Journal of Algebra and Related Topics
Vol. 6, No 2, (2018), pp 79-96

QUASI-BIGRADUATIONS OF MODULES, CRITERIA OF GENERALIZED ANALYTIC INDEPENDENCE

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Abstract. Let $\mathcal{R}$ be a ring. For a quasi-bigraduation $f = I_{(p,q)}$ of $\mathcal{R}$ we define an f^+-quasi-bigraduation of an $\mathcal{R}$-module $\mathcal{M}$ by a family $g = (G_{(m,n)})_{(m,n)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}}$ of subgroups of $\mathcal{M}$ such that $G_\infty = (0)$ and $I_{(p,q)}G_{(r,s)} \subseteq G_{(p+r,q+s)}$, for all (p,q) and all $(r,s) \in (\mathbb{N}\times\mathbb{N}) \cup \{\infty\}$.

Here we show that r elements of $\mathcal{R}$ are $J-$independent of order k with respect to the f^+quasi-bigraduation g if and only if the following two properties hold: they are $J-$independent of order k with respect to the $^+$quasi-bigraduation of ring $f_2(I_{(0,0)}, I)$ and there exists a relation of compatibility between g and g_I, where I is the sub-$\mathcal{A}-$module of $\mathcal{R}$ constructed by these elements. We also show that criteria of $J-$independence of compatible quasi-bigraduations of module are given in terms of isomorphisms of graded algebras.

1. Introduction

All rings are supposed to be commutative and unitary.

In 1954, D. G. Northcott and D. Rees [8] developed a theory of integral closure and reductions of ideals in a Noetherian local ring $(A, \mathfrak{M})$. In particular, they introduced two notions of analytic independence with respect to an ideal in a local ring and they proved that the reduction of an ideal in such a ring is minimal if and only if it has an analytically independent generating set.

MSC(2010): Primary: 13A02, 13A15; Secondary: 13B25, 13F20
Keywords: Quasi-bigraduations, modules, generalized analytic independence.
Received: 24 August 2018, Accepted: 11 December 2018.
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In 1970 one notion of independence is generalized by Valla [10] in a Noetherian commutative ring. He showed that the maximum number of independent elements in an ideal is bounded from above by its height.

Let $f = (I_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a filtration of an arbitrary commutative ring A and

$$R(A,f) = \bigoplus_{n\in\mathbb{N}} I_n X^n \quad \text{and} \quad \Re(A,f) = \bigoplus_{n\in\mathbb{Z}} I_n X^n$$

be its Rees rings. Let k be a positive integer which may be equal to $+\infty$ and let J be an ideal of A such that $J + I_k \neq A$. Take $u = X^{-1}$. Then the following numbers are known in the literature to be extensions to filtrations of the analytic spread, the last one being due to Y. M. Diagana [4] : the maximum number $\ell_J(f,k)$ of elements of the ideal J which are $J-$independent of order k with respect to f and the maximum number $\ell_J^a(f,k)$ of elements of the ideal J which are regularly $J-$independent of order k with respect to f.

That work generalized results of Okon [9] concerning the analytic spread of Noetherian filtrations and established comparisons of several extensions.

In [5] we studied theses notions for a $^+$quasi-graduation of a ring $\mathcal{R}$.

We recall that the family (G_n) of subgroups of $\mathcal{R}$ is a $^+$quasi-graduation of $\mathcal{R}$ if G_0 is a subring of $\mathcal{R}$, $G_{+\infty} = (0)$ and $G_pG_q \subseteq G_{p+q}$ $\forall p,q \in \mathbb{N}$.

Here we need the following concept of compatibility of a family of subgroups of $(\mathcal{R},+)$ with a given quasi-graduation (resp. $^+$quasi-graduation) f and we extend this concept to quasi-bigraduations.

Definition 1.1. Let $\mathcal{R}$ be a ring.

1) Let $f = (I_p)_{p\in\mathbb{Z}\cup\{+\infty\}}$ be a family of subgroups of $\mathcal{R}$. We say that f is a quasi-graduation (resp. $^+$quasi-graduation) of $\mathcal{R}$ if I_0 is a subring of $\mathcal{R}$, $I_\infty = (0)$ and $I_pI_q \subseteq I_{p+q}$ for all p and $q \in \mathbb{Z}\cup\{+\infty\}$ (resp. $\mathbb{N}\cup\{+\infty\}$).

2) Let $f = (I_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a quasi-graduation (resp. $^+$quasi-graduation) of $\mathcal{R}$ and let $g = (G_i)_{i\in\mathbb{Z}\cup\{+\infty\}}$ be a family of subgroups of an $\mathcal{R}-$module $\mathcal{M}$.

We say that g is an $\underline{f^+-\textbf{quasi-graduation}}$ of $\mathcal{M}$ or that g is a $^+$quasi-graduation of $\mathcal{M}$ compatible with f if $G_\infty = (0)$ and $I_pG_q \subseteq G_{p+q}$ for each p and $q \in \mathbb{N}$.

For a ring R, the construction of rings of polynomials $R[X_1,\ldots,X_n]$ of n indeterminates with coefficients in R have a critical importance to geometrical investigations, since geometrical objects (curves, surfaces, etc.) are described by equations in several variables.

Otherwise, in Database, stored values must be accessible concurrently but consistently by multiple users. This proves the importance of the Cartesian product in relational Algebra used in Relational Database. Hence there is an interest to replace $\mathbb{N}$ by $\mathbb{N}\times\mathbb{N}$ as set of indices.

We define the compatible $^{+}$quasi-bigraduation of a ring as follow:

Definition 1.2. Let $\mathcal{R}$ be a ring.

1) Let $f=\left(I_{(m,n)}\right)_{(m,n)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}}$ be a family of subgroups of $\mathcal{R}$ with the convention that $I_{(p,\infty)}$, $I_{(\infty,q)}$ and $I_{(\infty,\infty)}$ mean the same subgroup, denoted I_{∞}.

We say that f is a **quasi-bigraduation** (resp. $^{+}$**quasi-bigraduation**) of $\mathcal{R}$ if $I_{(0,0)}$ is a subring of $\mathcal{R}$, $I_{\infty}=(0)$ and $I_{(p,q)}I_{(r,s)}\subseteq I_{(p+r,q+s)}$ $\forall(p,q)$ and
$(r,s)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}$. (resp. $(\mathbb{N}\times\mathbb{N})\cup\{\infty\}$).

2) Let $f=\left(I_{(m,n)}\right)_{(m,n)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}}$ be a quasi-bigraduation (resp. $^{+}$quasi-bigraduation) of $\mathcal{R}$.

Let us construct the family $(S_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ as following :

$$\forall m\geq 0,\qquad S_m=\sum_{-m\leq n\leq 2m} I_{(n,m-n)},$$

$$\forall m\leq 0,\ S_m=\mathcal{A} \text{ and } S_{\infty}=(0)\,.$$

We have

$$S_pS_q\subseteq S_{p+q}\qquad \forall p,q\in\mathbb{N}\cup\{+\infty\}$$

Therefore, $(S_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ is a $^{+}$quasi-graduation of $\mathcal{R}$; it is called the $^{+}$*quasi-graduation of* $\mathcal{R}$ *deduced from* f.

In this paper we have two objectives :

To extend the notion of generalized analytic independence to compatible $^{+}$quasi-bigraduations of a module and to establish some characterizations of this notion by the mean of isomorphisms of graded algebras.

We will also extend the study to globally compatible quasi-bigraduations.

2. Compatible quasi-bigraduations of module

We define the compatible $^{+}$quasi-bigraduation of a module as follow:

2.1. **Compatible quasi-bigraduations of module.**

Definition 2.1. Let Δ be the abelian monoid $\mathbb{Z}^2 \cup \{\infty\}$ (resp. $\mathbb{N}^2 \cup \{\infty\}$).

Let $\mathcal{R}$ be a ring and $\mathcal{M}$ be an $\mathcal{R}$-module. Let $f = \left(I_{(m,n)}\right)_{(m,n)\in\Delta}$ be a $^+$quasi-bigraduation of $\mathcal{R}$.

Let $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in\Delta}$ be a family of subgroups of $\mathcal{M}$ with the convention that $\mathbb{G}_{(p,\infty)}$, $\mathbb{G}_{(\infty,q)}$ and $\mathbb{G}_{(\infty,\infty)}$ mean the same subgroup, denoted $\mathbb{G}_\infty$.

We say that $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)$ is an $f-$**quasi-bigraduation** (resp. an f^+-**quasi- bigraduation**) of $\mathcal{M}$ or that $\mathcal{H}$ is **a quasi-graduation (resp. a $^+$quasi-graduation) over Δ of $\mathcal{M}$ compatible with** f if $\mathbb{G}_\infty = (0)$ and $I_{(p,q)}\mathbb{G}_{(m,n)} \subseteq \mathbb{G}_{(p+m,q+n)}$ for each (m,n) and $(p,q) \in \Delta$.

2.2. **Globally compatible quasi-bigraduations of module.**

Let $\mathcal{R}$ be a ring, $\mathcal{A}$ be a subring of $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{R}$-module.

Let $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in\Delta}$ be a family of sub-$\mathcal{A}$-modules of $\mathcal{M}$ such that $\mathbb{G}_{(p,q)} = \mathbb{G}_{(0,0)}$ $\forall p,q$ verifying $p+q \leq 0$ and $\mathbb{G}_\infty = (0)$.

Let us construct the family $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ of sub-$\mathcal{A}$-modules of $\mathcal{M}$ as following:

$$\begin{aligned} \forall m \geq 0,\ \ \mathcal{N}_m &= \sum_{-m\leq n\leq 2m} \mathbb{G}_{(n,m-n)}, \\ \forall m \leq 0,\ \ \mathcal{N}_m &= \mathbb{G}_{(0,0)} \text{ and } \mathcal{N}_\infty = (0)\,. \end{aligned}$$

Definition 2.2. Let $\mathcal{R}$ be a ring and $\mathcal{M}$ be an $\mathcal{R}$-module.

Let $f = (I_{(s,t)})_{(s,t)\in\Delta}$ be a $^+$quasi-bigraduation of $\mathcal{R}$.

Let $\mathcal{A} = I_{(0,0)}$ and $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in\Delta}$ be a family of sub-$\mathcal{A}$-modules of $\mathcal{M}$.

Suppose that $I_{(p,q)} = \mathcal{A}$ $\forall p,q$ verifying $p+q \leq 0$ and $S = (S_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ is the quasi-graduation deduced from f (see 2) of Definition 1.2).

We say that $\mathcal{H}$ is a $^+$**quasi-bigraduation** of $\mathcal{M}$ **globally compatible with** f or that $\mathcal{H}$ is a **global** f^+-**quasi-bigraduation** of $\mathcal{M}$ if $\mathbb{G}_\infty = (0)$ and $S_p\mathcal{N}_q \subseteq \mathcal{N}_{p+q}$ for each p and $q \in \mathbb{N}$.

Remark 2.3. If $\mathcal{H} = (\mathbb{G}_n)_{n\in\Delta}$ is an f^+-quasi-bigraduation of $\mathcal{M}$, then $\mathcal{H}$ is a $^+$quasi-bigraduation of $\mathcal{M}$ globally compatible with f.

Indeed, we have

$$\forall p,q \in \mathbb{N},\ S_p\mathcal{N}_q = \left(\sum_{-p\leq n\leq 2p} I_{(n,p-n)}\right)\left(\sum_{-q\leq l\leq 2q} \mathbb{G}_{(l,q-l)}\right).$$

Therefore,

$$\begin{aligned} S_p\mathcal{N}_q &= \sum_{-p\le n\le 2p}\left(\sum_{-q\le l\le 2q} I_{(n,p-n)}\mathbb{G}_{(l,q-l)}\right) \\ \text{and } S_p\mathcal{N}_q &\subseteq \sum_{-p-q\le h\le 2(p+q)} \mathbb{G}_{(h,p+q-h)} = \mathcal{N}_{p+q}. \end{aligned}$$

Hence $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ is a S^+-quasi-graduation of $\mathcal{M}$.

One denotes $\mathsf{QG}(\mathcal{H}) = (\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ which is called the *S^+-quasi-graduation* of $\mathcal{M}$ deduced from $\mathcal{H}$.

3. Compatible quasi-bigraduations of module and generalized Analytic Independence

3.1. Generalized analytic independence.

Definition 3.1. Let $\mathcal{R}$ be a ring, $\mathcal{A}$ be a subring of $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{R}$-module.

Let $a_1,\dots,a_r$ be elements of $\mathcal{R}$. Let I be the sub-$\mathcal{A}$-module of $\mathcal{R}$ generated by $a_1,\dots,a_r$.

Let $f_2(\mathcal{A},I)$ be the $^+$quasi-bigraduation $(\mathcal{I}_{(m,n)})$ of $\mathcal{R}$ such that

$$\begin{cases} \mathcal{I}_{(m,n)} = \mathcal{A} \quad \text{if } m+n\le 0 \\ \mathcal{I}_{\infty} = (0) \quad \text{and} \\ \mathcal{I}_{(m,n)} = I^d \quad \text{if } m+n=d\ge 0. \end{cases}$$

Let (S_m) be the $^+$quasi-graduation of $\mathcal{R}$ deduced from $f_2(\mathcal{A},I)$. We have

$\forall m\ge 0$, $S_m = \sum\limits_{-m\le n\le 2m} I_{(n,m-n)} = I^m$, $S_\infty=(0)$ and $\forall m\le 0,\ S_m = \mathcal{A}$.

Let $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)$ be a $^+$quasi-bigraduation of $\mathcal{M}$ globally compatible with $(f_2(\mathcal{A},I))$ (i.e., for all $m\in\mathbb{N}$, $\mathcal{N}_m$ is a sub-$\mathcal{A}$-module of $\mathcal{M}$ and $I(\mathcal{N}_m)\subseteq\mathcal{N}_{m+1}$).

Suppose that $J\mathbb{G}_{(0,0)} + \mathcal{N}_k\cap\mathbb{G}_{(0,0)} \ne \mathbb{G}_{(0,0)}$.

The elements $a_1,\dots,a_r$ of $\mathcal{R}$ are said to be *$J-$ independent of order k with respect to $\mathcal{H}$* if for any homogeneous polynomial $\mathcal{F}$ of degree d in r indeterminates with coefficients in $\mathbb{G}_{(0,0)}$, the relation $\mathcal{F}(a_1,\dots,a_r)\in J\mathcal{N}_d+\mathcal{N}_{d+k}$ implies that $\mathcal{F}$ has all of its coefficients in $J\mathbb{G}_{(0,0)}+\mathcal{N}_k$.

Remark 3.2. Elements $a_1,\dots,a_r$ of $\mathcal{R}$ are $J-$ independent of order k with respect to $\mathcal{H}$ if they are $J-$ independent of order k with respect to the S^+-quasi-graduation $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ defined in section 1.2.

Proposition 3.3. *Let $\mathcal{R}$ be a ring, $\mathcal{A}$ be a subring of $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{R}$-module. Let $a_1, \ldots, a_r$ be elements of $\mathcal{R}$ and I be the sub-$\mathcal{A}$-module of $\mathcal{R}$ that they generate. Let $\mathcal{H} = (\mathbb{G}_n)_{n\in\Delta}$ be a global $(f_2(\mathcal{A}, I))^+$ −quasi-bigraduation of $\mathcal{M}$.*

Let J be an ideal of $\mathcal{A}$, $k \in \overline{\mathbb{N}^*}$ such that $J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$. Suppose that $a_1, \ldots, a_r$ are $J-$independent of order k with respect to $\mathcal{H}$.

(i) *If $J\mathbb{G}_{(0,0)} \supseteq \mathcal{N}_k \cap \mathbb{G}_{(0,0)}$, then the elements $a_1, \ldots, a_r$ are $J-$independent (of order $+\infty$) with respect to $\mathcal{H}$ and with respect to*

$$f_2\left(\mathbb{G}_{(0,0)}, I\right) \text{ and to } f\left(\mathbb{G}_{(0,0)}, I\right) = (\mathcal{I}_j).$$

(ii) *If there exists i such that $a_i \in J + S_k \cap \mathcal{A}$, then*

$$\left(I^p\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} = \left(S_p\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} \subseteq \mathcal{N}_p \cap \mathbb{G}_{(0,0)}$$

$$\text{and} \quad \mathcal{N}_p \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)} \ \forall p \geq 1.$$

Proof. *(i)* Let $x = F(a_1, \ldots, a_r)$, where F is a homogeneous polynomial of degree s in r indeterminates and with coefficients in $\mathbb{G}_{(0,0)}$.

Suppose that $J\mathbb{G}_{(0,0)}$ contains $\mathcal{N}_k \cap \mathbb{G}_{(0,0)}$ and put $I_m = I^m$. We have $\left[x \equiv 0\,(J\mathcal{N}_s) \text{ or } x \equiv 0 \ \left(JI_s\mathbb{G}_{(0,0)}\right)\right] \Rightarrow x \in J\mathcal{N}_s + \mathcal{N}_{s+k} \Rightarrow$
$F \in \left(J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)}\right)[X_1, \ldots, X_r]$.
Furthermore, $J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)} = J\mathbb{G}_{(0,0)}$ thus the elements $a_1, \ldots, a_r$ are $J-$independent (of order $+\infty$) with respect to $\mathcal{H}$ and with respect to $f_2\left(\mathbb{G}_{(0,0)}, I\right)$.

(ii) If $a_i \in J + S_k \cap \mathcal{A}$, then for each $p \geq 1$ and for each $y \in \mathcal{N}_p \cap \mathbb{G}_{(0,0)}$ we have

$$ya_i^p \in (J + S_k \cap \mathcal{A})\,\mathcal{N}_p \subseteq (J\mathcal{N}_p + \mathcal{N}_{p+k}).$$

The elements $a_1, \ldots, a_r$ being $J-$independent of order k with respect to $\mathcal{H}$, we have

$$y \in \left(J\mathbb{G}_{(0,0)} + \mathcal{N}_k\right) \cap \mathbb{G}_{(0,0)} = J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)}$$

therefore

$$\mathcal{N}_p \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)}$$

and we have

$$\left(I^p\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} = \left(S_p\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} \subseteq \mathcal{N}_p \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)}$$

$\forall p \geq 1$.

□

Proposition 3.4. *Let $\mathcal{R}$ be a ring, $\mathcal{A}$ be a subring of $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{R}$-module. Let $k \in \overline{\mathbb{N}^*}$ and J be an ideal of $\mathcal{A}$, $a_1, \ldots, a_r$ be elements of $\mathcal{R}$ and I be the sub-$\mathcal{A}$-module of $\mathcal{R}$ that they generate.*

Let $\mathcal{H} = \left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in\Delta}$ be a global $\left(f_2\left(\mathcal{A}, I\right)\right)^+ -$quasi-bigraduation of $\mathcal{M}$. Suppose that $J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$, $\left(\mathcal{N}_{i+k}\right)_{i\geq 0}$ is decreasing and that $a_1, \ldots, a_r$ are $J-$independent of order k with respect to $\mathcal{H}$.

If $\mathcal{N}_k \cap \mathbb{G}_{0,0} \subseteq J\mathbb{G}_{(0,0)} + \mathcal{N}_{pk} \cap \mathbb{G}_{(0,0)}$, then the elements $a_1^p, \ldots, a_r^p$ are $J-$independent of order k with respect to the $f_2\left(\mathcal{A}, I^p\right)^+ -$quasi-bigraduation $\mathcal{H}^{(p)} = \left(\mathbb{G}_{(pm,pn)}\right)$ for each $p \geq 1$.

Proof. This is the consequence of the fact that under the hypotheses we have

$$\begin{cases} \forall n \geq 0 \quad J\mathcal{N}_{pn} + \mathcal{N}_{p(n+k)} \subseteq J\mathcal{N}_{np} + \mathcal{N}_{np+k} \\ \text{and} \\ J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \mathcal{N}_{pk} \cap \mathbb{G}_{(0,0)} \end{cases}$$

Indeed, let F be a homogeneous polynomial of degree n in r indeterminates and with coefficients in $\mathbb{G}_{(0,0)}$.

$$F\left(a_1^p, \ldots, a_r^p\right) \in \left(J\mathcal{N}_{pn} + \mathcal{N}_{p(n+k)}\right) \Rightarrow G\left(a_1, \ldots, a_r\right) \in \left(J\mathcal{N}_{pn} + \mathcal{N}_{p(n+k)}\right)$$

where $G\left(X_1, \ldots, X_r\right) = F\left(X_1^p, \ldots, X_r^p\right)$ is homogeneous of degree np.

Thus $G\left(a_1, \ldots, a_r\right) \in J\mathcal{N}_{pn} + \mathcal{N}_{(pn+pk)} \subseteq J\mathcal{N}_{np} + \mathcal{N}_{np+k}$. Therefore G and F have their coefficients in $J\mathbb{G}_{(0,0)} + \mathcal{N}_k$. F has all of its coefficients in

$$\left(J\mathbb{G}_{(0,0)} + \mathcal{N}_k\right) \cap \mathbb{G}_{(0,0)} = J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \mathcal{N}_{pk} \cap \mathbb{G}_{(0,0)}.$$

□

3.2. **Criteria of $J-$independence :**

Let $\mathcal{R}$ be a ring, $\mathcal{A}$ be a subring of $\mathcal{R}$, and $\mathcal{M}$ be an $\mathcal{R}$-module.

3.2.1. *Preliminaries.*

Let (I_n) and (J_n) be two families of subgroups of $\mathcal{R}$ such that

$$(*) \quad \begin{cases} 1 \in I_0 \\ J_n \subseteq I_n \\ I_n I_m \subseteq I_{n+m} \\ I_n J_m \subseteq J_{n+m} \end{cases} \qquad \forall m, n \in \mathbb{Z} \text{ (resp. } \mathbb{N}).$$

Let $(\mathbb{P}_n)$ and $(\mathbb{K}_n)$ be two families of subgroups of $\mathcal{M}$ such that

$$(**) \quad \begin{cases} \mathbb{K}_n \subseteq \mathbb{P}_n \\ I_n \mathbb{P}_m \subseteq \mathbb{P}_{n+m} \\ J_n \mathbb{P}_m + I_n \mathbb{K}_m \subseteq \mathbb{K}_{n+m} \end{cases} \qquad \forall m, n \in \mathbb{Z} \text{ (resp. } \mathbb{N}).$$

Thus the group direct sum $\bigoplus\limits_{m} \dfrac{\mathbb{P}_m}{\mathbb{K}_m}$ is a graded $\bigoplus\limits_{n} \dfrac{I_n}{J_n}-$module with

$(a_n + J_n)(y_m + \mathbb{K}_m) = (a_n y_m + \mathbb{K}_{n+m})$ for each $a_n \in I_n$ and each $y_m \in \mathbb{P}_m$.

Let $\mathbb{P} = \bigoplus\limits_{m} \mathbb{P}_m X^m$ and $\mathbb{K} = \bigoplus\limits_{m} \mathbb{K}_m X^m$.

They are graded $F-$modules, where $F = \bigoplus\limits_{n} I_n X^n$ and we have

$$\frac{\mathbb{P}}{\mathbb{K}} = \frac{\bigoplus\limits_{m} \mathbb{P}_m X^m}{\bigoplus\limits_{m} \mathbb{K}_m X^m} \simeq \bigoplus_{m} \frac{\mathbb{P}_m}{\mathbb{K}_m}.$$

3.2.2. *Construction of morphisms relating to class of elements of a ring.*

We define the product for a subgroup I of $\mathcal{R}$ and an $\mathcal{R}-$module $\mathcal{N}$ by

$$I\mathcal{N} = \left\{ \sum_{i=1}^{s} a_i x_i : a_i \in I,\ b_i \in \mathcal{N} \text{ and } s \in \mathbb{N}^* \right\}.$$

Let $f = (I_n)_{n\in\mathbb{Z}\cup\{+\infty\}}$ be a quasi-graduation of $\mathcal{R}$ and $\mathcal{A} = I_0$.

Let J be an ideal of $\mathcal{A}$ and $J_n = I_n \cap (JI_n + I_{n+k})$ for all n.

Denote $f \ltimes f$ the quasi-bigraduation $(U_{i,j})_{(i,j)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}}$ of $\mathcal{R}$ such that

$U_{(m,n)} = I_m I_n$ for each $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ and $U_\infty = (0)$.

Let $a_1, \ldots, a_r$ be elements of $\mathcal{R}$ and I be the sub-$\mathcal{A}$-module of $\mathcal{R}$ that they generate. Put $I_n = I^n$ for all $n > 0$ and $I_n = \mathcal{A}$ for all $n \leq 0$.

Thus I_0 is a subring of $\mathcal{R}$ and the group direct sum $Q_J(f,k) = \bigoplus\limits_{n\in\mathbb{Z}} \dfrac{I_n}{J_n}$ is a graded ring.

Put $s_i = a_i + J_1$ $\forall i = 1, \ldots, r$.

Condition $(*)$ of 3.2.1 is satisfied.

Hence as in [5] there exists an isomorphism

$\psi_{1,k} : \bigoplus\limits_{n\geq 0} \dfrac{I_n}{J_n} \to \dfrac{R(\mathcal{A}, I)}{R(\mathcal{A}, I) \cap ((u^k, J)\,\Re\,(\mathcal{A}, I))}$ such that

$\psi_{1,k}(s_i) = a_i X + R(\mathcal{A}, I) \cap \left(\left(u^k, J\right)\Re\left(\mathcal{A}, I\right)\right)$ and $\psi_{1,k}(\alpha) = \alpha$ for $\alpha \in \dfrac{\mathcal{A}}{J + I^k \cap \mathcal{A}}$.

Furthermore, the products which follow are well defined :

For all $\alpha \in \mathcal{A}$ and $b_m \in I_m$ if $m = i_1 + \cdots + i_r$ then

$s_1^{i_1} \cdots s_r^{i_r} = \left(a_1^{i_1} \cdots a_r^{i_r} + J_m\right)$ and $(b_m + J_m)(\alpha + J_0) = b_m \alpha + J_m$.

Hence $s_1^{i_1}\cdots s_r^{i_r}\left(\alpha+J_0\right)=a_1^{i_1}\cdots a_r^{i_r}\alpha+J_m$.

Let $v_i=\psi_{1,k}(s_i)\ \ \forall i=1,\ldots,r$.

Since $I^mJ_0X^m\subseteq R(\mathcal{A},I)\cap\left(\left(u^k,J\right)\Re\left(\mathcal{A},I\right)\right)$, we have the products:

$$v_1^{i_1}\cdots v_r^{i_r}=a_1^{i_1}\cdots a_r^{i_r}X^m+R(\mathcal{A},I)\cap\left(\left(u^k,J\right)\Re\left(\mathcal{A},I\right)\right)\ \text{ and}$$

$$v_1^{i_1}\cdots v_r^{i_r}\left(\alpha+J_0\right)=a_1^{i_1}\cdots a_r^{i_r}\alpha X^m+R(\mathcal{A},I)\cap\left(\left(u^k,J\right)\Re\left(\mathcal{A},I\right)\right)$$

and we have the following commutative diagram:

$$\begin{tikzcd}
\dfrac{\mathcal{A}}{J_0}\left[X_1,\ldots,X_r\right] \arrow[r, "\varphi_{1,k}"] \arrow[dr, "\theta_{1,k}"'] & S_J(\mathcal{I},k)=\dfrac{\mathcal{A}}{J_0}\left[s_1,\ldots,s_r\right]=\bigoplus\limits_{m\geq 0}\dfrac{I^m}{J_m} \arrow[d, "\wr\ \psi_{1,k}"] \\
 & V_J(\mathcal{I},k)=\dfrac{\mathcal{A}}{J_0}\left[v_1,\ldots,v_r\right]
\end{tikzcd}$$

3.2.3. *Surjective morphisms relating to a global $(f\ltimes f)^+$ −quasi-bigraduation of module.*

1) Let $\mathcal{H}=\left(\mathbb{G}_{(i,j)}\right)_{(i,j)\in(\mathbb{Z}\times\mathbb{Z})\cup\{\infty\}}$ be a global $(f\ltimes f)^+$-quasi-bigraduation of $\mathcal{M}$ and $\mathcal{P}=(\mathbb{P}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}$ be a f^+−quasi-graduation of $\mathcal{R}$ with $\mathbb{P}_0=\mathbb{G}_{(0,0)}$ and $\mathbb{P}_m\subseteq S_m\mathbb{G}_{(0,0)}$, $\mathbb{K}_m=\mathbb{P}_m\cap(J\mathcal{N}_m+\mathcal{N}_{m+k})$, $\mathbb{J}_m=\mathbb{P}_m\cap\left(JS_m\mathbb{G}_{(0,0)}+S_{m+k}\mathbb{G}_{(0,0)}\right)\ \forall m\geq 0$ where $S=(S_m)_{m\in\mathbb{Z}\cup\{+\infty\}}=\mathsf{QG}\left(f\ltimes f\right)$ is the ${}^+$quasi-graduation of $\mathcal{R}$ deduced from $f\ltimes f$ and $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}=\mathsf{QG}\left(\mathcal{H}\right)$ is the S^+−quasi-graduation of $\mathcal{M}$ deduced from $\mathcal{H}$.

Consider $\mathbb{J}=\bigoplus\limits_m\mathbb{J}_mX^m$, $\mathcal{T}=\bigoplus\limits_d S_d\mathbb{G}_{(0,0)}X^d$, $\mathfrak{N}=\bigoplus\limits_d\mathcal{N}_dX^d$ and $Q_J\left(\mathcal{H},k\right)$ the graded $Q_J(f,k)$−module $\sum\limits_m\dfrac{\mathbb{P}_m}{\mathbb{K}_m}$, where $Q_J(f,k)=\bigoplus\limits_n\dfrac{I_n}{J_n}$.

Thus we have

$$I_m\mathbb{G}_{(0,0)}\cap\left(JS_m\mathbb{G}_{(0,0)}+S_{m+k}\mathbb{G}_{(0,0)}\right)\subseteq\mathbb{J}_m\subseteq\mathbb{P}_m\subseteq S_m\mathbb{G}_{(0,0)}\ ,$$

$$I_m\mathbb{G}_{(0,0)}\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)\subseteq\mathbb{K}_m=\mathbb{P}_m\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)\subseteq\mathbb{P}_m\subseteq\mathcal{N}_m$$

and

$$I_m\mathbb{G}_{(0,0)}\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)\subseteq I_m\mathbb{G}_{(0,0)}\subseteq\mathbb{P}_m\subseteq S_m\mathbb{G}_{(0,0)}\subseteq\mathcal{N}_m.$$

Conditions $(*)$ and $(**)$ of 3.2.1 are satisfied for (I_n) , (J_n) , $(\mathbb{P}_n)$ and $(\mathbb{K}_n)$, (resp. (I_n) , (J_n) , $(\mathbb{P}_n)$ and $(\mathbb{J}_n)$). Hence we have $u^k\mathfrak{N}=$

$\bigoplus_{d\geq 0}\mathcal{N}_d X^{d-k}$ and $\left(u^k,J\right)\mathfrak{N}=\bigoplus_{d\geq 0}\mathcal{N}_d X^{d-k}\oplus\bigoplus_{d\geq 0} J\mathcal{N}_d X^d$

$$\left(u^k,J\right)\mathfrak{N}=\left[\mathcal{N}_0 X^{-k}\bigoplus\mathcal{N}_1 X^{1-k}\bigoplus\ldots\bigoplus\mathcal{N}_{k-1}X^{-1}\right]\bigoplus\bigoplus_{d\geq 0}\left(J\mathcal{N}_d+\mathcal{N}_{d+k}\right)X^d.$$

$u^k\mathcal{T}=\bigoplus_{d\geq 0} S_d\mathbb{G}_{(0,0)}X^{d-k}$ and $\left(u^k,J\right)\mathcal{T}=\bigoplus_{d\geq 0} S_d\mathbb{G}_{(0,0)}X^{d-k}\oplus\bigoplus_{d\geq 0} JS_d\mathbb{G}_{(0,0)}X^d$

$$\left(u^k,J\right)\mathcal{T}=\left[\mathbb{G}_{(0,0)}X^{-k}\bigoplus S_1\mathbb{G}_{(0,0)}X^{1-k}\bigoplus\ldots\bigoplus S_{k-1}\mathbb{G}_{(0,0)}X^{-1}\right]\bigoplus\left[\left(u^k,J\right)\mathcal{T}\right]^+$$

where $\left[\left(u^k,J\right)\mathcal{T}\right]^+=\bigoplus_{d\geq 0}\left(JS_d+S_{d+k}\right)\mathbb{G}_{(0,0)}X^d$. From (2.2.1) we have

$$\frac{\mathbb{P}}{\mathbb{K}}=\frac{\bigoplus_{m\geq 0}\mathbb{P}_m X^m}{\bigoplus_{m\geq 0}\left[\mathbb{P}_m\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)\right]X^m}\simeq\bigoplus_{m\geq 0}\frac{\mathbb{P}_m}{\mathbb{P}_m\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)}$$

$$\frac{\mathbb{P}}{\mathbb{J}}=\frac{\bigoplus_{m\geq 0}\mathbb{P}_m X^m}{\bigoplus_{m\geq 0}\left[\mathbb{P}_m\cap\left(\left(JS_d+S_{d+k}\right)\mathbb{G}_{(0,0)}\right)\right]X^m}\simeq\bigoplus_{m\geq 0}\frac{\mathbb{P}_m}{\mathbb{P}_m\cap\left(\left(JS_d+S_{d+k}\right)\mathbb{G}_{(0,0)}\right)}.$$

2) Suppose that for each $m\geq 0$, $I_m=I^m$ and that $J\mathbb{G}_{(0,0)}+\mathcal{N}_k\cap\mathbb{G}_{(0,0)}\neq\mathbb{G}_{(0,0)}$ (resp. $J\mathbb{G}_{(0,0)}+\left(S_k\mathbb{G}_{(0,0)}\right)\cap\mathbb{G}_{(0,0)}\neq\mathbb{G}_{(0,0)}$), where $(\mathcal{N}_m)_{m\in\mathbb{Z}\cup\{+\infty\}}=\mathsf{QG}(\mathcal{H})$ is the S^+-quasi-graduation of $\mathcal{M}$ deduced from $\mathcal{H}$. Thus, for $m\geq 0$ we have

$\mathbb{P}_m=I^m\mathbb{G}_{(0,0)}$, $S_m=I^m$ and $\mathbb{J}_m=I^m\mathbb{G}_{(0,0)}\cap\left(JI^m\mathbb{G}_{(0,0)}+I^{m+k}\mathbb{G}_{(0,0)}\right)$
$\mathbb{K}_m=I^m\mathbb{G}_{(0,0)}\cap(J\mathcal{N}_m+\mathcal{N}_{m+k})$ and $Q_J(\mathcal{H},k)$ is the graded $Q_J(f,k)-$module $\sum_{m\geq 0}\frac{I^m\mathbb{G}_{(0,0)}}{\mathbb{K}_m}$. Hence we have

$$\frac{\bigoplus_{m\geq 0}I^m\mathbb{G}_{(0,0)}X^m}{\bigoplus_{m\geq 0}\left[I^m\mathbb{G}_{(0,0)}\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)\right]X^m}\simeq\bigoplus_{m\geq 0}\frac{I^m\mathbb{G}_{(0,0)}}{I^m\mathbb{G}_{(0,0)}\cap\left(J\mathcal{N}_m+\mathcal{N}_{m+k}\right)}$$

(resp.

$$\frac{\bigoplus_{m\geq 0}I^m\mathbb{G}_{(0,0)}X^m}{\bigoplus_{m\geq 0}\left[I^m\mathbb{G}_{(0,0)}\cap\left(JI^m+I^{m+k}\right)\mathbb{G}_{(0,0)}\right]X^m}\simeq\bigoplus_{m\geq 0}\frac{I^m\mathbb{G}_{(0,0)}}{I^m\mathbb{G}_{(0,0)}\cap\left(\left(JI^m+I^{m+k}\right)\mathbb{G}_{(0,0)}\right)})$$

Put $R\left(\mathbb{G}_{(0,0)},I\right)$ the graded $\mathcal{A}$-module $\bigoplus_{n\geq 0}I^n\mathbb{G}_{(0,0)}X^n$. We have

$$\begin{aligned}R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right] &= R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]^+\\ &= \bigoplus_{d\geq 0}\left[I^d\mathbb{G}_{(0,0)}\cap\left(J\mathcal{N}_d+\mathcal{N}_{d+k}\right)\right]X^d=\bigoplus_{d\geq 0}\mathbb{K}_d X^d\end{aligned}$$

(resp. $R\left(\mathbb{G}_{(0,0)}, I\right) \cap \Big[\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\Big] = R\left(\mathbb{G}_{(0,0)}, I\right) \cap \Big[\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\Big]^+$

$$= \bigoplus_{d \geq 0} \left[I^d \mathbb{G}_{(0,0)} \cap \left(\left(JI^d + I^{d+k}\right) \mathbb{G}_{(0,0)}\right)\right] X^d = \bigoplus_{d \geq 0} \mathbb{J}_d X^d).$$

3) Let us define the products as follows :
For all $\alpha \in \mathbb{G}_{(0,0)}$ and $b_m \in I_m$ if $m = i_1 + \cdots + i_r$ then

$s_1^{i_1} \cdots s_r^{i_r} = \left(a_1^{i_1} \cdots a_r^{i_r} + J_m\right)$ and $(b_m + J_m)(\alpha + \mathbb{J}_0) = b_m \alpha + \mathbb{J}_m$

where $\left(J + I^k\right) \mathbb{G}_{(0,0)} = J\mathbb{G}_{(0,0)} + \mathcal{N}_k \cap \mathbb{G}_{(0,0)}$.

Hence
$$s_1^{i_1} \cdots s_r^{i_r} \left(\alpha + \mathbb{K}_0\right) = a_1^{i_1} \cdots a_r^{i_r} \alpha + \mathbb{K}_m.$$

Furthermore,
$$v_1^{i_1} \cdots v_r^{i_r} = a_1^{i_1} \cdots a_r^{i_r} X^m + R(\mathcal{A}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathcal{A}, I\right)\right) \text{ and}$$

$$v_1^{i_1} \cdots v_r^{i_r} \left(\alpha + \mathbb{J}_0\right) = a_1^{i_1} \cdots a_r^{i_r} \alpha X^m + R(\mathbb{G}_{(0,0)}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\right).$$

Properties $(*)$ and $(**)$ of 3.2.1 show that the previous products are well defined.

4) Suppose that $\mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \left(I_k \mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)}$. Then

$$\mathbb{K}_0 = \mathbb{J}_0,\ I^m \mathbb{K}_0 X^m = I^m \mathbb{J}_0 X^m \subseteq R(\mathbb{G}_{(0,0)}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\right)$$

and the next product is well defined: for $\alpha \in \mathbb{G}_{(0,0)}$

$$v_1^{i_1} \cdots v_r^{i_r} \left(\alpha + \mathbb{K}_0\right) = a_1^{i_1} \cdots a_r^{i_r} \alpha X^m + R(\mathbb{G}_{(0,0)}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\right)$$

$$R(\mathbb{G}_{(0,0)}, I) \cap \left(\left(u^k, J\right) \Re\left(\mathbb{G}_{(0,0)}, I\right)\right) \subseteq R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right) \mathfrak{R}\right].$$

Put $S_J\left(\mathcal{H}, k\right) = \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[s_1, \ldots, s_r\right]$ and $V_J\left(\mathcal{H}, k\right) = \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[v_1, \ldots, v_r\right]$.

Let $\widetilde{\varphi}_{1,k} = \widetilde{\varphi}_{1,J}\left(\mathcal{H}, k\right)$ be the graded morphism of graded modules from $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[X_1, \ldots, X_r\right]$ onto $S_J\left(\mathcal{H}, k\right)$ such that $\widetilde{\varphi}_{1,k}(\overline{\alpha} X_i) = \overline{\alpha} s_i$ for each i and $\widetilde{\varphi}_{1,k}(\overline{\alpha}) = \overline{\alpha}$ for $\alpha \in \mathbb{G}_{(0,0)}$ and $\overline{\alpha} = \alpha + \mathbb{K}_0$.

There exists an isomorphism $\widetilde{\psi}_{1,k}$ of graded modules from $S_J\left(\mathcal{H}, k\right)$ onto $V_J\left(\mathcal{H}, k\right) = \dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right) \mathfrak{R}\left(\mathbb{G}_{(0,0)}, I\right)\right]}$ such that

$\widetilde{\psi}_{1,k}(\overline{\alpha} s_i) = \overline{\alpha} v_i = \alpha a_i X + R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right) \mathfrak{R}\left(\mathbb{G}_{(0,0)}, I\right)\right]$, $\widetilde{\psi}_{1,k}(\overline{\alpha}) = \overline{\alpha}$ for $\alpha \in \mathbb{G}_{(0,0)}$ and $\overline{\alpha} = \alpha + \mathbb{K}_0$.

Hence the following diagram commutes:

$$\begin{tikzcd}
\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\,[X_1,\ldots,X_r] \arrow[r, "\widetilde{\varphi}_{1,k}"] \arrow[dr, "\widetilde{\theta}_{1,k}"'] & \frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\,[s_1,\ldots,s_r] \arrow[d, "\wr\ \widetilde{\psi}_{1,k}"] \\
 & \frac{R(\mathbb{G}_{(0,0)},I)}{R(\mathbb{G}_{(0,0)},I)\cap(u^k,J)\Re(\mathbb{G}_{(0,0)},I)}
\end{tikzcd}$$

Proposition 3.5.

Suppose that $\mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \left(I_k\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)}$.

The following statements are equivalent :

(i) $a_1,\ldots,a_r$ *are* $J-$*independent of order* k *with respect to* $f_2\left(\mathbb{G}_{(0,0)}, I\right)$

(ii) $\widetilde{\varphi}_{1,k}$ *is an isomorphism of* $\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\,[X_1,\ldots,X_r]$ *over* $\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\,[s_1,\ldots,s_r]$

(iii) $\widetilde{\theta}_{1,k}$ *is an isomorphism of* $\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\,[X_1,\ldots,X_r]$ *over* $\frac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap(u^k,J)\,\Re\left(\mathbb{G}_{(0,0)},I\right)}$

(iv) The elements $s_1,\ldots,s_r$ *are algebraically independent over* $\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}$

(v) The elements $v_1,\ldots,v_r$ *are algebraically independent over* $\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}$.

Proof. See Theorem 2.3.1 of [6]. □

5) With the assumption that $\mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \left(I^k\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)} \neq \mathbb{G}_{(0,0)}$ put the canonical morphisms

$$V_J(\mathcal{H},k) = \frac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[(u^k,J)\,\Re\left(\mathbb{G}_{(0,0)},I\right)\right]} \overset{\widetilde{\theta}_{2,k}}{\to} \frac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[(u^k,J)\,\mathfrak{N}\right]}.$$

and $\delta_k = \widetilde{\theta}_{2,k}\circ\widetilde{\psi}_{1,k}$.

With $v_i = \psi_{1,k}(s_i) = a_iX + \left(R(\mathcal{A},I)\cap\left(\left(u^k,J\right)\Re\left(\mathcal{A},I\right)\right)\right)$ we have

$v_i\overline{\alpha} = a_i\alpha X + R(\mathbb{G}_{(0,0)},I)\cap\left(\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right)\right) = \widetilde{\psi}_{1,k}(s_i\overline{\alpha})$ for $\alpha\in\mathbb{G}_{(0,0)}$ and $\overline{\alpha} = \alpha+\mathbb{K}_0$.

Thus

$$\widetilde{\theta}_{2,k}\left(v_i\overline{\alpha}\right) = a_i\alpha X + R\left(\mathbb{G}_{(0,0)},I\right)\cap\left(u^k,J\right)\mathfrak{N}$$

Put

$$S_J\left(\mathcal{H},k\right)=\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[s_1,\ldots,s_r\right]=\left\{\sum_{i_1,\ldots,i_r}\left(\alpha_{i_1,\ldots,i_r}+\mathbb{K}_0\right)s_1^{i_1}\cdots s_r^{i_r}:\alpha_{i_1,\ldots,i_r}\in\mathbb{G}_{(0,0)}\right\}.$$

We have

$$\frac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right)}=\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[v_1,\ldots,v_r\right].$$

Put $\widetilde{\varphi}_{1,k}=\widetilde{\varphi}_{1,J}\left(\mathcal{H},k\right)$ and $\widetilde{\theta}_{1,k}=\widetilde{\psi}_{1,k}\circ\widetilde{\varphi}_{1,k}$.

The following diagram commutes:

$$\begin{tikzcd}
\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[X_1,\ldots,X_r] \arrow[r, "\widetilde{\varphi}_{1,k}"] \arrow[dr, "\widetilde{\theta}_{1,k}"'] \arrow[drr, "\widetilde{\theta}_k"] & \frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[s_1,\ldots,s_r] \arrow[r, "="] \arrow[d, "\wr\widetilde{\psi}_{1,k}"] \arrow[dr, "\delta_k"] & \bigoplus\limits_{m\geq 0}\frac{I^m\mathbb{G}_{(0,0)}}{\mathbb{K}_m} \\
\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[v_1,\ldots,v_r] \arrow[r, "="'] & \frac{R(\mathbb{G}_{(0,0)},I)}{R(\mathbb{G}_{(0,0)},I)\cap(u^k,J)\Re(\mathbb{G}_{(0,0)},I)} \arrow[r, "\widetilde{\theta}_{2,k}"] & \frac{R(\mathbb{G}_{(0,0)},I)}{R(\mathbb{G}_{(0,0)},I)\cap(u^k,J)\mathfrak{N}}
\end{tikzcd}$$

3.2.4. *Properties of independence.*
Under the previous hypotheses we show the following theorem as in [6] and [7]:

Theorem 3.6. *Under the notations and hypotheses of 2.2.3 and with the assumption that* $\mathcal{N}_k\cap\mathbb{G}_{(0,0)}\subseteq J\mathbb{G}_{(0,0)}+\left(I_k\mathbb{G}_{(0,0)}\right)\cap\mathbb{G}_{(0,0)}$ *the following assertions are equivalent:*
(i) The elements $a_1,\ldots,a_r$ *are* $J-$*independent of order* k *with respect to* $\mathcal{H}$.

$$\text{(ii)}\left\{\begin{array}{l}\text{a) The elements } a_1,\ldots,a_r \text{ are } J-\text{independent of order } k\\ \text{with respect to } f_2\left(\mathbb{G}_{(0,0)},I\right)\ \text{ and}\\ \text{b) } I_p\mathbb{G}_{(0,0)}\cap\left(J\mathcal{N}_p+\mathcal{N}_{p+k}\right)=JI_p\mathbb{G}_{(0,0)}+\left(I_{p+k}\cap I_p\right)\mathbb{G}_{(0,0)}\ \forall p\geq 0\end{array}\right.$$

$$\text{(iii)}\left\{\begin{array}{l}\text{a) The family } \{s_1,\ldots,s_r\} \text{ is algebraically free over } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\ \text{ and}\\ \text{b) } R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]=R\left(\mathbb{G}_{(0,0)},I\right)\cap\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right).\end{array}\right.$$

$$\text{(iv)} \begin{cases} a)\ \widetilde{\theta}_{1,k} \quad \textit{is an isomorphism from} \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[X_1,\ldots,X_r] \ \textit{onto} \\ \dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap (u^k, J)\,\Re\left(\mathbb{G}_{(0,0)}, I\right)} \quad \textit{and} \\ b)\ \widetilde{\theta}_{2,k} \ \textit{is an isomorphism from} \ \dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap (u^k, J)\,\Re\left(\mathbb{G}_{(0,0)}, I\right)} \ \textit{onto} \\ \dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap [(u^k, J)\,\mathfrak{N}]} \end{cases}$$

$$\text{(v)} \begin{cases} a)\ \widetilde{\varphi}_{1,k} \quad \textit{is an isomorphism from} \ \frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[X_1,\ldots,X_r] \ \textit{onto} \\ \frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[s_1,\ldots,s_r] \\ \quad \textit{and} \\ b)\ \delta_k \ \textit{is an isomorphism from} \ \frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[s_1,\ldots,s_r] \ \textit{onto} \ \frac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right)\cap\left[\left(u^k, J\right)\mathfrak{N}\right]} \end{cases}$$

(vi) $\widetilde{\theta}_k = \widetilde{\theta}_{2,k} \circ \widetilde{\theta}_{1,k}$ *is an isomorphism from* $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[X_1,\ldots,X_r]$ *onto*
$\dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap [(u^k, J)\,\mathfrak{N}]}$.

Proof. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)$\Leftrightarrow$ (v)

By Proposition 3.5, the elements $a_1,\ldots,a_r$ are $J-$independent of order k with respect to $f_2\left(\mathbb{G}_{(0,0)}, I\right)$ iff the family $\{s_1,\ldots,s_r\}$ is algebraically free over $\dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}$. It is equivalent to the fact that $\widetilde{\theta}_{1,k}$ (resp. $\widetilde{\varphi}_{1,k}$) is an isomorphism.

Moreover, $I_p\mathbb{G}_{(0,0)} \cap (J\mathcal{N}_p + \mathcal{N}_{p+k}) = JI_p\mathbb{G}_{(0,0)} + (I_{p+k}\cap I_p)\,\mathbb{G}_{(0,0)}\ \forall p \geq 0$ if and only if $R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left[\left(u^k, J\right)\mathfrak{N}\right] = R\left(\mathbb{G}_{(0,0)}, I\right) \cap \left(u^k, J\right)\Re\left(\mathbb{G}_{(0,0)}, I\right)$ if and only if $\widetilde{\theta}_{2,k}$ (resp. δ_k) is a graded ring isomorphism.

(iv) $\Leftrightarrow$ (vi)

We have: $\widetilde{\theta}_{1,k}$ and $\widetilde{\theta}_{2,k}$ are surjective and $\widetilde{\theta}_k = \widetilde{\theta}_{2,k} \circ \widetilde{\theta}_{1,k}$. Therefore $\widetilde{\theta}_k$ is an isomorphism if and only if both $\widetilde{\theta}_{1,k}$ and $\widetilde{\theta}_{2,k}$ are isomorphisms.

(i) $\Leftrightarrow$ (vi) As in Theorem 2.3.1 of [6],

[the elements $a_1,\ldots,a_r$ are $J-$independent of order k with respect to $\mathcal{H}$]
iff
$\widetilde{\theta}_k = \widetilde{\theta}_{2,k} \circ \widetilde{\theta}_{1,k}$ is an isomorphism from $\frac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}[X_1,\ldots,X_r]$ onto
$\dfrac{R\left(\mathbb{G}_{(0,0)}, I\right)}{R\left(\mathbb{G}_{(0,0)}, I\right) \cap [(u^k, J)\,\mathfrak{N}]}$. □

Corollary 3.7. *Under the notations and hypotheses of 2.2.3 and with the assumption that* $\mathcal{N}_k \cap \mathbb{G}_{(0,0)} \subseteq J\mathbb{G}_{(0,0)} + \left(I_k\mathbb{G}_{(0,0)}\right) \cap \mathbb{G}_{(0,0)}$ *the following assertions are equivalent :*

(i) $a_1,\ldots,a_r$ *are* $J-$*independent of order* k *with respect to* $\mathcal{H}$

$$(ii)\begin{cases} a)\ \ a_1,\ldots,a_r \text{ are } J-\text{independent of order } k \text{ with respect to } f_2\left(\mathbb{G}_{(0,0)},I\right) \\ b)\ \delta_k \text{ is an isomorphism of } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[s_1,\ldots,s_r\right] \text{ over } \dfrac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]} \end{cases}$$

$$(iii)\begin{cases} a)\ \widetilde{\varphi}_{1,k} \text{ is an isomorphism of } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[X_1,\ldots,X_r\right] \text{ over } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[s_1,\ldots,s_r\right] \\ b)\ \delta_k \text{ is an isomorphism of } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[s_1,\ldots,s_r\right] \text{ over } \dfrac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]} \end{cases}$$

$$(iv)\begin{cases} a)\ \ \widetilde{\theta}_{1,k} \text{ is an isomorphism of } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0}\left[X_1,\ldots,X_r\right] \text{ over} \\ \dfrac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right)} \\ b)\ \ \widetilde{\theta}_{2,k} \text{ is an isomorphism of } \dfrac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right)} \text{ over} \\ \dfrac{R\left(\mathbb{G}_{(0,0)},I\right)}{R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]} \end{cases}$$

$$(v)\begin{cases} a)\ \ \text{The elements } s_1,\ldots,s_r \text{ are algebraically independent over } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0} \\ b)\ \ R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]=R\left(\mathbb{G}_{(0,0)},I\right)\cap\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right) \end{cases}$$

$$(vi)\begin{cases} a)\ \ \text{The elements } v_1,\ldots,v_r \text{ are algebraically independent over } \dfrac{\mathbb{G}_{(0,0)}}{\mathbb{K}_0} \\ b)\ \ R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[\left(u^k,J\right)\mathfrak{N}\right]=R\left(\mathbb{G}_{(0,0)},I\right)\cap\left(u^k,J\right)\Re\left(\mathbb{G}_{(0,0)},I\right) \end{cases}.$$

Applying this result to case $k=\infty$ we have the following Corollary:

Corollary 3.8. *Suppose that $k=+\infty$. Under the notations and hypotheses of 2.2.3 the following assertions are equivalent :*

(i) The elements $a_1,\ldots,a_r$ are $J-$independent (of order $+\infty$) with respect to $\mathcal{H}$

$$(ii)\begin{cases} \text{The elements } a_1,\ldots,a_r \text{ are } J-\text{independent (of order } \infty) \\ \text{with respect to } f_2\left(\mathbb{G}_{(0,0)},I\right) \\ \text{and} \\ I_p\mathbb{G}_{(0,0)}\cap\left(J\mathcal{N}_p\right)=JI_p\mathbb{G}_{(0,0)} \ \text{ for all } p\geq 0 \end{cases}$$

$$(iii)\begin{cases} \text{The family } \{s_1,\ldots,s_r\} \text{ is algebraically free over } \dfrac{\mathbb{G}_{(0,0)}}{J\mathbb{G}_{(0,0)}} \\ \text{and} \\ R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[J\mathfrak{N}\right]=JR\left(\mathbb{G}_{(0,0)},I\right) \end{cases}$$

$$(iv)\begin{cases} \text{The family } \{v_1,\ldots,v_r\} \text{ is algebraically free over } \dfrac{\mathbb{G}_{(0,0)}}{J\mathbb{G}_{(0,0)}} \\ \text{and} \\ R\left(\mathbb{G}_{(0,0)},I\right)\cap\left[J\mathfrak{N}\right]=JR\left(\mathbb{G}_{(0,0)},I\right). \end{cases}.$$

Example 3.9. Let $\mathcal{R} = \mathbb{R}[X,Y]$ be the ring of polynomials of two indeterminates X and Y with coefficients in $\mathbb{R}$, $\mathcal{A} = \mathbb{Z}[X,Y]$ and $\mathcal{M} = i\mathbb{Q}[X,Y]$

Let $f = \left(I_{(m,n)}\right)$, where $I_{(m,n)} = (X^nY^m)\mathbb{Z}[X,Y]$ for all $m,n \in \mathbb{N}$.
Let $\mathcal{H} = \left(\mathbb{G}_{(p,q)}\right)$ such that $\mathbb{G}_{(p,q)} = (X^pY^q)\, i\mathbb{Q}[X,Y]$ for all $p,q \in \mathbb{N}\times\mathbb{N}$.
f is a quasi-bigraduation of $\mathcal{R}$ and a bifiltration of $\mathcal{R}$.
$\mathcal{H}$ is an f^+-quasi-bigraduation of $\mathcal{M}$.
Let $a_1 = X$ and $a_2 = Y$, $J = (X,Y)\mathbb{Z}[X,Y]$. We have
$\mathcal{N}_d = \left(X^d, X^{d-1}Y, X^{d-2}Y^2, \ldots, Y^d\right) i\mathbb{Q}[X,Y]$
$\mathcal{N}_1 = (X,Y)\, i\mathbb{Q}[X,Y]$ and $\mathcal{N}_0 = \mathbb{G}_{(0,0)} = i\mathbb{Q}[X,Y]$
$\alpha_{i,j} \in \mathcal{A} = \mathbb{Z}[X,Y]$. Put $k = +\infty$

1) Let $\mathcal{F} = \sum\limits_{i+j=d} \alpha_{i,j} X_1^i X_2^j \in \mathcal{A}[X_1,X_2]$ a homogeneous polynomial of degree d
$\mathcal{F}(a_1,a_2) = \sum\limits_{i+j=d} \alpha_{i,j} X^iY^j \in J\mathcal{N}_d \Rightarrow$
$\sum\limits_{i+j=d} \alpha_{i,j} X^iY^j \in (X,Y)\left(X^d, X^{d-1}Y, X^{d-2}Y^2, \ldots, Y^d\right) i\mathbb{Q}[X,Y] \Rightarrow$
$\alpha_{i,j} \in \mathcal{N}_1 = (X,Y)\, i\mathbb{Q}[X,Y] = (X,Y)\, i\mathbb{Z}[X,Y]\,\mathbb{Q}[X,Y] = J\mathbb{G}_{(0,0)}$
Thus $\alpha_{i,j} \in (J\mathcal{N}_0) \cap \mathbb{G}_{(0,0)} = J\mathbb{G}_{(0,0)}$.
Therefore, X and Y are $J-$independent with respect to $\mathcal{H}$.
We have:
X^2 and Y^2 are $J-$independent with respect to $\mathcal{H}$.

2) Put $a_3 = XY$ and $\mathcal{F}(X_1,X_2) = iYX_1 - iX_2 = \alpha_1 X_1 + \alpha_2 X_2$ where $\alpha_1 = iY$ and $\alpha_2 = -i \in \mathcal{A}$. Let $\mathcal{F} = iYX_1 - iX_2 \in \mathcal{N}_0[X_1,X_2]$.
$\mathcal{F}$ is a homogeneous polynomial of degree 1 and
$\mathcal{F}(a_1,a_3) = iYa_1 - ia_3 = iYX - iXY = 0 \in J\mathcal{N}_1$. But $\alpha_2 = -i \notin J\mathbb{G}_{(0,0)}$.
Therefore, X and XY aren't $J-$independent with respect to $\mathcal{H}$.

Example 3.10. Let $\mathcal{R} = \mathbb{Z}$ and let $p > 1$ and $q > 1$ be two integers.
Let $\left(I_{(m,n)}\right)_{(m,n)\in\Delta}$ be the family such that
* $I_{(m,n)} = (p^mq^n)\mathbb{Z}$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$
* $I_{(m,n)} = (q^n)\mathbb{Z}$ for all $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ with $m \le 0$ and $n \ge 0$
* $I_{(m,n)} = (p^m)\mathbb{Z}$ for all $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ with $n \le 0$ and $m \ge 0$.

Put (I_m) the family such that for $m \ge 0$ $I_m = (p^m)\mathbb{Z} = (p\mathbb{Z})^m$ and for $m < 0$ $I_m = \mathbb{Z}$ and (J_n) the family such that for $n \ge 0$ $J_n = (q^n)\mathbb{Z} = (q\mathbb{Z})^n$ and for $n < 0$ $J_n = \mathbb{Z}$.

We have $(I_m) = f_I$ where $I = p\mathbb{Z}$, $(J_n) = f_J$ where $J = q\mathbb{Z}$ and $I_{(m,n)} = I_mJ_n$. Thus the family $\left(I_{(m,n)}\right)_{(m,n)\in\Delta}$ is the bifiltration $f_I \ltimes f_J$.
* Put $\mathbb{G}_{(m,n)} = iI_{(m,n)}$ for all $(m,n) \in \Delta$.

The family $\mathcal{H} = \left(\mathbb{G}_{(m,n)}\right)_{(m,n)\in\Delta}$ is a $f-$quasi-bigraduation of the $\mathcal{R}$-module $\mathcal{M} = i\mathbb{Z}$. $\mathcal{N}_d = \sum\limits_{-d\le n\le -1} \mathbb{G}_{(n,d-n)} + \sum\limits_{0\le n\le d} \mathbb{G}_{(n,d-n)} + \sum\limits_{d+1\le n\le 2d} \mathbb{G}_{(n,d-n)}$
$\mathcal{N}_d = i\left(q^d\mathbb{Z} + pq^{d-1}\mathbb{Z} + \cdots + p^{d-1}q\mathbb{Z} + p^d\mathbb{Z}\right) = \sum\limits_{0\le n\le d} \mathbb{G}_{(n,d-n)}$.
$\mathcal{N}_0 = \mathbb{G}_{(0,0)} = i\mathbb{Z}$ and $\mathcal{N}_1 = i(p\mathbb{Z} + q\mathbb{Z}) = \delta i\mathbb{Z}$ where $\delta = \gcd(p,q)$.

Let $r > 1$ be an integer which is prime with $\mu = \mathrm{lcm}(p, q)$ and $J = r\mathbb{Z}$. We have

1) $a = p$ and $b = q$ aren't $J-$independent of order $+\infty$ (with respect to $\mathcal{H}$).
In fact, $J\mathcal{N}_0 = ri\mathbb{Z} \neq \mathbb{G}_{(0,0)}$, $qia - pib = 0 \in J\mathcal{N}_1$; but $qi \notin J\mathcal{N}_0 = ri\mathbb{Z}$.

2) If $k \in \mathbb{N}^*$ then p and q aren't $J-$independent of order k (with respect to $\mathcal{H}$).

In fact, $\mu \wedge r = 1 \Rightarrow \delta \wedge r = 1$ and $J\mathcal{N}_0 + \mathcal{N}_k = ri\mathbb{Z} + \delta^k i\mathbb{Z} = i\mathbb{Z} = \mathbb{G}_{(0,0)} = \mathcal{N}_0$.

3) $a = p$ is $J-$independent of order k (with respect to $\mathcal{H}$)

In fact, for $\lambda \in \mathcal{N}_0$, $\lambda a^d \in J\mathcal{N}_d \Rightarrow \lambda p^d \in r\mathbb{Z}i\delta^d\mathbb{Z} = r\delta^d i\mathbb{Z} \Rightarrow \exists t \in \mathbb{Z}$ such that

$\lambda p^d = rt\delta^d i$. Put $s = p/\delta$. We have $\lambda' s^d = rt$ where $\lambda' = -i\lambda$

As $\mu \wedge r = 1$, we have $\mu \wedge s = 1$; so r divide λ' and $\lambda \in ri\mathbb{Z} = J\mathcal{N}_0 \subseteq J\mathcal{N}_0 + \mathcal{N}_k$.

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