the set of all elements of $\mathcal{ODP}_n$ with zero fixed points. Here we give an alternative proof of the Proposition using the word *nilpotents* instead of *fixed points*, which (to our own opinion) will fit better into the context of our work.

Proposition 2.5. *Let N be the set of all nilpotents in $\mathcal{ODP}_n$. Then $|N| = 2^{n+1} - (2n + 1)$.*

Proof. Let $E(\mathcal{ODP}_n)$ be the set of all idempotents in $\mathcal{ODP}_n$. From Lemma (2.3) we have that

$$|\mathcal{ODP}_n| = |E(\mathcal{ODP}_n)| + |N| + 1,$$

it therefore follows from Theorem (1.4) and Lemma (2.2) that

$$|N| = 3(2^n) - 2(n + 1) - 2^n = 2^{n+1} - (2n + 1).$$

□

One of the interesting question one may ask pertaining the semigroup that has nilpotents is that; can some of those non-nilpotent elements be expressed as the product of nilpotents? The collection of all those elements that can be express as product of nilpotents, together with the nilpotent elements is what we called *nilpotent generated subsemigroup*. If every element of a semigroup S can be expressed as a product of some nilpotents, then S is called a semigroup generated by nilpotent elements. Otherwise, the nilpotent elements can only generate a proper subsemigroup. Next, we give the description of the subsemigroup $\langle N \rangle$ of $\mathcal{ODP}_n$ generated by nilpotent elements. Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \tag{2.1}$$

be an element of $\mathcal{ODP}_n$ with h$(\alpha) = r$ $(1 \leq r \leq n)$. We have the following theorem:

Theorem 2.6. *For $n \geq 2$, let α be defined by $a_i\alpha = b_i$ $(1 \leq i \leq r)$ be an element of $\mathcal{ODP}_n$ with $h(\alpha) < n$. Then α is a product of nilpotents if and only if it fails to satisfies the condition $a_1 = 1$ and $a_r = n$.*

Proof. Let α be defined as in Equation (2.1) be an element of $\mathcal{ODP}_n$ with $h(\alpha) < n$. By Lemma (2.3), we only need to consider when α is idempotent (i. e., $a_i = b_i \forall (1 \leq i \leq r)$.

Now, Suppose that α fail to satisfy the condition: $a_1 = 1$ and $a_r = n$. we shall consider two cases:

Case 1: $a_1 = 1\, (a_r \neq n)$. Then we can define a set $C = \{c_i : c_i = a_i + 1,\ (1 \leq i \leq n)\}$ and express α as

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ a_1 & a_2 & \dots & a_r \end{pmatrix}$$

a product of two nilpotents.

case 2: If $a_1 \neq 1$, then we can define the $C = \{c_i : c_i = a_i - 1, (\ 1 \leq i \leq n)\}$, and express α as product of two nilpotents just as in case 1.

Conversely, suppose for some $k \in \mathbb{N}$ that α can be written as

$$\alpha = \theta_1 \theta_2 \dots \theta_k \tag{2.2}$$

a product of k nilpotents and suppose by the way of contradiction that $a_1 = 1$ and $a_r = n$, then from Equation (2.2) we have that dom $\alpha \subseteq$ dom θ_1, and this implies that $\{1,\ n\} \subseteq dom\ \theta_1$ which by Lemma (1.3) contradicts our assumption that θ_1 is a nilpotent. □

Corollary 2.7. *Let α be defined as in Equation (2.1) then α is a product of nilpotent if and only if it satisfy one of the following conditions:*

(I) $a_1 \neq 1$ *and* $b_1 \neq 1$*;*
(II) $a_1 \neq 1$, $b_1 = 1$ *and* $b_r \neq n$*;*
(III) $a_1 = 1$, $b_1 \neq 1$ *and* $a_r \neq n$*;*
(IV) $a_1 = 1$, $b_1 = 1$ *and* $b_r = a_r \neq n$*;*

Remark 2.8. We can observe that;

(1) By Corollary (2.7) elements of the nilpotent generated subsemigroup $\langle N \rangle$ can be categorized into four different types. Henceforth, we shall say an element $\alpha \in \langle N \rangle$ is of type m (where m represents any of the conditions I, II, III, IV in the corollary) if α satisfies condition m.
(2) Elements of $\mathcal{ODP}_n$ that are not in $\langle N \rangle$ are partial identities of the form: $\begin{pmatrix} 1 & \dots & n \\ 1 & \dots & n \end{pmatrix}$
(3) For any r in $(2 \leq r \leq n)$, to select an element in item (2) above, we fix 1 and n and select $r-2$ out of $n-2$. Therefore, the total number of those elements is $\sum_{r=2}^{n} \binom{n-2}{r-2} = 2^{n-2}$.

The next theorem gives us the cardinality of the subsemigroup $\langle N \rangle$ generated by nilpotent elements.

Theorem 2.9. *Let N be the set of all nilpotents in $\mathcal{ODP}_n$, and $\langle N\rangle$ be the subsemigroup generated by those nilpotents. Then $|\langle N\rangle| = 11 \cdot 2^{n-2} - 2(n+1)$.*

Proof. Let $\langle N\rangle$ be the nilpotent generated subsemigroup of $\mathcal{ODP}_n$. From Theorem (1.4) and Remark (2.8) we have that;

$$\begin{aligned} |\langle N\rangle| &= 3 \cdot 2^n - 2(n+1) - \textstyle\sum_{r=2}^{n} \binom{n-2}{r-2} \\ &= 3 \cdot 2^n - 2(n+1) - 2^{n-2} \\ &= 11 \cdot 2^{n-2} - 2(n+1). \end{aligned}$$

□

3. Rank of Nilpotent Generated Subsemigroup

If a semigroup S is generated by nilpotent elements, then the cardinality of the smallest subset consisting of only nilpotent elements that generates S is called the nilpotent rank of S (written as $nilrank(S)$). The study of nilpotent elements was initiated by Sullivan [13], after which many other authors work on various subemigroups of $\mathcal{P}_n$. See for example [4, 5, 6].

Let N be the set of all nilpotents in $\mathcal{ODP}_n$ and $\langle N\rangle$ be the subsemigroup generated by those nilpotents. We have seen from Corollary (2.7) that, $\langle N\rangle$ generates only a proper subsemigroup of $\mathcal{ODP}_n$. Our aim in this section is to investigate the rank of $\langle N\rangle$. Let

$$L(n, r) = \{\alpha \in \mathcal{ODP}_n : |im\ \alpha| \leq r\} \quad (1 \leq r \leq n). \tag{3.1}$$

And let

$$K_r = L(n, r)\backslash L(n, r-1). \tag{3.2}$$

Then K_r is of the form $J_r \cup \{0\}$, where J_r is the set of all elements of $\mathcal{ODP}_n$ whose height is exactly r. The product of any two elements in K_r say α and β is of the form:

$$\alpha * \beta = \begin{cases} \alpha\beta, & \text{if } |h(\alpha\beta)| = r; \\ 0, & \text{if } |h(\alpha\beta)| < r \end{cases}$$

K_r is called the Rees quotient semigroup on $L(n, r)$. Also, let

$$M(n, r) = \{\alpha \in \langle N\rangle \ : |im\ \alpha| \leq r\} \tag{3.3}$$

be inverse subsemigroup of $\mathcal{ODP}_n$ generated by nilpotent elements of heights less than or equals to r. And let

$$W_r = M(n, r)\backslash M(n, r-1) \tag{3.4}$$

be the Rees quotient semigroup on $M(n, r)$. Observe that $M(n, r) \subseteq L(n, r)$, therefore $W_r \subseteq K_r$.

Let $X_n = \{1, 2, \ldots, n\}$ be a finite chain and $A = \{a_1, a_2, \ldots, a_r\}$ be a subset of X_n with r number of elements $(1 < r \leq n)$. Adopting the term used in [3], we shall say that A has *k-jumps* $(k \in \mathbb{N})$ between the elements a_i and a_{i+1} $(1 \leq i \leq r-1)$, if $a_{i+1} - a_i = k+1$. And the sum of all jumps in A is called the *total jumps* in A. Observe that if a_i and a_{i+1} are consecutive numbers, then k is zero in that case, and for any r in $(1 < r \leq n)$, subsets of the form $\{1, \ldots, r-1, n\}$ are subset that has the maximum number of jumps. Therefore, the number k is in the interval $0 \leq k \leq n - r$.

Example 3.1. consider the set $X_9 = \{1, 2, \ldots, 9\}$ and let $r = 5$, then the subset $\{1, 3, 4, 5, 8\}$ has 1-jump between it's first and second element and 2-jumps between it's forth and fifth element. While the subsets $\{1, 2, 3, 4, 5\}$ and $\{3, 4, 5, 6, 7\}$ has total jumps of zero each.

Before stating the main theorem of the section, we would first partition the set X_n using some set theoretic terminology, so as to characterize the elements of $\mathcal{ODP}_n$ according their respective domain sets and image sets.

Let $X_n = \{1, 2, \ldots, n\}$ and $A = \{a_1 < a_2 < \ldots < a_r\}$, $B = \{b_1 < b_2 < \ldots < b_r\}$ be any two subsets of X_n with same cardinality say $r(1 \leq r \leq n)$. Define an equivalence relation $\sim$ on X_n as

$$A \sim B \quad \text{if } \forall i, j \in \{1, 2, \ldots, r\},\ |a_i - a_j| = |b_i - b_j|. \tag{3.5}$$

Remark 3.2. It is clear from the definition $\sim$ that, if $A \sim B$ then:

(a) A and B must have the same jumps in the same respective positions. And
(b) one can always define an order preserving partial isometry mapping between A and B.

We shall denote the equivalent class of a subset A of X_n by $[A]$, that is

$$[A] = \{B \subseteq X_n \,:\, B \sim A\}. \tag{3.6}$$

And we denote the set of all equivalent classes on X_n by $X_n/\sim$.

From the definition of $\sim$, Equation (3.6) can be interpreted as

$[A] = \{B \subseteq X_n \,:\, \forall b_i, b_j \in B \text{ and } a_i, a_j \in A\ (i, j = \{1, 2, \ldots r\}),\ |b_i - b_j| = |a_i - a_j|\}$. But A and B are ordered subset of X_n, therefore, $|b_i - b_j| = |a_i - a_j| \Longrightarrow b_i - b_j = a_i - a_j$. Thus,

$$[A] = \{B \subseteq X_n \,: \forall b_i, b_j \in B \text{ and } a_i, a_j \in A\ b_i = a_i + (b_j - a_j)\}. \tag{3.7}$$

Let $l_j = b_j - a_j$ $(\forall j \in \{1, 2, \ldots r\})$. Then, $b_i - b_j = a_i - a_j \Longrightarrow b_i - a_i = b_j - a_j$. Hence,

$$l_i = l_j \quad (\forall i, j \in \{1, 2, \ldots r\}). \tag{3.8}$$

Since, $b_j \in B \subseteq X_n$, then $b_j \leq n$, which implies that $l_j = b_j - a_j \leq n - a_j$. Thus,

$$l_j \leq n - a_j. \tag{3.9}$$

Also, $1 \leq b_1 \Longrightarrow 1 - a_1 \leq b_1 - a_1 = l_1$. This implies

$$1 - a_1 \leq l_1. \tag{3.10}$$

From Equation (3.10), (3.9) (3.8), we have

$$1 - a_1 \leq l_j \leq n - a_r. \quad \forall j \in \{1, 2, \ldots r\}. \tag{3.11}$$

Lemma 3.3. *Let X_n be a chain and $\sim$ as defined in (3.5) be an equivalence relation on X_n. Then for any $A \subseteq X_n$,*

$$[A] = \{B \subseteq X_n \,:\, \forall b_i \in B;\, b_i = a_i + l, \quad 1 - a_1 \leq l \leq n - a_r\} \; (i = \{1, 2, \ldots r\})$$

Proof. It follows from Equation (3.7) and Equation (3.11) □

We shall call a given subset of X_n, a *1-subset* if such subset contains the element 1. Obviously, if an ordered subset contains 1, then this “1” will always be the the first element of the set.

Lemma 3.4. *Let $\sim$ as defined in (3.5) be an equivalence relation on X_n, then in each equivalent class of $\sim$, there exist a unique 1-subset.*

Proof. Let $A = \{a_1, a_2, \ldots, a_r\}$ with $a_i < a_{i+1}$ $(1 \leq i \leq r)$, be a subset of X_n. If A is a not a 1-subset, then a_1 must be equal to $(1 + c)$ for some $c \in \{1, 2, \ldots n - 1\}$. Define a set $B = \{b_i \,:\, b_i = a_i - c\}$, then clearly B is a 1-subset and $B \sim A$.

Next, we show that this 1-subset is unique in $[A]$. Suppose by the way of contradiction, that there exist another set say $D = \{1, d_2, d_3, \ldots d_r\}$ which is also a 1-subset in $[A]$. Then, $D \sim B$, implies that $\forall j \in \{1, 2, \ldots, r\}$, $|d_j - 1| = |b_j - 1| \Longrightarrow d_j - 1 = a_j - 1 \Longrightarrow d_j = a_j \, \forall j \in \{1, 2, \ldots, r\}$, which implies that $D = B$, hence a contradiction. Therefore B is unique in [A]. □

Remark 3.5. For a given 1-subset, say $A = \{1, a_2, \ldots a_r\}$, if A has total jump of zero (that is all the elements of A are consecutive in X_n), then $a_r = r$, and if A has total jump of 1 (that is there exist two numbers in A with a missing digit between them), then $a_r = r + 1$. In general, if a 1-subset A has a total jump of $m (0 \leq m \leq n - r)$, then it's last element a_r would be equals to $r + m$, therefore A can be written as $A = \{1, a_2, \ldots r + m\}$.

The next lemma tells us the number of subsets in each equivalent class of the equivalence relation defined in (3.5).

Lemma 3.6. *Let X_n be finite chain and $\sim$ be the equivalence relation on X_n. Then the number of subsets in each equivalent class is $n-(r+m)+1$, where r is the cardinality of the subsets and m is the total jump in each class.*

Proof. Let $A \subseteq X_n$ and $[A]$ be its equivalent class. Suppose without lost of generality we consider A to be the 1-subset in that class, then by Lemma (3.3) and Remark (3.5),

$$[A] = \{B \subseteq X_n \,:\, \forall b_i \in B;\, b_i = a_i + l, \quad 0 \leq l \leq n-(r+m)\}.$$

That is to say all the other elements of $[A]$ will be obtained by translating it with a constant l ($l = \{0, 2, \ldots, r\}$) from A. Therefore, the other of $[A]$ is $n-(r+m)+1$.

□

To see what we have been explaining clearly, consider the following example:

Example 3.7. Let $X_9 = \{1, 2, \ldots, 9\}$. Below is the list of subsets of X_9 with cardinality $r = 5$:

$$\begin{array}{lllll}
\{1,2,3,4,5\} & \{1,3,4,5,6\} \cdots \{1,2,3,4,6\} \cdots & \{1,2,3,4,9\} \cdots \{1,6,7,8,9\} \\
\{2,3,4,5,6\} & \{2,4,5,6,7\} \cdots \{2,3,4,5,7\} \vdots & \\
\{3,4,5,6,7\} & \{3,5,6,7,8\} \cdots \{3,4,5,6,8\} & \\
\{4,5,6,7,8\} & \{4,6,7,8,9\} \cdots \{4,5,6,7,9\} & \\
\{5,6,7,8,9\} & &
\end{array} \tag{3.12}$$

In the example above, each column of subsets represents a distinct equivalent class. The subsets in the first column has total jump of zero, while subsets in the second (third, fourth and fifth) up to next column have the total jump of 1 each. Thus, two different equivalent classes of $\sim$ can have the same total jumps. The following lemma will give us the exact number of equivalent classes with the same total jumps.

Lemma 3.8. *Let $X_n = \{1, 2, \ldots, n\}$. For a fix r ($1 < r \leq n$) and m ($0 \leq m \leq n-r$), the number of different equivalent classes with total jump of m and cardinality r is $\binom{r+m-2}{r-2}$.*

Proof. Let the $1-$subset $\{1, \ldots, r+m\}$ represents each equivalent class with total jump of m. Now, between 1 and $r+m$ we have $r+m-2$ elements in X_n and $r-2$ elements in the 1-subset. So, to select a particular $1-$subset we only need to select $r-2$ elements from this $r+m-2$. Therefore, the total number of those $1-$subsets is $\binom{r+m-2}{r-2}$. □

Remark 3.9. (a) Note that for $m = 0$, we have just 1 (class), while for $m = n - r$ (which is the maximum number m can reach) we have $\binom{n-2}{r-2}$ number of classes. And that is the number of subsets of the form $\{1, \ldots, n\}$ that characterized elements of $\mathcal{ODP}_n$ that are not in $\langle N \rangle$.

(b) Observe also that, by Lemma (3.6), those subsets $\{1, \ldots, n\}$ are the only elements in their own respective equivalent classes (i. e., the order of their equivalent classes is just one).

Now, take the 1-subset in each equivalent class and fixed it as domain and define an order-preserving partial isometry mapping with the remaining sets in that equivalent class, with the exception of the 1-subset itself (that is excluding the mapping of the 1-subset into itself). Then by Lemma (3.6) we will have $n - (r + m)$ number of order-preserving partial isometries in each class. It should be observed that, subsets of the form $\{1, \ldots, n\}$ define no mapping in their own equivalent classes, while in the remaining equivalent classes mappings defined there are all nilpotents.

Let G be the set of all mappings having 1-subset as domain, then we have the following lemma which we need in proving the proposition that follows it.

Lemma 3.10. *Let $\alpha \in G$. Then for any $\beta, \gamma \in W_r$, $\alpha = \beta\gamma$ implies that either β is in G and γ is not, or γ is in G and β is not.*

Proof. Let $\alpha \in G$ and $\beta, \gamma \in W_r$ such that $\alpha = \beta\gamma$. By the way of contradiction, suppose that neither $\beta \in G$ nor $\gamma \in G$. Then by our assumption that $\alpha = \beta\gamma$, and $\beta, \gamma \in W_r$ (having the same height), we have that

$$\begin{aligned} dom\ \alpha &= dom\ \beta \\ im\ \beta &= dom\ \gamma \\ im\ \gamma &= im\ \alpha. \end{aligned} \tag{3.13}$$

Now $dom\ \alpha = dom\ \beta$ (which is a 1-subset) and β not in G, implies that $dom\ \beta = im\ \beta$ which also implies from Equation(3.13) that $dom\ \gamma$ is a 1-subset. But γ also not in G, therefore $dom\ \gamma = im\ \gamma$. And so, we have: $dom\ \alpha = dom\ \beta = im\ \beta = dom\ \gamma = im\ \gamma = im\ \alpha$, which implies that $dom\ \alpha = im\ \alpha$ contradicting our assumption that α is in G. Therefore, if $\alpha = \beta\gamma$, then either β or γ must be in G.

Next, we show that both β and γ cannot be in G at the same time. It is clear that, if both β and γ are in G, then their domain sets must be 1-subset and their image sets are not. And so, $im\ \beta \neq dom\ \gamma$, which implies that $\beta\gamma \neq \alpha$. $\square$

The next Proposition will give us the minimal generating set for the subsemigroup W_r, an inverse subsemigroup of $M(n, r)$.

Proposition 3.11. *Let G be the set of all mappings having 1-subset as domain. For $n \geq 4$ and $1 < r \leq n-1$, the set G is the minimal generating set for W_r as an inverse semigroup.*

Proof. We first show that G is a generating set for W_r, we then show that it is indeed the minimal generating set. Let α be defined as in (2.1) be an element of W_r. Then by Remark (2.8) α must be one of the four types. Now, if α is of type I, then by Lemma (3.4), there exists a 1-subset say $\{1, c_2, \ldots, c_r\}$ which will be of the same class with $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_r\}$ and that

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ 1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} 1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix} = \alpha$$

If α is of type II then α^{-1} is in G and if α is of type III then α is in G. Now if α is of type IV, that is

$$\alpha = \begin{pmatrix} 1 & a_2 & \ldots & a_r \\ 1 & a_2 & \ldots & a_r \end{pmatrix},$$

$(a_r \neq n)$. Then, there exist a subset say $\{c_1, c_2, \ldots, c_r\}$ where $c_i = a_i + 1\,(i = 1, 2, \ldots, r)$, which is in the same equivalent class with $\{1, a_2, \ldots, a_r\}$ such that

$$\begin{pmatrix} 1 & a_2 & \ldots & a_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ 1 & a_2 & \ldots & a_r \end{pmatrix} = \alpha.$$

Now let $G' \subseteq W_r$ such that $\langle G' \rangle = W_r$ and suppose that there exist some $\alpha' \in G$ say

$$\alpha' = \begin{pmatrix} 1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$$

such that α' is not in G'. Then, since $\alpha' \in W_r$ and $\langle G' \rangle = W_r$ there exist some $\beta, \gamma \in G'$ such that $\beta\gamma = \alpha'$.

Claim: $\{\beta, \gamma\}$ generates no other non-zero element in W_r apart from α' and $(\alpha')^{-1}$.

Proof of the claim: Let $\beta, \gamma \in G'$ such that $\beta\gamma = \alpha'$, by Lemma (3.10) either β in G or γ in G, but not both.

Case I: $\beta \in G$; then β and γ must be of the form:

$$\beta = \begin{pmatrix} 1 & a_2 & \ldots & a_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix} \ (c_1 \neq 1).$$

Therefore, $\gamma\beta = \beta^{-1}\gamma = \beta^{-1}\gamma^{-1} = 0$ in W_r, and $\beta\gamma^{-1} = \alpha'$ if $\{c_1, c_2, \ldots, c_r\} = \{b_1, b_2, \ldots, b_r\}$, otherwise is 0, while $\gamma^{-1}\beta^{-1} = (\alpha')^{-1}$.

Case II: $\gamma \in G$; Then β and γ must be of the form:

$$\beta = \begin{pmatrix} 1 & a_2 & \dots & a_r \\ 1 & a_2 & \dots & a_r \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \ (b_1 \neq 1).$$

It follows that

$$\begin{aligned} \beta\gamma = \beta^{-1}\gamma = \alpha', \\ \beta\gamma^{-1} = \beta^{-1}\gamma^{-1} = 0 = \gamma\beta, \\ \gamma^{-1}\beta = \gamma^{-1}\beta^{-1} = (\alpha')^{-1}. \end{aligned}$$

In both cases, $\{\beta, \gamma\}$ generate only α' and $(\alpha')^{-1}$. Therefore, $|G'| \geq |G|$. Hence, G is the minimal generating set for W_r. $\square$

To find the generating set for $M(n, r)$, we need the following proposition:

Proposition 3.12. *For $n \geq 4$, let $J_r = \{\alpha \in \mathcal{ODP}_n : |im\ \alpha| = r\}$ be the set of all elements of $\mathcal{ODP}_n$ whose height is exactly r and N be the set of all nilpotents in $\mathcal{ODP}_n$. Then $\langle J_r \cap N\rangle \subseteq \langle J_{r+1} \cap N\rangle$ for all $1 < r \leq n-3$.*

Proof. Let $\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$ be an element of $\langle J_r \cap N\rangle$, essentially we consider two cases (that is when α is nilpotent and when α is idempotent).

Case (I): If α is an idempotent, then α is either of type or type IV. Suppose α is of type I, let

$$t = \max\{x : x \in X\backslash dom\ \alpha\}$$

and

$$s = \max\{x : x \in X\backslash dom\ \alpha \,\text{and}\, x \neq t\}$$

then $t \neq 1$ and $s \neq 1$ (since $|X\backslash dom\ \alpha| \geq 3$).

Now suppose without lost of generality that t is between a_i and a_{i+1} and s is between a_j and a_{j+1}. Define β and γ as

$$\beta = \begin{pmatrix} a_1 & a_2 & \dots & a_j & a_{j+1} & \cdots & a_i & t & a_{i+1} & \dots & a_r \\ b_1 & b_2 & \dots & b_j & b_{j+1} & \cdots & b_i & t & b_{i+1} & \dots & b_r \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b_1 & b_2 & \dots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \\ b_1 & b_2 & \dots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \end{pmatrix}$$

It is clear that both β and γ are elements of $\langle J_{r+1} \cap N\rangle$ and $\alpha = \beta\ \gamma$.

If α is of type IV, we let $t = \min\{x : x \in X\backslash dom\ \alpha\}$ and $s = \min\{x : x \in X\backslash dom\ \alpha \quad \text{and} \quad x \neq t\}$ (clearly $t \neq n$ and $s \neq n$) then β and γ are elements of $\langle J_{r+1} \cap N\rangle$ and $\alpha = \beta\gamma$.

Case (II) If α is a nilpotent: Suppose $a_i > b_i \, \forall i$, then by Corollary (2.7) α is either of type $I(a_1 \neq 1$ and $b_1 \neq 1)$, or type $II(a_1 \neq 1$, $b_1 = 1$ and $b_r \neq n)$. If α is of type I, we must have that $a_1 > b_1 > 1$. Define

$$\beta = \begin{pmatrix} a_1 - 1 & a_1 & a_2 & \dots & a_r \\ b_1 - 1 & b_1 & b_2 & \dots & b_r \end{pmatrix},$$

then $b_r < a_r \leq n$ and $1 \leq b_1 < a_1$. Therefore, β is in $\langle J_{r+1} \cap N \rangle$. Also, define γ as

$$\gamma = \begin{pmatrix} b_1 & b_2 & \dots & b_i & t & b_{i+1} & \dots & a_r \\ b_1 & b_2 & \dots & b_i & t & b_{i+1} & \dots & b_r \end{pmatrix}$$

where $t = \min\{x : x \in X \backslash im \ \beta\}$. It is clear that $t \neq n$ (since $|X \backslash im \ \beta| \geq 2$), therefore, γ is also in $\langle J_{r+1} \cap N \rangle$ and $\alpha = \beta\gamma$.

But if α is of type II, we first consider a case where α has 0 total jump and $a_r = n$ (note that $a_1 \geq 3$ since $\mathrm{h}(\alpha) \leq n - 3$) and define

$$\beta = \begin{pmatrix} a_1 - 1 & a_1 & a_2 & \dots & a_r \\ 1 & 2 & 3 & \dots & r+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} 2 & 3 & \dots & r+1 & r+2 \\ 1 & 2 & \dots & r & r+1 \end{pmatrix}.$$

Otherwise we let $t = \max\{x : x \in X \backslash dom \ \alpha\}$ and define

$$\beta = \begin{pmatrix} a_1 & a_2 & \dots & a_i & t & a_{i+1} & \dots & a_r \\ 1 & b_2 & \dots & b_i & t\beta & b_{i+1} & \dots & b_r \end{pmatrix}$$

where $t\beta = b_i - a_i + t$. And let $s = \min\{x : x \in X \backslash im \ \beta\}$ and define

$$\gamma = \begin{pmatrix} 1 & b_2 & \cdots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \\ 1 & b_2 & \cdots & b_j & s & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \end{pmatrix}$$

Note that $t\beta = b_i - a_i + t < t \leq n$. Therefore, $\beta, \gamma \in \langle J_{r+1} \cap N \rangle$ and $\alpha = \beta\gamma$.

Now suppose $a_i < b_i \, \forall i$, then α is either of type I or type III. If α is of type I, then α^{-1} is also of type I with $a_i < b_i$, and if α is of type III, then α^{-1} is of type II with $a_i < b_i$. In both cases $\alpha = \gamma^{-1}\beta^{-1}$ where β, γ are as defined in the respective previous cases.

$\square$

Remark 3.13. Observe that by the above Proposition, for $1 < r \leq n-2$ the set G is the minimal generating set for the whole $M(n, r)$ as well. Now, the task of finding the rank of $M(n, r)$ is reduced to just finding the cardinality of the set G.

But before then, we cite the following lemma from [1] which we need in proving the theorem.

Lemma 3.14 ([1]). *For $r, s, t \in \mathbb{N}$ such that $r > s$,*

$$\sum_{i=0}^{r-s} \binom{r-i}{s} \binom{i+t}{t} = \binom{r+t+1}{s+t+1}.$$

Theorem 3.15. *For $n \geq 4$ and $1 < r \leq n-2$, the rank of $M(n, r)$ as an inverse semigroup is*

$$\binom{n-1}{r}.$$

Proof. From Lemma (3.8) and paragraph after Remark (3.9) the number of elements of G for any given total jump of m is

$$(n-(r+m))\binom{r+m-2}{r-2}.$$

Observe that subsets that characterized elements in G have total jumps m ranging from 0 to $n-(r+1)$ (since subsets that have $m = n-r$ are subsets of the form $\{1, \ldots, n\}$ which are not in G). Therefore,

$$\begin{aligned} |G| &= \sum_{m=0}^{n-(r+1)} (n-(r+m))\binom{r+m-2}{r-2} \\ &= \sum_{m=0}^{(n-r)-1} \binom{(n-r)-m}{1}\binom{m+(r-2)}{r-2} \end{aligned}$$

Applying Lemma (3.14) we have $|G| = \binom{n-1}{r}$. □

Corollary 3.16. *The rank of $M(n, n-2)$ is $n-1$.*

Proof. Follows directly by substituting for $r = n-2$ in Theorem (3.15) above. □

Theorem 3.17. *Let $\langle N \rangle$ be the nilpotent generated subsemigroup of $\mathcal{ODP}_n$. Then, the rank of $\langle N \rangle$ as an inverse subsemigroup is n.*

Proof. Let $\langle N \rangle$ be the nilpotent generated subsemigroup of $\mathcal{ODP}_n$, from the above corollary, the rank of $M(n, n-2)$ is $n-1$, and elements of $\langle N \rangle$ that are not in $M(n, n-2)$ are those of $\langle N \rangle \cap J_{n-1}$ (since the only element of $\mathcal{ODP}_n$ with height n is the identity element which is clearly not in $\langle N \rangle$.

Let

$$\eta = \begin{pmatrix} 1 & 2 & \ldots & n-1 \\ 2 & 3 & \ldots & n \end{pmatrix}$$

Garba in [3] have shown that, the only elements of $\langle N \rangle \cap J_{n-1}$ are

$$\eta, \eta^{-1}, \eta^{-1}\eta, \text{ and } \eta\eta^{-1}.$$

which are generated by a single element η, therefore, the rank of $\langle N\rangle$ is $(n-1)+1$ which is equal to n. Hence the proof. $\square$

4. Rank of Ideals of $\mathcal{ODP}_n$.

We extend the result obtained in section three to compute the rank of ideals of $\mathcal{ODP}_n$. Although, Alkharousi [1] investigated the rank of $\mathcal{ODP}_n$ among other things and shown that; $\mathcal{ODP}_n$ as an inverse semigroup has rank n. We want to generalize this result to find the rank of the ideal $L(n,r)$ defined in Equation (3.1). To do that, we first study elements of $\mathcal{ODP}_n$ that are neither nilpotents nor can be expressed as product of nilpotents (i. e., elements of $\mathcal{ODP}_n$ that are not in $\langle N\rangle$). We have seen from Remark (2.8), these elements are partial identities of the form

$$\begin{pmatrix} 1 & \dots & n \\ 1 & \dots & n \end{pmatrix}.$$

It is clear that, multiplying any two element of this form will yield another element of the same form, as such we have the following lemma:

Lemma 4.1. *Let $E_N = \{\alpha \in \mathcal{ODP}_n : \alpha \text{ is not in } \langle N\rangle\}$, then E_N is a subsemigroup of $\mathcal{ODP}_n$.*

Now, for $(1 < r \leq n)$, let $D(n,r) = \{\alpha \in E_N : |im\ \alpha| \leq r\}$ and $Q_r = D(n,r)\backslash D(n,r-1)$ be the subsemigroup of K_r generated by the elements of E_N, then we have the following:

Lemma 4.2. *Let K_r be the Rees quotient semigroup on $L(n,r)$, and let W_r and Q_r be subsemigroups of K_r generated by nilpotent elements and elements of E_N respectively. Then K_r is a disjoint union of W_r and Q_r.*

Proof. It follows from Remark (2.8) and the fact that W_r and Q_r are complement to each other. $\square$

Suppose we extend the set G constructed in section three (i. e., the set of all nilpotent elements in W_r whose domain is a 1-subset) to another set $Z = G \cup Q_r$, (i. e., by adding elements of Q_r) then we have:

Lemma 4.3. *Let α be in Q_r then for any $\beta, \gamma \in K_r$, $\alpha = \beta\gamma$ if and only if $\alpha = \beta = \gamma$*

Proof. Let α be in Q_r such that $\alpha = \beta\gamma$ for some $\beta, \gamma \in K_r$, since $h(\alpha) = h(\beta\gamma)$ we have that

$$\begin{aligned} dom\ \alpha &= dom\ \beta \\ im\ \beta &= dom\ \gamma \\ im\ \gamma &= im\ \alpha. \end{aligned} \tag{4.1}$$

From the above equations, and the fact that $\alpha \in Q_r$, we have $\{1, n\} \subseteq dom(\beta)$, which implies from Lemma (1.3) that $dom\ \beta = im\ \beta$. Also, $\{1, n\} \subseteq im\ \gamma$ implies that $im\ \gamma = dom\ \gamma$. Therefore, $dom\ \alpha = dom\ \beta = im\ \beta = dom\ \gamma = im\ \gamma = im\ \alpha$, which implies that $\alpha = \beta = \gamma$.

Conversely suppose that $\alpha = \beta = \gamma$, then, since α is an idempotent; $\beta\gamma = \alpha\alpha = \alpha^2\alpha$. □

Next, we have a proposition analogue to Propostion(3.11)

Proposition 4.4. *For $n \geq 4$ and $(1 < r \leq n-1)$, let Z be the union of Q_r the set of all nilpotents in K_r whose domain is a 1-subset $(Q_r \cup G)$ then Z is a minimal generating set for K_r as an inverse semigroup.*

Proof. We first show that the set Z is a generating set for K_r; Let α be in K_r, by Lemma (4.2) α is either in W_r or in Q_r. If α is in W_r, then by Proposition (3.11), α is generated by elements of $G \subseteq Z$, and if α is in Q_r, then α is in Z

For the minimality of Z; we show that if Z' is any other generating set for K_r, then $|Z'| \geq |Z|$. Now let $Z' \subseteq K_r$ such that $\langle Z' \rangle = K_r$, suppose that there exist say $\delta \in Z$ such that δ is not in Z', then δ must come from W_r. For if δ is in Q_r, then by Lemma (4.3) we cannot find any other elements in $K_r \supseteq Z'$ that can generate δ, contradicting our assumption that Z' generates K_r. Now $\delta \notin Z'$ and $\langle Z' \rangle = K_r$ imply that $\delta = \eta\theta$ for some $\eta, \theta \in Z'$.

Claim: $\{\eta, \theta\}$ generates no other non-zero element in K_r apart from δ and δ^{-1}.

Proof of the claim: Let $\eta, \theta \in Z'$ such that $\eta\theta = \delta$. Since δ, η and θ are all in K_r, we must have $dom\ \delta = dom\ \eta$ and that implies that η cannot be in Q_r (since $\{1, n\} \nsubseteq dom\ \eta$). Also, (for the same reason) $im\ \delta = im\ \theta$ which implies that θ is not in Q_r (since $\{1, n\} \nsubseteq im\ \theta$). Therefore, η and θ must be elements of W_r. Thus δ, η and θ are all elements of W_r, hence, the result follows from Proposition (3.11). □

Next, we give an extension of Proposition (3.12) - where we will see that, the set Z constructed above generates not only K_r but also $L(n, r)$; for $r \leq n-1$.

Proposition 4.5. *For $n \geq 4$ and $r \leq n-2$, let $J_r = \{\alpha \in \mathcal{ODP}_n : |im\alpha| = r\}$ be the set of all elements of $\mathcal{ODP}_n$ whose height is exactly r. Then $\langle J_r \cap \mathcal{ODP}_n \rangle \subseteq \langle J_{r+1} \cap \mathcal{ODP}_n \rangle$.*

Proof. Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

be an element of $\langle J_{r-1} \cap \mathcal{ODP}_n \rangle$, then $|X \backslash dom\alpha| \geq 2$. Now let $t = \max\{x : x \in X \backslash dom\alpha\}$ and $s = \min\{x : x \in X \backslash dom\alpha\}$, suppose without lost of generality that t is between a_i and a_{i+1} and s is between a_j and a_{j+1}. Define β and γ as

$$\beta = \begin{pmatrix} a_1 & a_2 & \dots & a_j & a_{j+1} & \cdots & a_i & t & a_{i+1} & \dots & a_r \\ a_1 & a_2 & \dots & a_j & a_{j+1} & \cdots & a_i & t & a_{i+1} & \dots & a_r \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} a_1 & a_2 & \dots & a_j & s & a_{j+1} & \cdots & a_i & a_{i+1} & \dots & a_r \\ b_1 & b_2 & \dots & b_j & s\gamma & b_{j+1} & \cdots & b_i & b_{i+1} & \dots & b_r \end{pmatrix}$$

where

$$s\gamma = \begin{cases} b_j - a_j + s, & \text{if } s > a_1; \\ b_{j+1} - a_{j+1} + s, & \text{if } s < a_1 \end{cases}$$

then β and γ are elements of $\langle J_{r+1} \cap \mathcal{ODP}_n \rangle$ and $\beta\gamma = \alpha$. Hence α is in $\langle J_{r+1} \cap \mathcal{ODP}_n \rangle$. $\square$

Observe here also that, the task of finding the rank of $L(n, r)$ is reduced just to finding the cardinality of the set Z

Theorem 4.6. *For $n \geq 4$ and $1 < r \leq n - 1$, the rank of $L(n, r)$ as an inverse semigroup is*

$$\binom{n-1}{r} + \binom{n-2}{r-2}.$$

Proof. Since Z is a disjoint union of G and Q_r, it follows from Theorem (3.15) and the fact that cardinality of Q_r is $\binom{n-2}{r-2}$. Hence

$$|Z| = \binom{n-1}{r} + \binom{n-2}{r-2}.$$

$\square$

Corollary 4.7. [3, Theorem 3.2(a)] *Let 1_{X_n} be an identity element of $\mathcal{ODP}_n$. Then the rank of $\mathcal{ODP}_n \backslash \{1_{X_n}\}$ as an inverse semigroup is $n - 1$.*

Proof. Since the only element of $\mathcal{ODP}_n$ height n is the identity element $\{1_{X_n}\}$, therefore, $\mathcal{ODP}_n \backslash \{1_{X_n}\} = L(n, n-1)$, and computing the rank of $L(n, n-1)$ from Theorem (4.6) we obtain $n - 1$. $\square$

Corollary 4.8. [3, Theorem 3.2(b)] *The rank of $\mathcal{ODP}_n$ as an inverse semigroup is n.*

Proof. Follows from Corollary (4.7) above. $\square$

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References

1. F. Al-Kharousi, R. Kehinde and A. Umar, *Combinatorial results for certain semigroups of partial isometries of a finite chain*, Australas. J. Combin. (3) **58** (2014), 365–375.
2. F. Al-Kharousi, R. Kehinde and A. Umar, *On the semigroup of partial isometries of a finite chain*, Comm. Algebra, **44** (2016), 639–647.
3. G. U. Garba, *Nilpotents in semigroup of partial one-to-one order preserving mappings*, Semigroup Forum, **48** (1994), 37–49.
4. G. U. Garba, *Nilpotents in semigroups of partial order-preserving transformations*, Proc. Edinburgh Math. Soc. **37** (1994), 361–377.
5. G. U. Garba, *On the nilpotents rank of partial transformation semigroup*, Portugaliae Mathematica, **51** (1994), 163–172.
6. G. M. S. Gomes and J. M. Howie, *Nilpotents in finite symmetric inverse semigroups*, Proc. Edinburgh Math. Soc. **30** (1987), 383-395.
7. G. M. S. Gomes and J. M. Howie, *On the ranks of certain finite semigroups of transformations*, Math. Proc. Cambridge Phil. Soc. **101** (1987), 395-403.
8. Howie, J. M. and M. P. O. Marques-Smith, *Inverse semigroups generated by nilpotent transformations*, Proc.Royal Soc. Edinburgh Sect. A, **99** (1984), 153–162.
9. J. M. Howie, *Fundamental of semigroup theory.* London Mathematical Society, New series 12. The Clarendon Press, Oxford University Press, 1995.
10. S. Limpscomb, *Symmetric Inverse Semigroups*, Mathematical Surveys of The American mathematical Society, no. 46, Providence, R. I., 1996.
11. O. Ganyushkin and V. Mazorchuk. *Classical Finite Transformation Semigroups.* Springer−Verlag: London Limited, 2009.
12. P. M. Higgins. *Techniques of semigroup theory.* Oxford university Press, 1992.
13. R. P. Sullivan, *Semigroups generated by nilpotent transformations*, J. Algebra, **110** (1987), 324–343.
14. A. Umar, *On the semigroups of partial one-to-one order-decreasing finite transformations*, Proc. Roy. Soc. Edinburgh, **123A** (1993), 355–363.
15. A. Umar, *Some combinatorial problems in the theory of symmetric inverse semigroups*, Algebra and Discrete Math. **9** (2010), 115–126.

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