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SOME RESULTS ON A SUBGRAPH OF THE INTERSECTION GRAPH OF IDEALS OF A COMMUTATIVE RING

S. VISWESWARAN * AND P. VADHEL

ABSTRACT. The rings considered in this article are commutative with identity which admit at least one nonzero proper ideal. Let R be a ring. Let us denote the collection of all proper ideals of R by $\mathbb{I}(R)$ and $\mathbb{I}(R)\backslash\{(0)\}$ by $\mathbb{I}(R)^*$. With R, we associate an undirected graph denoted by $g(R)$, whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I_1, I_2 are adjacent in $g(R)$ if and only if $I_1 \cap I_2 \neq I_1I_2$. The aim of this article is to study the interplay between the graph-theoretic properties of $g(R)$ and the ring-theoretic properties of R.

1. Introduction

The rings considered in this article are commutative with identity which admit at least one nonzero proper ideal. Let R be a ring. An ideal I of R is said to be *nontrivial* if $I \notin \{(0), R\}$. As in [4], we denote the collection of all proper ideals of R by $\mathbb{I}(R)$ and the collection $\mathbb{I}(R)\backslash\{(0)\}$by $\mathbb{I}(R)^*$. Let R be a ring with identity which is not necessarily commutative and which admits at least one nonzero left ideal I with $I \neq R$. We denote the collection of all proper left ideals of R by $\mathbb{LI}(R)$ and $\mathbb{LI}(R)\backslash\{(0)\}$ by $\mathbb{LI}(R)^*$. Recall from [5] that the *intersection graph of ideals* of R, denoted by $G(R)$, is an undirected graph whose vertex set is $\mathbb{LI}(R)^*$ and distinct vertices I_1, I_2 are adjacent in $G(R)$ if and only if $I_1 \cap I_2 \neq (0)$. Let R be a commutative ring with identity. Note that $\mathbb{LI}(R)^* = \mathbb{I}(R)^*$. In this article, we try to study some graph-theoretic

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*Corresponding author.

properties of the graph $g(R)$, whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I_1, I_2 are adjacent in $g(R)$ if and only if $I_1 \cap I_2 \neq I_1I_2$. Observe that for any ideals I_1, I_2 of a ring R, $I_1I_2 \subseteq I_1 \cap I_2$. Thus if the ideals I_1, I_2 of a ring R are such that $I_1 \cap I_2 = (0)$, then $(0) = I_1 \cap I_2 = I_1I_2$. Therefore, if distinct nontrivial ideals I_1, I_2 are adjacent in $g(R)$, then $I_1 \cap I_2 \neq (0)$ and so, I_1 and I_2 are adjacent in $G(R)$. Hence, $g(R)$ is a subgraph of $G(R)$. The intersection graph of ideals of a ring was studied by several researchers (see, for example [1, 8, 10]). Let R be a ring. Motivated by the above mentioned articles on $G(R)$, in this article, we focus our study on investigating the interplay between the graph-theoretic properties of $g(R)$ and the ring-theoretic properties of R.

We first recall some relevant definitions and notations from commutative ring theory that are used in this article. The rings considered in this article are commutative with identity. Let R be a ring. We denote the nilradical of R by $nil(R)$ and the Jacobson radical of R by $J(R)$. A ring R is said to be *reduced* if $nil(R) = (0)$. We denote the set of all prime ideals of R by $Spec(R)$ and denote the set of all maximal ideals of R by $Max(R)$. A ring which admits a unique maximal ideal is referred to as a *quasilocal* ring. A ring which admits only a finite number of maximal ideals is referred to as a *semiquasilocal* ring. A Noetherian quasilocal (respectively, a semiquasilocal) ring is referred to as a *local* (respectively, *semilocal*) ring. A principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the only prime ideal of a SPIR R, then we denote it by mentioning that $(R, \mathfrak{m})$ is a SPIR. If $\mathfrak{m}$ is the only prime ideal of a SPIR R, then $\mathfrak{m}$ is principal and it follows from [2, Proposition 1.8] that $\mathfrak{m} = nil(R)$ and so, $\mathfrak{m}$ is nilpotent. It is useful to mention here that a quasilocal ring R with unique maximal ideal $\mathfrak{m}$ is a SPIR if and only if $\mathfrak{m}$ is principal and nilpotent. Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. Let $m \in \mathfrak{m}\backslash\{0\}$ be such that $\mathfrak{m} = Rm$. Let $n \geq 2$ be the least positive integer such that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $(iii) \Rightarrow (i)$ of [2, Proposition 8.8] that $\{\mathfrak{m}^i = Rm^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R. Therefore, $(R, \mathfrak{m})$ is a SPIR.

Let R be an integral domain and let K be its quotient field. Recall from [2, page 65] that R is a *valuation ring* of K if for each $\alpha \in K\backslash\{0\}$, either $\alpha \in R$ or $\alpha^{-1} \in R$. If R is a valuation ring of K, then it is well-known that the set of ideals of R is linearly ordered by inclusion. Hence, a valuation domain is necessarily quasilocal. Let K be a field. Recall from [2, page 94] that a *discrete valuation* on K is a mapping v from $K^* = K\backslash\{0\}$ onto $\mathbb{Z}$ such that (1) $v(\alpha\beta) = v(\alpha) + v(\beta)$ and (2) $v(\alpha + \beta) \geq min(v(\alpha), v(\beta))$. It is useful to recall from [2, page

94] that an integral domain R is said to be a *discrete valuation ring* if there exists a discrete valuation v of its quotient field K such that $R = \{0\} \cup \{\alpha \in K^* | v(\alpha) \geq 0\}$.

Let R be a ring. Recall from [6, Exercise 7, page 184] that R is a *chained ring* if the set of ideals of R is linearly ordered by inclusion. If R is a chained ring, then it is clear that R is quasilocal.

Let R be a ring. Recall from [6, Exercise 16, page 111] that R is said to be *von Neumann regular* if for each $a \in R$, there exists $b \in R$ such that $a = a^2b$. The Krull dimension of a ring R is simply denoted by $dimR$. We denote the set of all units of a ring R by $U(R)$. If A and B are sets and if A is properly contained in B, then we denote it symbolically by $A \subset B$. The cardinality of a set A is denoted by $|A|$.

We next recall some definitions and notations from graph theory that we use in this article. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $a, b \in V$, $a \neq b$. Recall that the *distance* between a and b, denoted by $d(a, b)$, is defined as the length of a shortest path in G between a and b if there exists such a path in G; otherwise, we define $d(a, b) = \infty$. We define $d(a, a) = 0$. Recall from [3] that the *diameter* of G, denoted by $diam(G)$, is defined as $diam(G) = sup\{d(a, b) | a, b \in V\}$. A graph $G = (V, E)$ is said to be *connected*, if for any distinct $a, b \in V$, there exists a path in G between a and b [3]. Let $G = (V, E)$ be a connected graph. Let $a \in V$. Recall from [3] that the *eccentricity* of a denoted by $e(a)$, is defined as $e(a) = sup\{d(a, b) | b \in V\}$. The *radius* of G, denoted by $r(G)$, is defined as $r(G) = min\{e(a) | a \in V\}$. A simple graph $G = (V, E)$ is said to be *complete* if every pair of distinct vertices of G are adjacent in G [3, Definition 1.1.11]. Let $n \in \mathbb{N}$. A complete graph on n vertices is denoted by K_n. A graph $G = (V, E)$ is said to be *bipartite* if V can be partitioned into nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other end in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. A complete bipartite graph $G = (V, E)$ with $V = V_1 \cup V_2$ is said to be *star* if $|V_i| = 1$ for at least one $i \in \{1, 2\}$ [3, Definition 1.1.12].

Let $G = (V, E)$ be a graph such that G contains a cycle. Recall from [3, page 159] that the *girth* of G, denoted by $girth(G)$, is equal to the length of a shortest cycle in G. If a graph G does not contain any cycle, then we define $girth(G) = \infty$. Let $G = (V, E)$ be a graph. Recall from [3, Definition 1.2.2] that a *clique* of G is a complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$, is defined as the largest integer $n \geq 1$ such that G contains a clique on n vertices [3, page 185]. We set $\omega(G) = \infty$ if G contains a clique on n vertices for all $n \geq 1$. Recall from [3, page 129] that a *vertex coloring* of G is a map $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper* if adjacent vertices of G receive different colors of S; that is, if a and b are adjacent vertices of G, then $f(a) \neq f(b)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed for a proper vertex coloring of G [3, Definition 7.1.2]. It is well-known that for any graph G, $\omega(G) \leq \chi(G)$.

For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. A subgraph H of G is said to be a *spanning subgraph* of G if $V(H) = V(G)$. Observe that for any ring R with $|\mathbb{I}(R)^*| \geq 1$, $g(R)$ is a spanning subgraph of $G(R)$.

Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. In Section 2 of this article, some basic properties of $g(R)$ are proved. It is proved in Proposition 2.1 that $g(R)$ is connected and $diam(g(R)) \leq 2$. Let $(V, \mathfrak{m})$ be a valuation domain which is not a field. It is shown in Proposition 2.4 that $g(V) = G(V)$ if and only if $g(V)$ is complete if and only if V is a discrete valuation ring. Let $(R, \mathfrak{m})$ be a chained ring which is not an integral domain. It is proved in Proposition 2.6 that $g(R) = G(R)$ if and only if $g(R)$ is complete if and only if $(R, \mathfrak{m})$ is a SPIR. Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. If $g(R)$ does not contain any infinite clique, then it is shown in Proposition 2.13 that R is Artinian. From this result, it is deduced that $girth(g(R)) = 3$ if R is not Artinian. Let $(R, \mathfrak{m})$ be a local Artinian ring which is not a field. It is proved in Proposition 2.15 that $girth(g(R)) \in \{3, \infty\}$. Let $(R, \mathfrak{m})$ be a SPIR which is not a field. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. It is verified in Corollary 2.17 that $girth(g(R)) = \infty$ if and only if $n \in \{2, 3\}$. Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal. It is shown in Theorem 2.18 that $girth(g(R)) = \infty$ if and only if $g(R) = G(R)$ and $g(R)$ is a star graph. In Theorem 2.20, quasilocal rings $(R, \mathfrak{m})$ are characterized such that $G(R)$ is bipartite and it is proved that in the case when $G(R)$ is a bipartite graph, $g(R) = G(R)$ is a star graph. For a local Artinian ring $(R, \mathfrak{m})$ with $dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = n \geq 3$, it is shown in Theorem 2.21 that $\omega(g(R)) < \infty$ if and only if R is finite. Several examples are given to illustrate the results proved in this section.

Let R be a ring such that $|Max(R)| \geq 2$. In Section 3 of this article, some basic properties of $g(R)$ are proved. If $dim R = 0$, then it is proved in Proposition 3.4 that $g(R)$ is not connected. If R is an integral domain, then it is shown in Proposition 3.6 that $g(R)$ is connected and $diam(g(R)) = 2$. If $J(R) = (0)$, then it is verified in Corollary 3.7 that $r(g(R)) = 2$. If R is a Noetherian domain with $dim R = 1$, then it is proved in Theorem 3.9 that $r(g(R)) = 1$ if and only if R is semilocal. For a ring R with $|Max(R)| \geq 2$, some results on $girth(g(R))$ are also proved in this section. It is observed that if $dim R > 0$, then $girth(g(R)) = 3$ (see the remark in the paragraph just preceding the statement of Theorem 3.10). If $dim R = 0$ and R is reduced (that is, equivalently, if R is von Neumann regular), then it is shown in Proposition 3.5 that $g(R)$ has no edges and so, $girth(g(R)) = \infty$. Let R be such that $dim R = 0$ and R is not reduced. If $Max(R)$ is infinite, then it is proved in Theorem 3.10 that $\omega(g(R)) = \infty$ and in such a case, it is noted in Corollary 3.11 that $girth(g(R)) = 3$. If R is semiquasilocal and $dim R = 0$, then it is shown in Proposition 3.13 that if $g(R)$ does not contain any infinite clique, then R is necessarily Artinian. If R is Artinian which is not reduced and if $|Max(R)| \geq 3$, then it is proved in Corollary 3.15 that $girth(g(R)) = 3$. Let R be an Artinian ring such that $|Max(R)| = 2$ and R is not reduced. It is shown in Theorem 3.16 that $girth(g(R)) \in \{3, \infty\}$ and moreover, in Theorem 3.16, Artinian rings R with $|Max(R)| = 2$ are characterized such that $g(R)$ does not contain any cycle. Some examples are provided to illustrate the results proved in this section.

2. Some basic results in the case, where R is quasilocal

Let $(R, \mathfrak{m})$ be a quasilocal ring with $\mathfrak{m} \neq (0)$. The aim of this section is to investigate some graph-theoretic properties of $g(R)$.

Proposition 2.1. *Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. Then $g(R)$ is connected and $diam(g(R)) \leq 2$.*

Proof. Let $I, J \in V(g(R))$ be such that $I \neq J$. We claim that there exists a path of length at most two between I and J in $g(R)$. We can assume that I and J are not adjacent in $g(R)$. We consider the following cases.

Case(1): $I \cap J \neq (0)$.

Note that for any nonzero proper ideal A of R and $a \in A \backslash \{0\}$, $Ra \neq Aa$. For if $Ra = Aa$, then $a = ba$ for some $b \in A$. This implies that $a(1 - b) = 0$. As $1 - b \in U(R)$, we obtain that $a = 0$. This is a contradiction and so, $Ra \neq Aa$. Thus $Ra = Ra \cap A \neq Aa$ and so, A and Ra are adjacent in $g(R)$ if $A \neq Ra$. Let $x \in I \cap J, x \neq 0$. Since $I \neq J$, it follows that either $I \neq Rx$ or $J \neq Rx$. Without loss of generality, we can assume that $I \neq Rx$. As $x \in I \backslash \{0\}$, we get that I and Rx are adjacent in $g(R)$. Since we are assuming that I and J are not adjacent in $g(R)$, it follows that $J \neq Rx$. As $x \in J \backslash \{0\}$, we obtain that Rx and J are adjacent in $g(R)$. Therefore, we obtain that

$I - Rx - J$ is a path of length two between I and J in $g(R)$.

Case(2): $I \cap J = (0)$.

From $I \cap J = (0)$, it follows that $IJ = (0)$. Let $a \in I\backslash\{0\}$ and $b \in J\backslash\{0\}$. Let us denote the ideal $Ra + Rb$ by A. It is clear that $A \in V(g(R))$. As $a \notin J$ and $b \notin I$, it follows that $A \notin \{I, J\}$. Note that $a \in I \cap A$ and it follows from $Ib = (0)$ that $IA = Ia$. As $a \notin Ia$, it follows that $I \cap A \neq IA$. Hence, I and A are adjacent in $g(R)$. Similarly, note that $b \in A \cap J$ and it follows from $Ja = (0)$ that $AJ = bJ$. From $b \notin bJ$, it follows that $A \cap J \neq AJ$. Hence, A and J are adjacent in $g(R)$. Therefore, $I - A - J$ is a path of length two between I and J in $g(R)$.

This proves that $g(R)$ is connected and $diam(g(R)) \leq 2$. □

We next try to determine quasilocal rings $(R, \mathfrak{m})$ with $\mathfrak{m} \neq (0)$ such that $g(R)$ is complete. In Lemma 2.2, we provide some necessary conditions in order that $g(R)$ is complete.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. If $g(R)$ is complete, then the following hold.*

(i) Either $dim R = 0$ or R is an integral domain with $dim R = 1$.

(ii) $I \neq I^2$ for any $I \in V(g(R))$.

Proof. We are assuming that $g(R)$ is complete.

(i) Suppose that $dim R > 0$. Let $\mathfrak{p} \in Spec(R)$ be such that $\mathfrak{p} \subset \mathfrak{m}$. We claim that $\mathfrak{p} = (0)$. Suppose that $\mathfrak{p} \neq (0)$. Then $\mathfrak{p} \in V(g(R))$. Let $m \in \mathfrak{m}\backslash\mathfrak{p}$. It is clear that $Rm \in V(g(R))$ and $\mathfrak{p} \neq Rm$. If $x \in \mathfrak{p} \cap Rm$, then $x = rm \in \mathfrak{p}$ for some $r \in R$. As $m \notin \mathfrak{p}$, we get that $r \in \mathfrak{p}$. Hence, $x \in \mathfrak{p}m$. This shows that $\mathfrak{p} \cap Rm \subseteq \mathfrak{p}m$ and so, $\mathfrak{p} \cap Rm = \mathfrak{p}m$. This implies that $\mathfrak{p}$ and Rm are not adjacent in $g(R)$. This is a contradiction. Therefore, $\mathfrak{p} = (0)$. Thus if $g(R)$ is complete, then either $dim R = 0$ or R is an integral domain and $dim R = 1$.

(ii) Let $I \in V(g(R))$. If $I^2 = (0)$, then it is clear that $I \neq I^2$. Hence, we can assume that $I^2 \neq (0)$. Therefore, there exists $a \in I$ such that $Ia \neq (0)$. It is already noted in the proof of Proposition 2.1 that $a \notin Ia$ and so, $I \neq Ia$. Now, $Ia \in V(g(R))$ and $I \cap Ia = Ia$. Since I and Ia are adjacent in $g(R)$, we obtain that $Ia = I \cap Ia \neq I^2a$ and so, $I \neq I^2$. □

Remark 2.3. Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. Then $G(R)$ is connected with $diam(G(R)) \leq 2$ and if $|\mathbb{I}(R)^*| \geq 2$, then $r(G(R)) = 1$.

Proof. Let $I_1, I_2 \in \mathbb{I}(R)^*$ be such that $I_1 \neq I_2$. Suppose that I_1 and I_2 are not adjacent in $G(R)$. Hence, $I_1 \cap I_2 = (0)$. Note that $I_1 - \mathfrak{m} - I_2$ is a path of length two between I_1 and I_2 in $G(R)$. This shows that $G(R)$ is connected and $diam(G(R)) \leq 2$. If $|\mathbb{I}(R)^*| \geq 2$, then $d(\mathfrak{m}, I) = 1$ in $G(R)$ for each $I \in \mathbb{I}(R)^* \backslash \{\mathfrak{m}\}$ and so, $e(\mathfrak{m}) = 1$ in $G(R)$. Hence, $r(G(R)) = 1$. □

Let R be an integral domain which is not a field (R is not necessarily quasilocal). Then for any $I_1, I_2 \in \mathbb{I}(R)^*$, it is clear that $I_1 \cap I_2 \in \mathbb{I}(R)^*$ and so, $G(R)$ is complete. We proceed to discuss some results regarding the status of this result for $g(R)$, where R is a quasilocal integral domain which is not a field. Let $(V, \mathfrak{m})$ be a valuation domain which is not a field. In Proposition 2.4, we characterize valuation domains V such that $g(V)$ is complete. In Example 2.5, we provide an example of a valuation domain V such that $diam(g(V)) = 2$.

Proposition 2.4. *Let $(V, \mathfrak{m})$ be a valuation domain which is not a field. The following statements are equivalent:*
(i) $g(V) = G(V)$;
(ii) $g(V)$ is complete;
(iii) V is a discrete valuation ring.

Proof. $(i) \Rightarrow (ii)$ As V is an integral domain which is not a field, $G(V)$ is complete. Hence, from $g(V) = G(V)$, it follows that $g(V)$ is complete.
$(ii) \Rightarrow (iii)$ We are assuming that $g(V)$ is complete. Hence, we obtain from Lemma 2.2(i) that $dimV = 1$. We know from Lemma 2.2(ii) that $\mathfrak{m} \neq \mathfrak{m}^2$. Let $m \in \mathfrak{m} \backslash \mathfrak{m}^2$. We claim that $\mathfrak{m} = Vm$. It is clear that $Vm \subseteq \mathfrak{m}$. Let $a \in \mathfrak{m}$. We want to prove that $Va \subseteq Vm$. Since the ideals of V are comparable under the inclusion relation, it follows that either $Va \subseteq Vm$ or $Vm \subseteq Va$. There is nothing to prove if $Va \subseteq Vm$. Hence, we need to consider the case in which $Vm \subseteq Va$. If $Vm \subseteq Va$, then $m = va$ for some $v \in V$. As $m \notin \mathfrak{m}^2$, it follows that v is a unit in V and so, $a = v^{-1}m \in Vm$. This proves that $\mathfrak{m} \subseteq Vm$ and so, $\mathfrak{m} = Vm$. Let I be any nonzero proper ideal of V. We assert that $I = \mathfrak{m}^n = Vm^n$ for some $n \geq 1$. We can assume that $I \neq \mathfrak{m}$. Since $dimV = 1$, it follows from [2, Proposition 1.14] that $\sqrt{I} = \mathfrak{m} = Vm$. Hence, $\mathfrak{m}^n \subseteq I$ for some $n \geq 1$. As $I \neq \mathfrak{m}$, it follows that $n \geq 2$. Let $R = \frac{V}{\mathfrak{m}^n}$. Note that $\mathfrak{n} = \frac{\mathfrak{m}}{\mathfrak{m}^n}$ is the unique maximal ideal of R, $\mathfrak{n}$ is principal, and $\mathfrak{n}^n = (0 + I)$. Hence, we obtain from the proof of $(iii) \Rightarrow (i)$ of [2, Proposition 8.8] that $\frac{I}{\mathfrak{m}^n} = \frac{\mathfrak{m}^i}{\mathfrak{m}^n}$ for some i such that $2 \leq i \leq n$. Therefore, $I = \mathfrak{m}^i$. Now, it follows from $(v) \Rightarrow (i)$ of [2, Proposition 9.2] that V is a discrete valuation ring.
$(iii) \Rightarrow (i)$ We are assuming that V is a discrete valuation ring. Hence, we obtain from $(i) \Rightarrow (vi)$ of [2, Proposition 9.2] that there exists

$m \in \mathfrak{m}$ such that $\mathfrak{m} = Vm$ and $\{Vm^n | n \in \mathbb{N}\}$ is the set of all nonzero proper ideals of V. Let I, J be distinct nonzero proper ideals of V. Note that $I = Vm^i$ and $J = Vm^j$ for some distinct $i, j \in \mathbb{N}$. We can assume without loss of generality that $i < j$. Now, $I \cap J = J$ and $IJ = Vm^{i+j}$. It is clear that $I \cap J = J \neq IJ = Vm^{i+j}$. This shows that I and J are adjacent in $g(V)$ for any distinct nonzero proper ideals I, J of V. Therefore, we get that $g(V)$ is complete. Since $g(V)$ is a spanning subgraph of $G(V)$ and as $g(V)$ is complete, we obtain that $g(V) = G(V)$. □

Example 2.5. Consider the totally ordered abelian group $(\mathbb{Q}, +)$. We know from [2, Exercise 33, page 72] that it is possible to construct a field K and a valuation v of K such that the value group of v is $(\mathbb{Q}, +)$. Let V be the valuation ring of v. Then $diam(g(V)) = 2$, $g(V) \neq G(V)$, and $r(g(V)) = 1$.

Proof. Let $\mathfrak{m}$ denote the unique maximal ideal of V. We know from Proposition 2.1 that $g(V)$ is connected and $diam(g(V)) \leq 2$. As $|V(g(V))| \geq 2$, it follows that $diam(g(V)) \geq 1$. Since the value group of v is $(\mathbb{Q}, +)$, it follows that $\mathfrak{m} = \mathfrak{m}^2$. Therefore, we obtain from Lemma 2.2 (ii) that $diam(g(V)) \geq 2$ and so, $diam(g(V)) = 2$. Since $G(V)$ is complete, it follows that $g(V) \neq G(V)$. Let $m \in \mathfrak{m}, m \neq 0$. Let $A = Vm$. We claim that $e(A) = 1$ in $g(V)$. Let $I \in V(g(V)), I \neq A$. Then either $A \subset I$ or $I \subset A$. Suppose that $A \subset I$. Then $A \cap I = A$. Note that $AI = Im$. If $m \in Im$, then $m = am$ for some $a \in I$. This implies that $m(1 - a) = 0$. Since $1 - a \in U(V)$, we obtain that $m = 0$. This is a contradiction and so, $m \notin Im$. Hence, $A = A \cap I \neq AI$. Suppose that $I \subset A$. Then $A \cap I = I$. If $A \cap I = AI$, then we obtain that $I = Im$. This implies that $I = Im^n$ for all $n \in \mathbb{N}$. Hence, $I \subseteq \cap_{n=1}^{\infty} Vm^n$. Let $a \in I \backslash \{0\}$. Note that for each $n \in \mathbb{N}$, there exists $v_n \in V$ such that $a = v_n m^n$. This implies that $v(a) \geq nv(m)$ and so, $\frac{v(a)}{v(m)} \geq n$ for each $n \in \mathbb{N}$. This is impossible since $\frac{v(a)}{v(m)}$ is a positive rational number. Hence, $A \cap I \neq AI$. This shows that $d(A, I) = 1$ in $g(V)$ for any $I \in \mathbb{I}(V)^*$ with $I \neq A$. Therefore, $e(A) = 1$ in $g(V)$ and so, $r(g(V)) = 1$. □

Let R be a chained ring which is not an integral domain, In Proposition 2.6, we determine necessary and sufficient conditions for $g(R)$ to be complete.

Proposition 2.6. *Let $(R, \mathfrak{m})$ be a chained ring which is not an integral domain. The following statements are equivalent:*
(i) *$g(R) = G(R)$;*
(ii) *$g(R)$ is complete;*

(iii) $(R, \mathfrak{m})$ *is a SPIR.*

Proof. $(i) \Rightarrow (ii)$ Let $I_1, I_2 \in \mathbb{I}(R)^*$ be such that $I_1 \neq I_2$. Since the set of ideals of R is linearly ordered by inclusion, it follows that either $I_1 \subset I_2$ or $I_2 \subset I_1$ and so, $I_1 \cap I_2 \neq (0)$. Hence, I_1 and I_2 are adjacent in $G(R)$. This shows that $G(R)$ is complete. As we are assuming that $g(R) = G(R)$, we get that $g(R)$ is complete.
$(ii) \Rightarrow (iii)$ We are assuming that $g(R)$ is complete. Hence, we obtain from Lemma 2.2(i) that $dim R = 0$. Hence, $\mathfrak{m}$ is the only prime ideal of R. Therefore, we obtain from [2, Proposition 1.8] that $nil(R) = \mathfrak{m}$. We know from Lemma 2.2(ii) that $\mathfrak{m} \neq \mathfrak{m}^2$. Since the ideals of R are comparable under the inclusion relation, it follows as in the proof of $(ii) \Rightarrow (iii)$ of Proposition 2.4 that $\mathfrak{m} = Rm$ for any $m \in \mathfrak{m} \backslash \mathfrak{m}^2$. As $\mathfrak{m} = nil(R)$, we obtain that there exists $n \geq 2$ least with the property that $\mathfrak{m}^n = Rm^n = (0)$. It now follows from the proof of $(iii) \Rightarrow (i)$ of [2, Proposition 8.8] that $\{Rm^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R. Therefore, $(R, \mathfrak{m})$ is a SPIR.
$(iii) \Rightarrow (i)$ We are assuming that $(R, \mathfrak{m})$ is a SPIR. Let $m \in \mathfrak{m} \backslash \{0\}$ be such that $\mathfrak{m} = Rm$. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Observe that $\{Rm^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R. It can be shown as in the proof of $(iii) \Rightarrow (i)$ of Proposition 2.4 that $g(R)$ is complete. Since $g(R)$ is a spanning subgraph of $G(R)$ and $g(R)$ is complete, we obtain that $g(R) = G(R)$. □

Example 2.7. Let $(V, \mathfrak{m})$ be the valuation domain considered in Example 2.5. Let $m \in \mathfrak{m}, m \neq 0$. Let $R = \frac{V}{mV}$. Then $diam(g(R)) = 2$, $g(R) \neq G(R)$, and $r(g(R)) = 1$.

Proof. Observe that R is a chained ring with $\mathfrak{n} = \frac{\mathfrak{m}}{mV}$ as its unique maximal ideal. It is already noted in Example 2.5 that $\mathfrak{m} = \mathfrak{m}^2$. Hence, $\mathfrak{n} = \mathfrak{n}^2$. It can be shown as in the proof of Example 2.5 that $diam(g(R)) = 2$. Since R is a chained ring, we know from the proof of $(i) \Rightarrow (ii)$ of Proposition 2.6 that $G(R)$ is complete. Therefore, $g(R) \neq G(R)$. Let $y \in \mathfrak{m} \backslash mV$. Let $A = \frac{yV}{mV}$. Let $B \in \mathbb{I}(R)^*$ with $B \neq A$. Then either $A \subset B$ or $B \subset A$. Observe that $B = \frac{I}{mV}$ for some $I \in \mathbb{I}(V)^*$ with $mV \subset I$. We claim that $A \cap B \neq AB$. Suppose that $A \cap B = AB$. It is clear that $AB = \frac{Iy+mV}{mV}$. If $A \subset B$, then $A \cap B = A$ and in such a case, we obtain that $yV = Iy + mV$. Note that $m = yw$ for some $w \in \mathfrak{m}$. Hence, $y = y(a + wv)$ for some $a \in I$ and $v \in V$. This implies that $y(1 - a - wv) = 0$. Since $1 - a - wv \in U(V)$, we get that $y = 0$. This is a contradiction. If $B \subset A$, then $A \cap B = B$. It follows from the assumption $A \cap B = AB$ that $B = AB$ and so, $B = A^n B$ for each $n \in \mathbb{N}$. This implies that $I = Iy^n + mV$ for each $n \in \mathbb{N}$. Since the value group of V is isomorphic to $(\mathbb{Q}, +)$, it follows from [2, Exercise 32, page 72] that $\mathfrak{m}$ is the only nonzero prime ideal of V. Therefore, we obtain from [2, Proposition 1.14] that $\sqrt{yV} = \sqrt{mV} = \mathfrak{m}$. Hence, $y^n \in mV$ for some $n \in \mathbb{N}$. It follows from $I = Iy^n + mV$ that $I \subseteq mV$. This is impossible since $mV \subset I$. Therefore, $A \cap B \neq AB$ and so, A and B are adjacent in $g(R)$ for any $B \in \mathbb{I}(R)^*$ with $B \neq A$. This shows that $e(A) = 1$ in $g(R)$ and so, we get that $r(g(R)) = 1$. □

Lemma 2.8. *Let* $(R, \mathfrak{m})$ *be a quasilocal ring such that* $|V(g(R))| \geq 2$. *If* $\cap_{n=1}^{\infty} \mathfrak{m}^n = (0)$, *then* $r(g(R)) = 1$.

Proof. We know from Proposition 2.1 that $g(R)$ is connected. We claim that $e(\mathfrak{m}) = 1$ in $g(R)$. By hypothesis, $|V(g(R)| \geq 2$. Let $I \in V(g(R))$ be such that $I \neq \mathfrak{m}$. Note that $I \cap \mathfrak{m} = I$. If $I = I\mathfrak{m}$, then we obtain that $I = I\mathfrak{m}^n$ for all $n \geq 1$ and this implies that $I \subseteq \cap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. This is impossible since $I \neq (0)$. Therefore, $I = I \cap \mathfrak{m} \neq I\mathfrak{m}$. Hence, $d(\mathfrak{m}, I) = 1$ for each $I \in V(g(R))$ with $I \neq \mathfrak{m}$ and so, $e(\mathfrak{m}) = 1$ in $g(R)$. This proves that $r(g(R)) = 1$. □

Proposition 2.9. *Let* $(R, \mathfrak{m})$ *be a quasilocal reduced ring which is not an integral domain. Then* $diam(g(R)) = 2$. *If* $\cap_{n=1}^{\infty} \mathfrak{m}^n = (0)$, *then* $r(g(R)) = 1$.

Proof. We know from Proposition 2.1 that $g(R)$ is connected and $diam(g(R)) \leq 2$. Since R is not an integral domain, there exist $x, y \in R\backslash\{0\}$ such that $xy = 0$. As R is reduced, it follows that $Rx \neq Ry$ and $Rx \cap Ry = (0)$. Thus $Rx \cap Ry = Rxy = (0)$. Hence, Rx and Ry are not adjacent in $g(R)$. Indeed, Rx and Ry are not adjacent in $G(R)$.(This part of the proof does not use the hypothesis that R is quasilocal.) Therefore, we obtain that $diam(g(R)) \geq 2$ and so, $diam(g(R)) = 2$. If $\cap_{n=1}^{\infty} \mathfrak{m}^n = (0)$, then we obtain from Lemma 2.8 that $r(g(R)) = 1$. □

Let $(R, \mathfrak{m})$ be a quasilocal reduced ring which is not an integral domain. We know from Remark 2.3 that $G(R)$ is connected and $diam(G(R)) \leq 2$. As R is not an integral domain, it follows that $diam(G(R)) \geq 2$ (see the proof of Proposition 2.9) and so, $diam(G(R)) = 2$. Since $|\mathbb{I}(R)^*| \geq 2$, we obtain from Remark 2.3 that $r(G(R)) = 1$. We provide in Example 2.10, an example of a local reduced ring $(R, \mathfrak{m})$ which is not an integral domain such that $G(R) \neq g(R)$.

Example 2.10. Let $T = K[[X, Y]]$ be the power series in two variables X, Y over a field K. Let us denote the ideal $TX \cap TY$ by I. Let $R = \frac{T}{I}$. Then R is a local reduced ring, R is not an integral domain, and is such that $G(R) \neq g(R)$.

Proof. We know from [2, Exercise 5(iv), page 11] that $\mathfrak{m} = TX + TY$ is the only maximal ideal of T. We know from [9, Theorem 71] that T is Noetherian. Hence, we obtain that $(T, \mathfrak{m})$ is local. Therefore, R is local with $\mathfrak{n} = \frac{\mathfrak{m}}{I}$ as its unique maximal ideal. Observe that $\mathfrak{p}_1 = TX$ and $\mathfrak{p}_2 = TY$ are prime ideals of T and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Hence, $R = \frac{T}{I}$ is reduced. Observe that $X \notin TY$, $Y \notin TX$, and $I = TXY$. Let us denote $X + I$ by x and $Y + I$ by y. Note that x and y are nonzero elements of R. Since $XY \in I$, we get that $xy = 0 + I$. Therefore, R is not an integral domain. As is mentioned in the introduction, we know that $g(R)$ is a spanning subgraph of $G(R)$. Since TX and $T(X+Y)$ are incomparable prime ideals of T, we obtain that $TX \cap T(X+Y) = T(X^2 + XY)$. Hence, $Rx, R(x+y) \in \mathbb{I}(R)^*$ are such that $Rx \cap R(x+y) = R(x^2+xy) = Rx^2$. Therefore, Rx and $R(x+y)$ are not adjacent in $g(R)$. As $x^2 \neq 0 + I$, it follows that Rx and $R(x+y)$ are adjacent in $G(R)$. This proves that $G(R) \neq g(R)$. It is noted in the paragraph just preceding the statement of Example 2.10 that $diam(G(R)) = 2$ and $r(G(R)) = 1$. Since $(R, \mathfrak{n})$ is a local ring, we obtain from [2, Corollary 10.20] that $\cap_{n=1}^{\infty} \mathfrak{n}^n = (0 + I)$. Therefore, we obtain from Proposition 2.9 that $diam(g(R)) = 2$ and $r(g(R)) = 1$. □

Proposition 2.11. *Let R be a ring such that $dim R > 0$. Then $g(R)$ contains an infinite clique. In particular, if $(R, \mathfrak{m})$ is a quasilocal ring such that $\mathfrak{m} \neq nil(R)$, then $g(R)$ contains an infinite clique.*

Proof. By hypothesis, $dim R > 0$. Hence, there exist prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ of R such that $\mathfrak{p}_1 \subset \mathfrak{p}_2$. Let $a \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$. Then $a^n \notin \mathfrak{p}_1$ for each $n \in \mathbb{N}$ and so, $a^n \neq 0$ for all $n \in \mathbb{N}$. Let $i, j \in \mathbb{N}$ with $i \neq j$. We claim that $Ra^i \neq Ra^j$. We can assume that $i < j$. Suppose that $Ra^i = Ra^j$. Then $a^i = ra^j$ for some $r \in R$. This implies that $a^i(1 - ra^{j-i}) = 0$. As $a^i \notin \mathfrak{p}_1$, we obtain that $1 - ra^{j-i} \in \mathfrak{p}_1 \subset \mathfrak{p}_2$. Since $a \in \mathfrak{p}_2$, it follows that $1 = 1 - ra^{j-i} + ra^{j-i} \in \mathfrak{p}_2$. This is impossible and so, $Ra^i \neq Ra^j$. Let $t, k \in \mathbb{N}$ with $t \neq k$. Note that $Ra^t \cap Ra^k = Ra^{max(t,k)} \neq Ra^{t+k} = (Ra^t)(Ra^k)$. Hence, Ra^t and Ra^k are adjacent in $g(R)$. Therefore, the subgraph of $g(R)$ induced on $\{Ra^n | n \in \mathbb{N}\}$ is an infinite clique.

We next verify the in particular statement of this Proposition. If $(R, \mathfrak{m})$ is a quasilocal ring with $\mathfrak{m} \neq nil(R)$, then it follows from [2, Proposition 1.8] that there exists $\mathfrak{p} \in Spec(R)$ such that $\mathfrak{p} \subset \mathfrak{m}$ and so, $dim R > 0$. Therefore, it follows as in the previous paragraph that $g(R)$ contains an infinite clique. □

Proposition 2.12. *Let R be a ring. If there exists an ideal I of R with $I \subseteq J(R)$ such that I is not finitely generated, then $g(R)$ contains an* *infinite clique. In particular, if* $(R, \mathfrak{m})$ *is a quasilocal ring such that* I *is not finitely generated for some proper ideal* I *of* R*, then* $g(R)$ *contains an infinite clique.*

Proof. Since we are assuming that there exists an ideal $I \subseteq J(R)$ such that I is not finitely generated, there exists $x_n \in J(R)\backslash\{0\}$ for each $n \in \mathbb{N}$ such that $Rx_1 + \cdots + Rx_{n-1} \subset Rx_1 + Rx_2 + \cdots + Rx_n$ for all $n \geq 2$. For each $n \in \mathbb{N}$, let us denote the ideal $Rx_1 + \cdots + Rx_n$ by I_n. We claim that the subgraph of $g(R)$ induced on $\{I_n | n \in \mathbb{N}\}$ is a clique. Let $i, j \in \mathbb{N}$ with $i \neq j$. We can assume that $i < j$. As $I_i \subset I_j$, it follows that $I_i \cap I_j = I_i$. Observe that $I_i \cap I_j \neq I_i I_j$. For if $I_i \cap I_j = I_i I_j$, then we get that $I_i = I_i I_j$. As I_i is finitely generated and $I_j \subseteq J(R)$, we obtain from Nakayama's lemma [2, Proposition 2.6] that $I_i = (0)$. This is a contradiction. Therefore, $I_i \cap I_j \neq I_i I_j$. Hence, I_i and I_j are adjacent in $g(R)$ for all distinct $i, j \in \mathbb{N}$. This shows that the subgraph of $g(R)$ induced on $\{I_n | n \in \mathbb{N}\}$ is a clique and so, we obtain that $g(R)$ contains an infinite clique.

We next verify the in particular statement of this Proposition. Suppose that $(R, \mathfrak{m})$ is a quasilocal ring such that I is not finitely generated for some proper ideal I of R. Observe that $I \subseteq \mathfrak{m}$, $J(R) = \mathfrak{m}$, and so, we obtain as in the previous paragraph that $g(R)$ contains an infinite clique. □

Proposition 2.13. *Let* $(R, \mathfrak{m})$ *be a quasilocal ring which is not a field. If* $g(R)$ *does not contain any infinite clique, then* R *is Artinian. In particular, if* R *is not Artinian, then* $girth(g(R)) = 3$.

Proof. Since we are assuming that $g(R)$ does not contain any infinite clique, we obtain from Proposition 2.11 that $dim R = 0$ and it follows from Proposition 2.12 that each ideal of R is finitely generated. Therefore, R is Noetherian. Thus R is Noetherian and $dim R = 0$. Hence, it follows from [2, Theorem 8.5] that R is Artinian.

We next verify the in particular statement of this Proposition. Suppose that R is not Artinian. Then it follows that $g(R)$ contains an infinite clique and so, $girth(g(R)) = 3$. □

Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. In view of Proposition 2.13, in determining $girth(g(R))$, we can assume that R is Artinian. If $(R, \mathfrak{m})$ is a local Artinian ring, then we show in Proposition 2.15 that $girth(g(R)) \in \{3, \infty\}$.

Lemma 2.14. *Let* $(R, \mathfrak{m})$ *be a local ring. Let* $I, J \in \mathbb{I}(R)^*$ *be such that* $I \subset J$*. Then* I *and* J *are adjacent in* $g(R)$*.*

Proof. As $I \subset J$, it follows that $I \cap J = I$. Since R is Noetherian, I is finitely generated. Now, $I \neq (0)$, $J \subseteq \mathfrak{m} = J(R)$ and so, we obtain from Nakayama's lemma [2, Proposition 2.6] that $I \neq IJ$. Therefore, $I \cap J \neq IJ$. Therefore, I and J are adjacent in $g(R)$. $\square$

Proposition 2.15. *Let $(R, \mathfrak{m})$ be a local Artinian ring which is not a field. The following statements are equivalent:*
(i) $g(R)$ contains a cycle;
(ii) $girth(g(R)) = 3$.

Proof. Let $I \in \mathbb{I}(R)^*$ be such that $I \neq \mathfrak{m}$. Since R is Artinian, we know from [2, Theorem 8.5] that R is Noetherian. Hence, we obtain from Lemma 2.14 that I and $\mathfrak{m}$ are adjacent in $g(R)$.
$(i) \Rightarrow (ii)$ We are assuming that $g(R)$ contains a cycle. Hence, there exist $I_1, I_2 \in \mathbb{I}(R)^* \backslash \{\mathfrak{m}\}$ such that I_1 and I_2 are adjacent in $g(R)$. As $\mathfrak{m}$ and I are adjacent in $g(R)$ for any $I \in \mathbb{I}(R)^* \backslash \{\mathfrak{m}\}$, we get that $I_1 - I_2 - \mathfrak{m} - I_1$ is a cycle of length three in $g(R)$. Therefore, we obtain that $girth(g(R)) = 3$.
$(ii) \Rightarrow (i)$ This is clear. $\square$

We next try to characterize local Artinian rings $(R, \mathfrak{m})$ which are not fields such that $g(R)$ does not contain any cycle. First, we assume that $\mathfrak{m}$ is principal. In such a case, we know from the proof of $(iii) \Rightarrow (i)$ of [2, Proposition 8.8] that $(R, \mathfrak{m})$ is SPIR.

Lemma 2.16. *Let $(R, \mathfrak{m})$ be a SPIR which is not a field. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then $\omega(g(R)) = \omega(G(R)) = n - 1$.*

Proof. Note that $\mathfrak{m}$ is principal and $n \geq 2$ is least with the property that $\mathfrak{m}^n = (0)$. Therefore, we obtain from the proof of $(iii) \Rightarrow (i)$ of [2, Proposition 8.8] that $\{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of R. Moreover, we know from $(iii) \Rightarrow (ii)$ of Proposition 2.6 that $g(R)$ is complete. Hence, we obtain that $\omega(g(R)) = n - 1$. From $(iii) \Rightarrow (i)$ of Proposition 2.6, we get that $g(R) = G(R)$ and so, $\omega(G(R)) = n - 1$. $\square$

Corollary 2.17. *Let $(R, \mathfrak{m})$ be a SPIR which is not a field. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then $girth(g(R)) = girth(G(R)) = \infty$ if and only if $n \in \{2, 3\}$.*

Proof. We know from the proof of Lemma 2.16 that $g(R) = G(R)$ is a complete graph on $n - 1$ vertices. Therefore, it is clear that $girth(g(R)) = girth(G(R)) = \infty$ if and only if $n - 1 \in \{1, 2\}$, that is, if and only if $n \in \{2, 3\}$. $\square$

Theorem 2.18. *Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal. Then the following statements are equivalent:*

(*i*) *girth*$(G(R)) = \infty$;
(*ii*) *girth*$(g(R)) = \infty$;
(*iii*) *Each* $I \in \mathbb{I}(R)^*\backslash\{\mathfrak{m}\}$ *is a minimal ideal of* R;
(*iv*) $g(R) = G(R)$ *and* $g(R)$ *is a star graph.*

Proof. $(i) \Rightarrow (ii)$ We are assuming that $G(R)$ does not contain any cycle. As $g(R)$ is a spanning subgraph of $G(R)$, it follows that $g(R)$ does not contain any cycle and so, $girth(g(R)) = \infty$.
$(ii) \Rightarrow (iii)$ Let $I \in \mathbb{I}(R)^*$ be such that $I \neq \mathfrak{m}$. If I is not a minimal ideal of R, then there exists $J \in \mathbb{I}(R)^*$ such that $J \subset I$. Then it follows from Lemma 2.14 that $I - J - \mathfrak{m} - I$ is a cycle of length 3 in $g(R)$. This is in contradiction to the assumption that $girth(g(R)) = \infty$. Therefore, I is a minimal ideal of R.
$(iii) \Rightarrow (iv)$ By hypothesis, $\mathfrak{m}$ is not principal. Hence, there are ideals $I \in \mathbb{I}(R)^*$ such that $I \neq \mathfrak{m}$. Note that $V(g(R)) = V(G(R)) = \{\mathfrak{m}\} \cup \{I \in \mathbb{I}(R)^* | I \neq \mathfrak{m}\}$. We know from Lemma 2.14 that in $g(R)$, $\mathfrak{m}$ is adjacent to any $I \in \mathbb{I}(R)^*$ such that $I \neq \mathfrak{m}$ and so, $\mathfrak{m}$ is adjacent to any $I \in \mathbb{I}(R)^*$ with $I \neq \mathfrak{m}$ in $G(R)$. Let $I, J \in \mathbb{I}(R)^*\backslash\{\mathfrak{m}\}$ with $I \neq J$. By hypothesis, I, J are minimal ideals of R. As $I \neq J$, we obtain that $I \cap J = (0)$. Therefore, I and J are not adjacent in $G(R)$ and so, they are not adjacent in $g(R)$. This shows that $g(R) = G(R)$ and $g(R)$ is a star graph.
$(iv) \Rightarrow (i)$ Since $G(R)$ is a star graph, we get that $girth(G(R)) = \infty$. □

We provide some examples in Example 2.19 to illustrate Theorem 2.18.

Example 2.19. (*i*) Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $I = \mathfrak{m}^2$, where $\mathfrak{m} = TX + TY$. Let $R = \frac{T}{I}$. Then $(R, \frac{\mathfrak{m}}{I})$ is a local Artinian ring with $girth(g(R)) = \infty$ and $g(R) = G(R)$.
(*ii*) Let T be as in (*i*) and let $J = TX^2 + TY^2$. Let $R = \frac{T}{J}$. Then $(R, \frac{\mathfrak{m}}{J})$, where $\mathfrak{m}$ is as in (*i*), is a local Artinian ring with $girth(g(R)) = 3$ and $g(R) \neq G(R)$.

Proof. (*i*) Note that by Hilbert's basis theorem [2, Theorem 7.5], T is Noetherian. As $\mathfrak{m} \in Max(T)$, it follows that $\frac{\mathfrak{m}}{I} = \frac{\mathfrak{m}}{\mathfrak{m}^2}$ is the only prime ideal of R. Hence, $R = \frac{T}{I}$ is a Noetherian ring and $dim R = 0$. Therefore, we obtain from [2, Theorem 8.5] that R is Artinian and so, $(R, \frac{\mathfrak{m}}{I})$ is a local Artinian ring. It is convenient to denote $X + I$ by x and $Y + I$ by y. Observe that $\frac{\mathfrak{m}}{I} = Rx + Ry$ is not principal and it is not hard to verify that $\mathbb{I}(R)^* = \{Rx, Ry, R(x + \alpha y), Rx + Ry | \alpha \in K\backslash\{0\}\}$ and so, each nonzero proper ideal of R other than its unique maximal

ideal is a minimal ideal of R. Therefore, we obtain from $(iii) \Rightarrow (ii)$ of Theorem 2.18 that $girth(g(R)) = \infty$. Moreover, we obtain from $(iii) \Rightarrow (iv)$ of Theorem 2.18 that $g(R) = G(R)$.
(ii) Since $\mathfrak{m} = TX + TY \in Max(T)$, it follows that $\frac{\mathfrak{m}}{J} = \frac{\mathfrak{m}}{TX^2+TY^2}$ is the only prime ideal of $R = \frac{T}{J}$. Hence, we obtain that $(R, \frac{\mathfrak{m}}{J})$ is a local Artinian ring. It is convenient to denote $X + J$ by x and $Y + J$ by y. Observe that $\frac{\mathfrak{m}}{J} = Rx + Ry$ is not principal, $xy \neq 0 + J$, $Rxy \subset Rx$, and $Rxy \subset Ry$. Therefore, R admits nonzero proper ideals other than its unique maximal ideal such that they are not minimal ideals of R. Hence, we obtain from $(ii) \Rightarrow (iii)$ of Theorem 2.18 that $g(R)$ contains a cycle and so, it follows from $(i) \Rightarrow (ii)$ of Proposition 2.15 that $girth(g(R)) = 3$. Indeed, it follows from the proof of $(i) \Rightarrow (ii)$ of Proposition 2.15 that $Rx - Rxy - Rx + Ry - Rx$ is a cycle of length three in $g(R)$. Let $I_1 = Rx$ and let $I_2 = R(x+y)$. It is clear that $I_1, I_2 \in \mathbb{I}(R)^*$ and $I_1 \neq I_2$. Observe that $I_1 \cap I_2 = Rxy$ and since $x^2 = 0 + J$, it follows that $I_1 \cap I_2 = Rxy = (Rx)(R(x+y)) = I_1 I_2$. Hence, I_1 and I_2 are not adjacent in $g(R)$. However, as $I_1 \cap I_2 \neq (0 + J)$, we get that I_1 and I_2 are adjacent in $G(R)$. Therefore, $g(R) \neq G(R)$. □

Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. We prove in Theorem 2.20 that $G(R)$ is a bipartite graph if and only if $g(R) = G(R)$ is a star graph.

Theorem 2.20. *Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. The following statements are equivalent:*

(i) $G(R)$ is a bipartite graph;

(ii) Either $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^3 = (0)$ but $\mathfrak{m}^2 \neq (0)$ or $(R, \mathfrak{m})$ is a local Artinian ring such that $\mathfrak{m}$ is not principal and any $I \in \mathbb{I}(R)^* \backslash \{\mathfrak{m}\}$ is a minimal ideal of R;

(iii) $g(R) = G(R)$ is a star graph.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $G(R)$ is a bipartite graph. Since $g(R)$ is a spanning subgraph of $G(R)$, we obtain that $g(R)$ is also a bipartite graph. Hence, $|\mathbb{I}(R)^*| \geq 2$ and $\omega(g(R)) \leq 2$. Therefore, we obtain from Proposition 2.13 that $(R, \mathfrak{m})$ is a local Artinian ring. We consider the following cases.

Case(1): $\mathfrak{m}$ is principal.

In this case $(R, \mathfrak{m})$ is a SPIR. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Observe that $\mathbb{I}(R)^* = \{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$. Since $|\mathbb{I}(R)^*| \geq 2$, it follows that $n \geq 3$. As $g(R)$ is a bipartite graph, we obtain from $(i) \Rightarrow (ii)$ of Proposition 2.15 that $girth(g(R)) = \infty$ and so, it follows from Corollary 2.17 that $n = 3$. Therefore, $\mathfrak{m}^3 = (0)$ but

$\mathfrak{m}^2 \neq (0)$.
Case(2): $\mathfrak{m}$ is not principal.
Since $g(R)$ is a bipartite graph, it follows from $(i) \Rightarrow (ii)$ of Proposition 2.15 that $girth(g(R)) = \infty$. Therefore, we obtain from $(ii) \Rightarrow (iii)$ of Theorem 2.18 that any $I \in \mathbb{I}(R)^*\backslash\{\mathfrak{m}\}$ is a minimal ideal of R.
$(ii) \Rightarrow (iii)$ Suppose that $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^3 = (0)$ but $\mathfrak{m}^2 \neq (0)$. Note that $g(R) = G(R)$ is a complete graph on the vertex set $\{\mathfrak{m}, \mathfrak{m}^2\}$. Hence, $g(R) = G(R)$ is a star graph. Suppose that $(R, \mathfrak{m})$ is a local Artinian ring such that $\mathfrak{m}$ is not principal and any $I \in \mathbb{I}(R)^*\backslash\{\mathfrak{m}\}$ is a minimal ideal of R. Then we obtain from $(iii) \Rightarrow (iv)$ of Theorem 2.18 that $g(R) = G(R)$ and $g(R)$ is a star graph.
$(iii) \Rightarrow (i)$ This is clear. $\square$

Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. If $g(R)$ does not contain any infinite clique, then it is shown in Proposition 2.13 that R is Artinian. In Example 2.22 (i), we provide an example of a local Artinian ring $(R, \mathfrak{m})$ such that $g(R)$ contains an infinite clique.

Theorem 2.21. *Let $(R, \mathfrak{m})$ be a local Artinian ring such that $dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = n \geq 3$. The following statements are equivalent:*
(i) $\chi(G(R)) < \infty$;
(ii) $\omega(G(R)) < \infty$;
(iii) $\omega(g(R)) < \infty$;
(iv) $g(R)$ does not contain any infinite clique;
(v) R is finite.

Proof. $(i) \Rightarrow (ii)$ If $\chi(G(R)) < \infty$, then as $\omega(G(R)) \leq \chi(G(R))$, we obtain that $\omega(G(R)) < \infty$.
$(ii) \Rightarrow (iii)$ By assumption, $\omega(G(R)) < \infty$. Since $g(R)$ is a spanning subgraph of $G(R)$, it follows that $\omega(g(R)) < \infty$.
$(iii) \Rightarrow (iv)$ This is clear.
$(iv) \Rightarrow (v)$ Let $\{x_1, x_2, x_3, \ldots, x_n\} \subseteq \mathfrak{m}$ be such that $\{x_1 + \mathfrak{m}^2, x_2 + \mathfrak{m}^2, x_3 + \mathfrak{m}^2, \ldots, x_n + \mathfrak{m}^2\}$ is a basis of $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as a vector space over $\frac{R}{\mathfrak{m}}$. On applying [2, Proposition 2.8] with $M = \mathfrak{m}$, we obtain that $\mathfrak{m} = \sum_{i=1}^{n} Rx_i$. Let us denote the ideal $\sum_{i=1}^{n-1} Rx_i + \mathfrak{m}^2$ by I. Note that $dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{I}) = 1$. Let us denote the collection consisting of all proper ideals W of R such that $W \supseteq \mathfrak{m}^2$ and $dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{W}) = 1$ by $\mathcal{C}$. It is clear that $I \in \mathcal{C}$ and so, $\mathcal{C}$ is nonempty. As $dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = n \geq 3$ by hypothesis, it follows that $\mathfrak{m}^2 \subset W$ for any $W \in \mathcal{C}$. We claim that the subgraph of $g(R)$ induced on $\mathcal{C}$ is a clique. Let $W_1, W_2 \in \mathcal{C}$ be such that $W_1 \neq W_2$. Since W_1 and W_2 are not comparable under the inclusion relation, we obtain that $W_1 + W_2 = \mathfrak{m}$. Observe that $dim_{\frac{R}{\mathfrak{m}}}(\frac{W_i}{\mathfrak{m}^2}) = n - 1$

for each $i \in \{1, 2\}$. It is convenient to denote $\frac{W_i}{\mathfrak{m}^2}$ by N_i for each $i \in \{1, 2\}$. It is clear that $N_1 \cap N_2 = \frac{W_1 \cap W_2}{\mathfrak{m}^2}$ and $N_1 + N_2 = \frac{W_1 + W_2}{\mathfrak{m}^2} = \frac{\mathfrak{m}}{\mathfrak{m}^2}$. Note that $dim_{\frac{R}{\mathfrak{m}}}(N_1 \cap N_2) = dim_{\frac{R}{\mathfrak{m}}} N_1 + dim_{\frac{R}{\mathfrak{m}}} N_2 - dim_{\frac{R}{\mathfrak{m}}}(N_1 + N_2) = n - 1 + n - 1 - n = n - 2 \geq 1$. Therefore, we get that $W_1 \cap W_2 \supset \mathfrak{m}^2$. As $W_1 W_2 \subseteq \mathfrak{m}^2$, it follows that $W_1 \cap W_2 \neq W_1 W_2$. Hence, W_1 and W_2 are adjacent in $g(R)$ and this proves that the subgraph of $g(R)$ induced on $\mathcal{C}$ is a clique. Since we are assuming that $g(R)$ does not contain any infinite clique, we obtain that $\mathcal{C}$ is a finite collection. Let the elements $x_1, x_2, x_3, \ldots, x_n \in \mathfrak{m}$ be as mentioned in the beginning of the proof of $(iv) \Rightarrow (v)$ of this Theorem. Let $r \in R$. Observe that the ideal $A(r) = Rx_1 + \cdots + R(x_{n-1} + rx_n) + \mathfrak{m}^2 \in \mathcal{C}$. Let $r, s \in R$ be such that $r - s \notin \mathfrak{m}$. We assert that $A(r) \neq A(s)$. Suppose that $A(r) = A(s)$. Then $x_{n-1} + rx_n, x_{n-1} + sx_n \in A(r) = A(s)$. Hence, $(r - s)x_n \in A(r)$. Since $r - s$ is a unit in R, it follows that $x_n \in A(r)$ and so, $x_i \in A(r)$ for each $i \in \{1, 2, \ldots, n\}$. Therefore, $A(r) = \sum_{i=1}^{n} Rx_i = \mathfrak{m}$. This is a contradiction. Therefore, $A(r) \neq A(s)$. It follows from $\mathcal{C}$ is finite that $\frac{R}{\mathfrak{m}}$ is finite. Since $(R, \mathfrak{m})$ is a local Artinian ring, we obtain from [2, Proposition 8.4] that $\mathfrak{m}$ is nilpotent. Let $k \geq 2$ be least with the property that $\mathfrak{m}^k = (0)$. Let $j \in \{1, \ldots, k - 1\}$. As $\frac{\mathfrak{m}^j}{\mathfrak{m}^{j+1}}$ is a finite-dimensional vector space over the finite field $\frac{R}{\mathfrak{m}}$, we get that $\frac{\mathfrak{m}^j}{\mathfrak{m}^{j+1}}$ is finite. Therefore, we obtain that $\mathfrak{m}$ is finite. Now, $\mathfrak{m}, \frac{R}{\mathfrak{m}}$ are finite and so, R is finite.
$(v) \Rightarrow (i)$ Since R is finite, $\mathbb{I}(R)^*$ is a finite collection, and so, $\chi(G(R))$ is finite. □

We provide some examples in Example 2.22 to illustrate Theorem 2.21.

Example 2.22. (i) Let K be an infinite field. Let $T = K[X, Y, Z]$ be the polynomial ring in three variables X, Y, Z over K and let $I = \mathfrak{m}^2$, where $\mathfrak{m} = TX + TY + TZ$. Let $R = \frac{T}{I}$. Then $g(R)$ contains an infinite clique.
(ii) Let K be an infinite field. Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over K. Let $I = \mathfrak{m}^2$, where $\mathfrak{m} = TX + TY$. Let $R = \frac{T}{I}$. Then $\omega(g(R)) = 2$.

Proof. (i) Observe that R is a local Artinian ring with $\frac{\mathfrak{m}}{I}$ as its unique maximal ideal. It is convenient to denote $\frac{\mathfrak{m}}{I}$ by $\mathfrak{n}$. Let us denote the field $\frac{R}{\mathfrak{n}}$ by k. Note that $dim_k(\frac{\mathfrak{n}}{\mathfrak{n}^2}) = 3$. As K is infinite, we obtain that R is infinite. Therefore, we obtain from $(iv) \Rightarrow (v)$ of Theorem 2.21 that $g(R)$ contains an infinite clique.
(ii) Note that R is a local Artinian ring with $\frac{\mathfrak{m}}{I}$ as its unique maximal

ideal. Let us denote $\frac{\mathfrak{m}}{I}$ by $\mathfrak{n}$ and the field $\frac{R}{\mathfrak{n}}$ by k. Observe that $dim_k(\frac{\mathfrak{n}}{\mathfrak{n}^2}) = 2$. Hence, $\mathfrak{n}$ is not principal. It is verified already in Example 2.19(i) that each nonzero proper ideal of R other than $\mathfrak{n}$ is a minimal ideal of R. Therefore, we obtain from $(iii) \Rightarrow (iv)$ of Theorem 2.18 that $g(R)$ is a star graph. Hence, we obtain that $\omega(g(R)) = 2$. Since K is infinite, it follows that R is infinite. Thus this example illustrates that $(iv) \Rightarrow (v)$ of Theorem 2.21 can fail to hold if the hypothesis that the unique maximal ideal of the local Artinian ring requires at least three generators is omitted. $\square$

3. Some results in the case, where R is not quasilocal

Let R be a ring such that $|Max(R)| \geq 2$. The aim of this section is to investigate some graph-theoretic properties of $g(R)$. We first try to determine R such that $g(R)$ is connected.

Lemma 3.1. *Let $n \geq 2$. Let R_i be a nonzero ring for each $i \in \{1, 2, \ldots, n\}$. If $R = R_1 \times R_2 \times \cdots \times R_n$, then $g(R)$ is not connected.*

Proof. It is clear that $R_1 \times (0) \times \cdots \times (0) \in V(g(R))$. We claim that $R_1 \times (0) \times \cdots \times (0)$ is an isolated vertex of $g(R)$. Suppose that $R_1 \times (0) \times \cdots \times (0) - A$ is an edge of $g(R)$. As $A \in \mathbb{I}(R)^*$, it follows that $A = I_1 \times I_2 \times \cdots \times I_n$, where I_i is an ideal of R_i for each $i \in \{1, 2, \ldots, n\}$ with $I_1 \times I_2 \times \cdots \times I_n \notin \{R_1 \times R_2 \times \cdots \times R_n, (0) \times (0) \times \cdots \times (0)\}$. Note that $(R_1 \times (0) \times \cdots \times (0)) \cap (I_1 \times I_2 \times \cdots \times I_n) = I_1 \times (0) \times \cdots \times (0) = (R_1 \times (0) \times \cdots \times (0))(I_1 \times I_2 \times \cdots \times I_n)$. This implies that $R_1 \times (0) \times \cdots \times (0)$ and $I_1 \times I_2 \times \cdots \times I_n = A$ are not adjacent in $g(R)$. This is a contradiction and so, we get that $R_1 \times (0) \times \cdots \times (0)$ is an isolated vertex of $g(R)$. As $|V(g(R)| \geq 2$, we obtain that $g(R)$ is not connected. $\square$

Proposition 3.2. *Let R be a ring such that $|Max(R)| \geq 2$. Then the following statements are equivalent:*
(i) *$G(R)$ is connected;*
(ii) *$\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$ for any two distinct $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(R)$.*
Moreover, if (i) or (ii) holds, then $diam(G(R)) \leq 2$.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $G(R)$ is connected. Suppose that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$ for some distinct $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(R)$. Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, we obtain from the Chinese remainder theorem [2, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2}$ defined by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2)$ is an isomorphism of rings. Let us denote the field $\frac{R}{\mathfrak{m}_i}$ by F_i for each $i \in \{1, 2\}$. Let us denote the ring $F_1 \times F_2$ by T. Note that $R \cong T$ as rings. Since we are assuming that $G(R)$ is connected, it follows that $G(T)$ is connected. However, observe that $\mathbb{I}(T)^* = \{(0) \times F_2, F_1 \times (0)\}$. Since $(0) \times F_2) \cap (F_1 \times (0)) = (0) \times (0)$, we get that $G(T)$ has no edges. Therefore, $G(T)$ is not connected. This is a contradiction. Hence, $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$ for any two distinct $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(R)$.

$(ii) \Rightarrow (i)$ Let $I_1, I_2 \in \mathbb{I}(R)^*$ be such that $I_1 \neq I_2$. We prove that there exists a path of length at most two between I_1 and I_2 in $G(R)$. We can assume that I_1 and I_2 are not adjacent in $G(R)$. Hence, $I_1 \cap I_2 = (0)$. We consider the following cases.

Case(1): $I_1 + I_2 \neq R$.

Let $\mathfrak{m} \in Max(R)$ be such that $I_1 + I_2 \subseteq \mathfrak{m}$. Then $I_i \cap \mathfrak{m} = I_i \neq (0)$ for each $i \in \{1, 2\}$ and so, $I_1 - \mathfrak{m} - I_2$ is a path of length two between I_1 and I_2 in $G(R)$.

Case(2): $I_1 + I_2 = R$.

Let $i \in \{1, 2\}$. Let $\mathfrak{m}_i \in Max(R)$ be such that $I_i \subseteq \mathfrak{m}_i$. It follows from $I_1 + I_2 = R$ that $\mathfrak{m}_1 \neq \mathfrak{m}_2$. By hypothesis, $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$. Let $x \in \mathfrak{m}_1 \cap \mathfrak{m}_2, x \neq 0$. From $R = I_1 + I_2$, we obtain that $Rx = I_1x + I_2x$. Therefore, either $I_1x \neq (0)$ or $I_2x \neq (0)$. Suppose that $I_1x \neq (0)$. Then $I_1\mathfrak{m}_2 \neq (0)$ and it is clear that $I_2 \cap \mathfrak{m}_2 = I_2 \neq (0)$. Hence, $I_1 - \mathfrak{m}_2 - I_2$ is a path of length two between I_1 and I_2 in $G(R)$. Suppose that $I_2x \neq (0)$. Then $I_2\mathfrak{m}_1 \neq (0)$. Observe that $I_1 \cap \mathfrak{m}_1 = I_1 \neq (0)$. Hence, in this case, $I_1 - \mathfrak{m}_1 - I_2$ is a path of length two between I_1 and I_2 in $G(R)$.

This proves that $G(R)$ is connected and $diam(G(R)) \leq 2$.

The moreover part of this Proposition is already verified in the proof of $(ii) \Rightarrow (i)$ of this Proposition. □

Let R be a ring with $|Max(R)| \geq 2$. We are interested in knowing the status of Proposition 3.2 in the case of $g(R)$. We prove in Proposition 3.4 that for $g(R)$ to be connected, it is necessary that $dim R > 0$.

Lemma 3.3. *Let R be a von Neumann regular ring which is not a field. Then R admits at least one nontrivial idempotent element.*

Proof. Since R is not a field, it is possible to find $a \in R \backslash \{0\}$ such that a is not a unit in R. From the hypothesis that R is von Neumann regular, it follows that that there exists $b \in R$ such that $a = a^2b$. Therefore, $ab = a^2b^2 = (ab)^2$. It is clear that $e = ab$ is a nontrivial idempotent element of R. □

Proposition 3.4. *Let R be a ring with $dim R = 0$. If $|Max(R)| \geq 2$, then $g(R)$ is not connected.*

Proof. Let us denote the ring $\frac{R}{nil(R)}$ by T. Note that $dim T = 0$ and T is reduced. Hence, we obtain from $(d) \Rightarrow (a)$ of [6, Exercise 16, page 111] that T is von Neumann regular. Observe that $|Max(T)| = |Max(R)| \geq 2$ and so, T is not a field. Therefore, we obtain from Lemma 3.3 that T admits at least one nontrivial idempotent. Let $r \in R$ be such that $r + nil(R)$ is a nontrivial idempotent element of T. Since $nil(R)$ is a nil ideal of R, it follows from [7, Proposition 7.14] that there exists a unique idempotent element e of R such that $r + nil(R) = e + nil(R)$. It is clear that e is nontrivial. Observe that the mapping $f : R \to Re \times R(1-e)$ defined by $f(x) = (xe, x(1-e))$ is an isomorphism of rings. Let us denote the ring Re by R_1 and $R(1-e)$ by R_2. Note that R_1 and R_2 are nonzero rings and $R \cong R_1 \times R_2$ as rings. We know from Lemma 3.1 that $g(R_1 \times R_2)$ is not connected and so, we obtain that $g(R)$ is not connected. □

Let R be a ring such that $dimR = 0$ and R is reduced. We know from $(a) \Leftrightarrow (d)$ of [6, Exercise 16, page 111] that a ring R is von Neumann regular if and only if $dimR = 0$ and R is reduced. We prove in Proposition 3.5 that if R is a von Neumann regular ring with $|Max(R)| \geq 2$, then $g(R)$ has no edges.

Proposition 3.5. *Let R be a ring with $|Max(R)| \geq 2$. If R is von Neumann regular, then $g(R)$ has no edges.*

Proof. Suppose that R is von Neumann regular. Let $a \in R$. We know from $(1) \Rightarrow (3)$ of [6, Exercise 29, page 113] that there exists a unit u of R and an idempotent element e of R such that $a = ue$. Using this fact, it follows easily that each proper ideal I of R is a radical ideal of R. Let $I_1, I_2 \in \mathbb{I}(R)^*$ be such that $I_1 \neq I_2$. We know from [2, Exercise 1.13(*iii*), page 9] that $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$. Since each ideal of R is a radical ideal of R, we obtain that $I_1 \cap I_2 = \sqrt{I_1 \cap I_2} = \sqrt{I_1 I_2} = I_1 I_2$. Therefore, I_1 and I_2 are not adjacent in $g(R)$. This shows that $g(R)$ has no edges. □

Let R be an integral domain which is not a field. Irrespective of the size of $Max(R)$, it is well-known that $G(R)$ is complete. In Proposition 3.6, we discuss the status of this result in the case of $g(R)$, where R is an integral domain with $|Max(R)| \geq 2$.

Proposition 3.6. *Let R be an integral domain with $|Max(R)| \geq 2$. Then $g(R)$ is connected and $diam(g(R)) = 2$.*

Proof. Let $I_1, I_2 \in \mathbb{I}(R)^*$ be such that $I_1 \neq I_2$. We prove that there exists a path of length at most two between I_1 and I_2 in $g(R)$. We can assume that I_1 and I_2 are not adjacent in $g(R)$. For each $i \in \{1, 2\}$, let $a_i \in I_i \backslash \{0\}$. Since R is an integral domain $a_1 a_2 \neq 0$. Let us denote the ideal $Ra_1 a_2$ by A. It is clear that $A \in \mathbb{I}(R)^*$. Let $i \in \{1, 2\}$.

Since $A \subseteq I_i$, it follows that $A \cap I_i = A$. We claim that $A \neq AI_i$. For if $A = AI_i$, then $a_1a_2 = a_1a_2b_i$ for some $b_i \in I_i$. This implies that $a_1a_2(1 - b_i) = 0$. This is impossible since $a_1a_2, 1 - b_i \in R\backslash\{0\}$ and R is an integral domain. Therefore, $A = A \cap I_i \neq AI_i$ for each $i \in \{1, 2\}$. Hence, A and I_i are adjacent in $g(R)$ for each $i \in \{1, 2\}$ and so, $I_1 - A - I_2$ is a path of length two between I_1 and I_2 in $g(R)$. This shows that $g(R)$ is connected and $diam(g(R)) \leq 2$. Since $|Max(R)| \geq 2$ by assumption, there exist $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(R)$ such that $\mathfrak{m}_1 \neq \mathfrak{m}_2$. It follows from $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathfrak{m}_1\mathfrak{m}_2$. Hence, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are not adjacent in $g(R)$ and so, we obtain that $diam(g(R)) \geq 2$. Therefore, $diam(g(R)) = 2$. □

Corollary 3.7. *Let R be an integral domain with $|Max(R)| \geq 2$. If $J(R) = (0)$, then $diam(g(R)) = r(g(R)) = 2$.*

Proof. We know from Proposition 3.6 that $g(R)$ is connected and $diam(g(R)) = 2$. (For this part of the proof, we do not need the assumption that $J(R) = (0)$.) Suppose that $J(R) = (0)$. Let $I \in V(g(R)) = \mathbb{I}(R)^*$. From $J(R) = (0)$, it follows that $I \not\subseteq \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$. Hence, $I + \mathfrak{m} = R$ and so, $I \cap \mathfrak{m} = I\mathfrak{m}$. Therefore, I and $\mathfrak{m}$ are not adjacent in $g(R)$. This shows that $d(I, \mathfrak{m}) \geq 2$ in $g(R)$. It follows from $diam(g(R)) = 2$ that $e(I) = 2$. Thus for any $I \in \mathbb{I}(R)^*$, $e(I) = 2$ in $g(R)$ and so, we obtain that $r(g(R)) = 2$. □

Corollary 3.8. *Let R be an integral domain. Then $diam(g(R[X])) = r(g(R[X])) = 2$, where $R[X]$ is the polynomial ring in one variable X over R.*

Proof. Note that $R[X]$ is an integral domain. Let $\mathfrak{m} \in Max(R)$. Observe that $\frac{R[X]}{\mathfrak{m}[X]} \cong \frac{R}{\mathfrak{m}}[X]$, the polynomial ring in one variable X over the field $\frac{R}{\mathfrak{m}}$. Hence, $\frac{R[X]}{\mathfrak{m}[X]}$ has an infinite number of maximal ideals and so, $Max(R[X])$ is infinite. We know from [2, Exercise 4, page 11] that $J(R[X])) = nil(R[X]) = (0)$. Therefore, we obtain from Corollary 3.7 that $diam(g(R[X])) = r(g(R[X])) = 2$. □

Let R be an integral domain with $|Max(R)| \geq 2$. It is clear that $r(g(R)) \geq 1$. We are not able to characterize integral domains R such that $r(g(R)) = 1$. In Theorem 3.9, we characterize Noetherian domains R with $dimR = 1$ such that $r(g(R)) = 1$.

Theorem 3.9. *Let R be a Noetherian domain with $dimR = 1$ and $|Max(R)| \geq 2$. The following statements are equivalent:*
(i) $r(g(R)) = 1$;
(ii) R is semilocal.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $r(g(R)) = 1$. It follows from Corollary 3.7 that $J(R) \neq (0)$. Let $a \in J(R)\backslash\{0\}$. Since R is Noetherian, we know from [2, Theorem 7.13] that Ra admits a primary decomposition. Let $Ra = \cap_{i=1}^{n}\mathfrak{q}_i$ be an irredundant primary decomposition of Ra, where $\mathfrak{q}_i$ is a $\mathfrak{p}_i$ -primary ideal of R for each $i \in \{1, \ldots, n\}$. Since $dim R = 1$, it follows that any nonzero prime ideal of R is maximal. Hence, $\mathfrak{p}_i \in Max(R)$ for each $i \in \{1, \ldots, n\}$. Let $\mathfrak{m} \in Max(R)$. Now, as $a \in \mathfrak{m}$, we get that $\mathfrak{m} \supseteq Ra = \cap_{i=1}^{n}\mathfrak{q}_i$. Therefore, we obtain from [2, Proposition 1.11(ii)] that $\mathfrak{m} \supseteq \mathfrak{q}_i$ for some $i \in \{1, \ldots, n\}$ and so, $\mathfrak{m} \supseteq \sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$. Hence, $\mathfrak{m} = \mathfrak{p}_i$ for some $i \in \{1, \ldots, n\}$. This shows that $Max(R) = \{\mathfrak{p}_i | i \in \{1, \ldots, n\}\}$ and therefore, we obtain that R is semilocal.
$(ii) \Rightarrow (i)$ We are assuming that R is a Noetherian domain, $|Max(R)| \geq 2$, and $Max(R)$ is finite. Let $Max(R) = \{\mathfrak{m}_i | i \in \{1, 2, \ldots, n\}\}$. Note that $J(R) = \cap_{i=1}^{n}\mathfrak{m}_i$. We claim that $e(J(R)) = 1$ in $g(R)$. (We prove this claim without assuming that $dim R = 1$.) Let $I \in \mathbb{I}(R)^*$ be such that $I \neq J(R)$. Observe that $I \subseteq \mathfrak{m}_i$ for some $i \in \{1, 2, \ldots, n\}$. Hence, $I_{\mathfrak{m}_i} \subseteq (\mathfrak{m}_i)_{\mathfrak{m}_i}$. Now, since $(\mathfrak{m}_k)_{\mathfrak{m}_i} = R_{\mathfrak{m}_i}$ for each $k \in \{1, 2, \ldots, n\}\backslash\{i\}$, we obtain from [2, Proposition 3.11(v)] that $(J(R))_{\mathfrak{m}_i} = (\cap_{k=1}^{n}\mathfrak{m}_k)_{\mathfrak{m}_i} = (\mathfrak{m}_i)_{\mathfrak{m}_i}$. Note that $(I \cap J(R))_{\mathfrak{m}_i} = I_{\mathfrak{m}_i} \cap J(R)_{\mathfrak{m}_i} = I_{\mathfrak{m}_i} \cap (\mathfrak{m}_i)_{\mathfrak{m}_i} = I_{\mathfrak{m}_i}$. It follows from [2, Proposition 3.11(v)] that $(IJ(R))_{\mathfrak{m}_i} = I_{\mathfrak{m}_i}(\mathfrak{m}_i)_{\mathfrak{m}_i}$. We verify that $I \cap J(R) \neq IJ(R)$. Suppose that $I \cap J(R) = IJ(R)$. Then $(I \cap J(R))_{\mathfrak{m}_i} = (IJ(R))_{\mathfrak{m}_i}$. This implies that $I_{\mathfrak{m}_i} = I_{\mathfrak{m}_i}(\mathfrak{m}_i)_{\mathfrak{m}_i}$. We know from [2, Example 1, page 38] that $(\mathfrak{m}_i)_{\mathfrak{m}_i}$ is the unique maximal ideal of $R_{\mathfrak{m}_i}$. Since R is Noetherian, we obtain from [2, Corollary 7.4] that $R_{\mathfrak{m}_i}$ is Noetherian. Hence, $R_{\mathfrak{m}_i}$ is a local domain. As $I_{\mathfrak{m}_i} = I_{\mathfrak{m}_i}(\mathfrak{m}_i)_{\mathfrak{m}_i}$, we obtain from Nakayama's lemma [2, Proposition 2.6] that $I_{\mathfrak{m}_i} = (0)$ and so, $I = (0)$. This is a contradiction. Therefore, $I \cap J(R) \neq IJ(R)$ for any $I \in \mathbb{I}(R)^*$ with $I \neq J(R)$. This shows that $J(R)$ is adjacent to any $I \in \mathbb{I}(R)^*$ with $I \neq J(R)$ in $g(R)$. Hence, $e(J(R)) = 1$ in $g(R)$ and so, we get that $r(g(R)) = 1$. $\square$

Let R be a ring such that $|Max(R)| \geq 2$. Our aim is to determine $girth(g(R))$. If $dim R > 0$, then we know from Proposition 2.11 that $g(R)$ contains an infinite clique and so, $girth(g(R)) = 3$. If there exists an ideal I of R with $I \subseteq J(R)$ such that I is not finitely generated, then we know from Proposition 2.12 that $g(R)$ contains an infinite clique and so, $girth(g(R)) = 3$. Hence, in determining $girth(g(R))$, we can assume that $dim R = 0$ and all the ideals I of R with $I \subseteq J(R)$ are finitely generated. If R is reduced, then R is von Neumann regular and we know from Proposition 3.5 that $g(R)$ has no edges and so, $girth(g(R)) = \infty$. Hence, in determining $girth(g(R))$, we can assume that $dim R = 0$

and R is not reduced. With the hypothesis that $dim R = 0$ and R is not reduced and $Max(R)$ is infinite, we prove in Theorem 3.10 that $\omega(g(R)) = \infty$.

Theorem 3.10. *Let R be a ring such that $dim R = 0$ and R is not reduced. If $Max(R)$ is infinite, then $\omega(g(R)) = \infty$.*

Proof. Let $m \geq 1$. We claim that there exist nonzero rings $R_1, R_2, \ldots, R_{m+1}$ such that $dim R_i = 0$ for each $i \in \{1, 2, \ldots, m+1\}$ and $R \cong R_1 \times R_2 \times \cdots \times R_{m+1}$ as rings. We are assuming that $Max(R)$ is infinite. Hence, we obtain from the proof of Proposition 3.4 that there exist nonzero rings R_{11} and R_{12} such that $R \cong R_{11} \times R_{12}$ as rings. It is clear that $dim R_{1j} = 0$ for each $j \in \{1, 2\}$. Since $Max(R)$ is infinite by assumption, it follows that either $Max(R_{11})$ is infinite or $Max(R_{12})$ is infinite. Without loss of generality, we can assume that $Max(R_{11})$ is infinite. Again it follows from the proof of Proposition 3.4 that there exist nonzero rings $R_{11}^{(1)}$ and $R_{11}^{(2)}$ such that $R_{11} \cong R_{11}^{(1)} \times R_{11}^{(2)}$ as rings. It is clear that $dim R_{11}^{(1)} = dim R_{11}^{(2)} = 0$ and $R \cong R_{11}^{(1)} \times R_{11}^{(2)} \times R_{12}$ as rings. The above argument can be repeated and it is clear that there exist nonzero rings $R_1, R_2, \ldots, R_{m+1}$ with $dim R_i = 0$ for each $i \in \{1, 2, \ldots, m+1\}$ and $R \cong R_1 \times R_2 \times \cdots \times R_{m+1}$ as rings. Let us denote the ring $R_1 \times R_2 \times \cdots \times R_{m+1}$ by T. We are assuming that R is not reduced. Hence, it follows that T is not reduced and so, R_i is not reduced for at least one $i \in \{1, 2, \ldots, m+1\}$. Without loss of generality, we can assume that R_1 is not reduced. Let $a \in R_1 \backslash \{0\}$ be such that $a^2 = 0$. Let us denote the ideal $R_1 a$ of R_1 by I. Consider the collection $\mathcal{C} = \{I \times I_2 \times \cdots \times I_{m+1} | I_i \in \mathbb{I}(R_i) \cup \{R_i\} \text{ for each } i \in \{2, \ldots, m+1\}\}$. It is clear that $\mathcal{C} \subseteq \mathbb{I}(T)^*$ and $\mathcal{C}$ contains at least 2^m elements. Let A_1, A_2 be any distinct members of $\mathcal{C}$. Note that $A_1 = I \times I_2 \times \cdots \times I_{m+1}$ and $A_2 = I \times J_2 \times \cdots \times J_{m+1}$, where $I_i, J_i \in \mathbb{I}(R_i) \cup \{R_i\}$ for each $i \in \{2, \ldots, m+1\}$. Observe that $A_1 \cap A_2 = I \times (I_2 \cap J_2) \times \cdots \times (I_{m+1} \cap J_{m+1})$ and it follows from $I^2 = (0)$ that $A_1 A_2 = (0) \times I_2 J_2 \times \cdots \times I_{m+1} J_{m+1}$. From $I \neq (0)$, we obtain that $A_1 \cap A_2 \neq A_1 A_2$. Hence, A_1 and A_2 are adjacent in $g(T)$. This shows that the subgraph of $g(T)$ induced on $\mathcal{C}$ is a clique. As $\mathcal{C}$ contains at least 2^m elements, we get that $\omega(g(T)) \geq 2^m \geq m+1$. Therefore, $\omega(g(T)) \geq m+1$ and since $R \cong T$ as rings, we obtain that $\omega(g(R)) \geq m+1$. This is true for all $m \geq 1$ and so, $\omega(g(R)) = \infty$. $\square$

Corollary 3.11. *Let R be a ring such that $dim R = 0$, R is not reduced, and $Max(R)$ is infinite. Then $girth(g(R)) = 3$.*

Proof. We know from the proof of Theorem 3.10 that for each $m \geq 1$, there exists a clique of $g(R)$ containing at least $m+1$ elements. Therefore, it follows that $girth(g(R)) = 3$. □

Let R be a ring such that $dim R = 0$, R is not reduced, and R has at least two maximal ideals. In view of Corollary 3.11, in determining $girth(g(R))$, we can assume that R is semiquasilocal.

Lemma 3.12. *Let R be a semiquasilocal ring with $dim R = 0$. Suppose that $|Max(R)| = n$. Then for each $i \in \{1, \ldots, n\}$, there exists a quasilocal ring $(R_i, \mathfrak{n}_i)$ with $dim R_i = 0$ such that $R \cong R_1 \times \cdots \times R_n$ as rings.*

Proof. This is well-known. For the sake of completeness, we include a proof of this lemma. There is nothing to prove if $|Max(R)| = n = 1$. Hence, we can assume that $n \geq 2$. Let $\{\mathfrak{m}_i | i \in \{1, 2, \ldots, n\}\}$ denote the set of all maximal ideals of R. For each $i \in \{1, 2, \ldots, n\}$, let $f_i : R \to R_{\mathfrak{m}_i}$ denote the homomorphism of rings defined by $f_i(r) = \frac{r}{1}$. It follows from $dim R = 0$ that $\sqrt{Ker f_i} = \mathfrak{m}_i$ for each $i \in \{1, 2, \ldots, n\}$ and it follows from $(iii) \Rightarrow (i)$ of [2, Proposition 3.8] that $\cap_{i=1}^{n} Ker f_i = (0)$. Let $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Since $\mathfrak{m}_i + \mathfrak{m}_j = R$, it follows from [2, Proposition 1.16] that $Ker f_i + Ker f_j = R$. Now, it follows from the Chinese remainder theorem [2, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{Ker f_1} \times \frac{R}{Ker f_2} \times \cdots \times \frac{R}{Ker f_n}$ defined by $f(r) = (r + Ker f_1, r + Ker f_2, \ldots, r + Ker f_n)$ is an isomorphism of rings. Let $i \in \{1, 2, \ldots, n\}$ and let us denote the ring $\frac{R}{Ker f_i}$ by R_i. It is clear that R_i is quasilocal with $\mathfrak{n}_i = \frac{\mathfrak{m}_i}{Ker f_i}$ as its unique maximal ideal, $dim R_i = 0$, and $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings. □

Proposition 3.13. *Let $n \geq 2$ and let for each $i \in \{1, 2, \ldots, n\}$, $(R_i, \mathfrak{m}_i)$ be a quasilocal ring with $dim R_i = 0$. Let $R = R_1 \times R_2 \cdots \times R_n$. If $g(R)$ does not contain any infinite clique, then R is Artinian.*

Proof. We are assuming that $g(R)$ does not contain any infinite clique. Note that to prove R is Artinian, it is enough to show that R_i is Artinian for each $i \in \{1, 2, \ldots, n\}$. First, we verify that R_1 is Artinian. If $\mathfrak{m}_1 = (0)$, then it is clear that R_1 is a field. Hence, we can assume that $\mathfrak{m}_1 \neq (0)$. Consider the mapping $f : \mathbb{I}(R_1)^* \to \mathbb{I}(R)^*$ defined by $f(I) = I \times R_2 \times \cdots \times R_n$. Observe that the mapping f is one-one and $I, J \in \mathbb{I}(R_1)^*$ are adjacent in $g(R_1)$ if and only if $f(I)$ and $f(J)$ are adjacent in $g(R)$. This implies that $g(R)$ contains a subgraph isomorphic to $g(R_1)$. From the assumption that $g(R)$ does not contain any infinite clique, it follows that $g(R_1)$ does not contain any infinite clique. Hence, we obtain from Proposition 2.13 that R_1 is Artinian.

Similarly, it can be shown that R_i is Artinian for each $i \in \{2, \ldots, n\}$ and so, it follows that R is Artinian. $\square$

Let R be a ring such that R is semiquasilocal with $|Max(R)| \geq 2$ and $dim R = 0$. If R is not Artinian, then it follows from Lemma 3.12 and Proposition 3.13 that $g(R)$ contains an infinite clique and so, $girth(g(R)) = 3$. Hence, in determining $girth(g(R))$, we can assume that R is Artinian.

Lemma 3.14. *Let T_1, T_2 be rings such that T_1 is not reduced and T_2 is not a field. Let $T = T_1 \times T_2$. Then $girth(g(T)) = 3$.*

Proof. Since T_1 is not a reduced ring, it is possible to find $t_1 \in T_1 \backslash \{0\}$ such that $t_1^2 = 0$. As T_2 is not a field by assumption, there exists at least one $J \in \mathbb{I}(T_2)^*$. Let us denote the ideal $T_1 t_1$ by I. Observe that $I \times J - I \times T_2 - I \times (0) - I \times J$ is a cycle of length 3 in $g(T)$ and so, $girth(g(T)) = 3$. $\square$

Corollary 3.15. *Let R be an Artinian ring with $|Max(R)| = n \geq 3$. If R is not reduced, then $girth(g(R)) = 3$.*

Proof. We know from [2, Theorem 8.7] that there exist Artinian local rings $(R_1, \mathfrak{m}_1), (R_2, \mathfrak{m}_2), (R_3, \mathfrak{m}_3), \ldots, (R_n, \mathfrak{m}_n)$ such that $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$ as rings. Since R is not reduced by assumption, we obtain that R_i is not reduced for at least one $i \in \{1, 2, 3, \ldots, n\}$. Without loss of generality, we can assume that R_1 is not reduced. Let us denote the ring $R_1 \times R_2 \times R_3 \times \cdots \times R_n$ by T. Note that $R \cong T$ as rings. Since R_1 is not reduced and $R_2 \times R_3 \times \cdots \times R_n$ is not a field, it follows from Lemma 3.14 that $girth(g(T)) = 3$ and so, we obtain that $girth(g(R)) = 3$. $\square$

Let R be an Artinian ring with $|Max(R)| = 2$ and R is not reduced. In Theorem 3.16, we describe $girth(g(R))$ and moreover, we characterize rings R such that $g(R)$ does not contain any cycle.

Theorem 3.16. *Let R be an Artinian ring with $|Max(R)| = 2$. Suppose that R is not reduced. Then $girth(g(R)) \in \{3, \infty\}$.*

Moreover, $girth(g(R)) = \infty$ if and only if $R \cong R_1 \times F$ as rings, where F is a field and $(R_1, \mathfrak{m}_1)$ is an Artinian ring which is not a field satisfying one of the following conditions:

(i) $(R_1, \mathfrak{m}_1)$ is a SPIR and if k is the least positive integer such that $\mathfrak{m}_1^k = (0)$, then $k \in \{2, 3\}$.

(ii) $\mathfrak{m}_1$ is not principal and any $I \in \mathbb{I}(R_1)^*$ with $I \neq \mathfrak{m}_1$ is a minimal ideal of R_1.

Proof. We know from [2, Theorem 8.7] that there exist Artinian local rings $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ such that $R \cong R_1 \times R_2$ as rings. Since R

is not reduced by assumption, it follows that R_i is not reduced for at least one $i \in \{1, 2\}$. Without loss of generality, we can assume that R_1 is not reduced. Let us denote the ring $R_1 \times R_2$ by T. We consider the following cases.

Case(1): R_2 is not reduced.

In such a case, we obtain from Lemma 3.14 that $girth(g(T)) = 3$ and since $R \cong T$ as rings, we obtain that $girth(g(R)) = 3$.

Case (2): R_2 is reduced.

Note that $\mathfrak{m}_2 = (0)$ and so, R_2 is a field. Let us denote R_2 by F. Now, $T = R_1 \times F$. Since F and (0) are the only ideals of F, $F \cap (0) = (0) = F(0)$, $F \cap F = F = FF$, we obtain that any edge of $g(T)$ is of the form $I_1 \times J_1 - I_2 \times J_2$ with $I_1 - I_2$ is an edge of $g(R_1)$. Thus $g(T)$ contains a cycle if and only if $g(R_1)$ contains a cycle. If $g(R_1)$ contains a cycle, then we know from $(i) \Rightarrow (ii)$ of Proposition 2.15 that $girth(g(R_1)) = 3$. Thus if $g(T)$ contains a cycle, then $girth(g(T)) = 3$. Hence, $girth(g(R)) = 3$.

It is clear from the above discussion that $girth(g(R)) \in \{3, \infty\}$. Note that $girth(g(R)) = \infty$ if and only if $R \cong R_1 \times F$ as rings, where F is a field and $(R_1, \mathfrak{m}_1)$ is a nonreduced Artinian local ring with $girth(g(R_1)) = \infty$. It follows from Corollary 2.17 and $(ii) \Leftrightarrow (iii)$ of Theorem 2.18 that $girth(g(R_1)) = \infty$ if and only if the Artinian local ring $(R_1, \mathfrak{m}_1)$ satisfies one of the conditions (i) , (ii) stated in the statement of Theorem 3.16. □

We mention some examples in Example 3.17 to illustrate Theorem 3.16.

Example 3.17. (i) Let $S = R \times F$, where R is as in Example 2.19(i) and F is a field. Then $girth(g(S)) = \infty$.
(ii) Let $S = R \times F$, where R is as in Example 2.19(ii) and F is a field. Then $girth(g(S)) = 3$.

Proof. (i) Let $T, \mathfrak{m}, I$ be as in Example 2.19(i). It is noted in Example 2.19(i) that $(R, \frac{\mathfrak{m}}{I})$ is a local Artinian ring with $girth(g(R)) = \infty$. Observe that S is Artinian, $|Max(S)| = 2$, and $Max(S) = \{\frac{\mathfrak{m}}{I} \times F, R \times (0)\}$. It is noted in the proof of Theorem 3.16 that $g(S)$ contains a cycle if and only if $g(R)$ contains a cycle. From $girth(g(R)) = \infty$, it follows that $girth(g(S)) = \infty$.
(ii) Let $T, \mathfrak{m}, J$ be as in Example 2.19(ii). It is observed in Example 2.19(ii) that $(R, \frac{\mathfrak{m}}{J})$ is a local Artinian ring with $girth(g(R)) = 3$. Note that S is Artinian, $|Max(S)| = 2$, and $Max(S) = \{\frac{\mathfrak{m}}{J} \times F, R \times (0)\}$. From $girth(g(R)) = 3$, we obtain that $girth(g(S)) = 3$. □

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Subramanian Visweswaran
Department of Mathematics,
Saurashtra University, P.O.Box 360005,
Rajkot, India.
Email:s_visweswaran2006@yahoo.co.in

Pravin Vadhel
Department of Mathematics,
Saurashtra University, P.O.Box 360005,
Rajkot, India.
Email:pravin_2727@yahoo.com