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E-LIFTING MODULES RELATIVE TO FULLY INVARIANT SUBMODULES

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ABSTRACT. In this paper, we introduce the notion FI-e-lifting modules which is proper generalization of lifting (e-lifting) modules. Then we give some characterizations and properties of e-lifting and FI-e-lifting modules. We provide a decomposition of any e-lifting modules. It is shown that every finite direct sum of FI-e-lifting modules is FI-e-lifting.

1. Introduction

Throughout this paper R is an associative ring with unity and all modules are unital right R-modules. By $N \leq M$, we mean that N is a submodule of M. A submodule N of a module M is called *essential* in M, if for every nonzero submodule L of M, we have $N \cap L \neq 0$ (denoted by $N \leq_e M$). As a dual concept a submodule N of a module M is called *small* in M, if for every proper submodule L of M, $N + L \neq M$ (denoted by $N \ll M$). Also M is called a small module, if there exists a module T such that $M \ll T$. Recall that the *singular* submodule $Z(M)$ of a module M is the set of $m \in M$ with $mI = 0$ for some essential right ideal I of R. If $Z(M) = M$ ($Z(M) = 0$), then M is called a singular (*nonsingular*) module. Let K, N be submodules of M. Following [13], as a generalization of small submodules, N is called *δ-small* in M, if $M = N + K$ with M/K singular implies $M = K$ (denoted by $N \ll_\delta M$). A submodule K of M is called *fully invariant* if $\varphi(K) \subseteq K$ for every endomorphism φ of M. An R-module M is called

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lifting if for every submodule A of M there exists a direct summand N of M with $N \subseteq A$ and $A/N \ll M/N$ (see [3]). As a generalization concept of lifting modules, introduced the notion *FI-lifting* modules (see [6], [9]). An R-module M is said to be FI-lifting if every fully invariant submodule A of M contains a direct summand N of M with $A/N \ll M/N$. Following [7], Kosan defined *δ-lifting* modules, The module M is called δ-lifting if for every submodule A of M there exists a direct summand N of M with $N \subseteq A$ and $A/N \ll_\delta M/N$. In [11] authors defined and consider *FI-δ-lifting* modules. A module M is FI-δ-lifting if every fully invariant submodule A of M contains a direct summand N of M such that $A/N \ll_\delta M/N$.

Following [14], a submodule N of M is called *e-small* in M (denoted by $N \ll_e M$), if $N + L = M$ with L essential in M implies $L = M$. We say, a module M is called a e-small module if there exists a module T such that $M \ll_e T$. It is clear that if N is a δ-small submodule of M then N is an e-small submodule of M. Some basic characterizations of e-small submodules are obtained in [14]. Recently, several authors used the small and e-small notions to study some characterizations of rings and modules ([8], [12],...). Using this notion, Quynh-Hong Tin [8] introduced a generalization of lifting modules. A module M is said to be *e-lifting* if for every submodule A of M, there exists a direct summand N of M with $N \subseteq A$ and $A/N \ll_e M/N$. Also, we introduce the notion of *FI-e-lifting modules*. We call a right R-module M FI-e-lifting if for every fully invariant submodule A of M, there exists a direct summand N of M with $N \subseteq A$ and $A/N \ll_e M/N$.

In Section 2, we study some properties of e-small submodules and e-lifting modules. We provide decompositions for e-lifting module in term of its special submodules. We show that if M is e-lifting module. Then there exist a semisimple submodule M_1 and a submodule M_2 of M such that $M = M_1 \oplus M_2$ and every nonzero submodule of M_2 contains a nonzero e-small submodule (see Proposition 2.11).

We define and investigate FI-e-lifting modules in Section 3, which were motivated by definitions of FI-lifting modules. It is shown that, every finite direct sum of FI-e-lifting modules is FI-e-lifting module (see Theorem 3.9).

2. Some properties of e-lifting modules

This section devoted to study about properties of e-small submodule and e-lifting modules.

The following lemma, which characterizes e-small, is taken from [14].

Lemma 2.1. *Let M be a module. Then*

(1) *If* $N \ll_e M$ *and* $K \leq N$*, then* $K \ll_e M$ *and* $N/K \ll_e M/K$.
(2) *Let* $N \ll_e M$ *and* $M = X + N$. *Then* $M = X \oplus Y$ *for a semisimple submodule* Y *of* M.
(3) *Let* $N, K \leq M$. *Then* $N + K \ll_e M$ *if and only if* $N \ll_e M$ *and* $K \ll_e M$.
(4) *If* $K \ll_e M$ *and* $f : M \to N$ *is a homomorphism, then* $f(K) \ll_e N$. *In particular, if* $K \ll_e M \leq N$*; then* $K \ll_e N$.
(5) *Let* $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ *and* $M = M_1 \oplus M_2$. *Then* $K_1 \oplus K_2$ *is e-small in* $M_1 \oplus M_2$ *if and only if* $K_1 \ll_e M_1$ *and* $K_2 \ll_e M_2$.
(6) *Let* $N, K \leq M$ *such that* $N \subseteq K$. *If* K *is a direct summand of* M *and* $N \ll_e M$*, then* $N \ll_e K$.

The next proposition shows an equivalent statement of e-small submodules.

Proposition 2.2. *A submodule* N *of* R*-module* M *is e-small if and only if for each submodule* X *of* M*, if* $N + X = M$*, then* X *is a direct summand of* M.

Proof. ($\Rightarrow$) Let N be a e-small submodule of M and suppose that X a submodule of M such that $N + X = M$. Let X' be a relative complement of X in M. By [4, Page 6], $X \oplus X'$ is an essential submodule in M. Since $N + X = M$, it follows that $K + X + X' = M$. Since N is a e-small submodule of M, $X + X' = M$ and hence $M = X \oplus X'$. Thus X is a direct summand of M.

($\Leftarrow$) Let for each submodule X of M, if $N + X = M$, then X is a direct summand of M. Now, let X be an essential submodule of M. By hypothesis, X is a direct summand in M. But M is the only essential direct summand in M, so $X = M$ and hence K is an e-small submodule in M.

□

By definitions every δ-small submodule of a module is e-small in that module and by above lemma, every e-small submodule of a projective module is δ-small. The following example shows that the class of e-small submodules contains properly the class of δ-small submodules.

Example 2.3. (see Example 2.2 [14]) Assume that $R = \mathbb{Z}$, $M = \mathbb{Z}_6$, $N = \{0, 3\}$ and $K = \{0, 2, 4\}$. Then N is e-small in M. But M/K is singular and $N + K = M$. So N is neither δ-small nor small in M.

Proposition 2.4. *Let* M *be a nonsingular* R*-module. A proper submodule* N *of* M *is e-small if and only if it is* δ*-small.*

Proof. We know, every δ-small submodule of M is a e-small submodule of M. Now, let $N \ll_e M$, Suppose that $N + K = M$ with M/K

is singular. Since M is nonsingular, then by [5, Proposition 1.21], $K \leq_e M$. But $N \ll_e M$ so $K = M$. Thus $N \ll_\delta M$ □

Example 2.5. Let R be a right semisimple ring and M a nonzero right R-module. Then M is semisimple and nonsingular. For any nonzero $N \leq M$, N is a direct summand of M and hence is not small in M, but every submodule of M(even M itself) is δ-small in M and so e-small.

For an R-module M, $\delta(M) = \sum\{N \leq M \mid L \ll_\delta M\}$ and $Rad_e(M) = \sum\{N \leq M \mid N \ll_e M\}$ ([14]). Clearly $Rad(M) \subseteq \delta(M) \subseteq Rad_e(M)$.

Corollary 2.6. *Let M be a semisimple module, then $Rad_e(M) = M$.*

But the converse above corollary is not true. Let $\mathbb{Z}$-module $M = \mathbb{Q}$. Since $Rad(M) = M$, so $Rad_e(M) = M$, but M is not semisimple.

Let M be an R-module. Recall that, a pair (P, p) is a projective δ-cover of M in case P is a projective R-module and $P \stackrel{p}{\to} M \to 0$ is epimorphism and $Kerp \ll_\delta P$. Also a ring R is called δ-semiperfect, if every simple R-module has a projective δ-cover ([13]).

We know, every lifting module is e-lifting, but next example shows that the converse is not true.

Example 2.7. Let R be a δ-semiperfect ring. Then by [13, Theorem 3.6], for any right ideal I of R, $I = eR \oplus S$, where $e^2 = e \in R$ and $S \leq \delta(R) \subseteq Rad_e(M)$. Hence by [8, Lemma 2], R is a e-lifting right R-module. If R were lifting, R would be perfect and so semiperfect. But by [13, Example 4.1], A δ-semiperfect ring is not necessarily semiperfect. Thus, R is not lifting.

Proposition 2.8. *Let M be e-lifting module. Then the module $M/Rad_e(M)$ is semisimple.*

Proof. By [8, Lemma 7]. □

Corollary 2.9. *Let R be a ring such that every simple R-module is e-small and M an e-lifting module. Then $Rad_e(M)$ is an essential submodule of M.*

Proof. Let N be any submodule of M such that $N \cap Rad_e(M) = 0$. So N can be embedded in $M/Rad_e(M)$. By Proposition 2.8, N is semisimple, so that, by hypothesis, $N \subseteq Rad_e(M)$. Hence $N = 0$. Thus $Rad_e(M)$ is an essential submodule of M. □

Lemma 2.10. *Let M be a e-lifting module and N be any submodule of M. Then N contains a nonzero e-small submodule or N is a semisimple direct summand of M.*

Proof. Suppose that N does not contain a e-small. Let P be any submodule of N. By [8, Lemma 2], $P = K \oplus L$ for some direct summand K of M and e-small submodule L of M. But $L = 0$, and hence, $P = K$. By [1, Theorem 9.6], N is a semisimple direct summand of M. □

The following provide a decomposition of e-lifting modules in term of its special submodules.

Proposition 2.11. *Let M be a e-lifting module. Then there exist a semisimple submodule M_1 and a submodule M_2 of M such that $M = M_1 \oplus M_2$ and every nonzero submodule of M_2 contains a nonzero e-small submodule.*

Proof. Let $\mathcal{A} = \{N \leq M$ such that N does not contain a non-zero e-small submodule$\}$. By Zorn's Lemma, $\mathcal{A}$ contains a maximal element M_1. By Lemma 2.10, M_1 is a semisimple direct summand of M. So there exists a submodule M_2 such that $M = M_1 \oplus M_2$. Let N be a non-zero submodule of M_2. Then $M_1 \oplus N$ contains a non-zero e-small submodule K, by the choice of M_1. Note that $K \cap M_1$ is a e-small submodule and hence $K \cap M_1 = 0$. Thus K can be embedded in N and hence N contains a non-zero e-small submodule. □

Recall that an R-module M is called *extending*, provided for every submodule A of M there exists a direct summand B of M such that $A \leq_e B$ ([4]).

Proposition 2.12. *Let M be an extending module. Then M is e-lifting if and only if every submodule of M is a direct sum of an extending module and a e-small module.*

Proof. Suppose that M is e-lifting. Let $N \leq M$. Then $N = N_1 \oplus N_2$ where N_1 is a direct summand of M and N_2 is e-small. It follows that N_1 is extending. Conversely, Suppose that every submodule of M is a direct sum of an extending module and a e-small module. Let L be any submodule of M. Then $L = L_1 \oplus L_2$ for some extending module L_1 and e-small module L_2. Since L_1 is extending, there exists a direct summand K of L such that $L_1 \leq_e K$. It follows that $K \cap L_2 = 0$ and $L = K \oplus L_2$. Hence M is e-lifting. □

By analogy with [10, Proposition 2.8], we get the following proposition.

Proposition 2.13. *Let R be a ring. An injective right R-module M is e-lifting if and only if every submodule of M is a direct sum of an injective module and a e-small module.*

Proof. Suppose that M be a e-lifting injective module. By [8, Lemma 2], every submodule of M is a direct sum of an injective module and a e-small module. Conversely, suppose that every submodule of M is a direct sum of an injective module and a e-small module. Since an injective submodule is a direct summand, M is e-lifting. □

Proposition 2.14. *The following are equivalent for a ring R.*

(1) *Every extending right R-module is e-lifting;*

(2) *Every quasi-injective right R-module is e-lifting;*

(3) *Every injective right R-module is e-lifting;*

(4) *Every right R-module is a direct sum of an extending module and a e-small module;*

(5) *Every right R-module is a direct sum of an injective module and a e-small module.*

Proof. (3) $\Longleftrightarrow$ (5) By Proposition 2.13. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) Clear. (1) $\Longleftrightarrow$ (4) By Proposition 2.12. □

Let R be a ring. Recall that R is a right *Harada ring* (*H-ring* for short), if every injective right R-module is lifting. R is a right H-ring if and only if every right R-module can be expressed as a direct sum of a small R-module and an injective module. Also R is a *Quasi-Frobenius* ring (*QF-ring* for short), if every injective module is projective if and only if every projective module is injective. Therefore, if R is H-ring or QF-ring then by Proposition 2.14, for every right R-module, can be written a decomposition (See [3]).

Let M be an R-module. We say that M satisfies the condition $(*)$, if for every direct summands M_1 and M_2 of M with $M_1 \cap M_2 \ll_e M$, then $M_1 \cap M_2 = 0$

Example 2.15. (1) $\mathbb{Z}$-module $\mathbb{Z}_6$ satisfies $(*)$ condition.

(2) Consider $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ as a $\mathbb{Z}$-module, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ does not satisfies $(*)$ condition. Because $A = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{0}, \overline{2}), (\overline{1}, \overline{3})\}$ and $B = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{1}), (\overline{0}, \overline{2}), (\overline{0}, \overline{3})\}$ are direct summands of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, but $A \cap B \neq 0$.

Lemma 2.16. *Let M be an R-module satisfies $(*)$ condition, then every direct summand of M satisfies $(*)$ condition.*

Proof. Let A be a direct summand of M and let A_1 and A_2 be direct summands of A with $A_1 \cap A_2 \ll_e A$. So A_1 and A_2 are direct summands of M with $A_1 \cap A_2 \ll_e M$. Since M satisfies $(*)$ condition, then $A_1 \cap A_2 = 0$. Thus A satisfies $(*)$ condition. □

Proposition 2.17. *Let M be a e-lifting module satisfies $(*)$ condition. If M_1 and M_2 are direct summands of M, then $M_1 \cap M_2$ is a direct summand of M.*

Proof. Let $M_1 \cap M_2 \neq 0$. Since M is a e-lifting module, then there is a submodule A of $M_1 \cap M_2$ such that $M = A \oplus B$ and $(M_1 \cap M_2) \cap B \ll_e B$ and so $(M_1 \cap M_2) \cap B \ll_e M$. We show $M_1 \cap B$ are $B \cap M_2$ direct summands of B. It is easy that, $M_1 = M \cap M_1 = M_1 \cap (A \oplus B) = A \oplus (M_1 \cap B)$. Since M_1 is a direct summand of M, then $M_1 \cap B$ is a direct summand of B. Similarly $M_2 \cap B$ is a direct summand of B. But by Lemma 2.16, B satisfies $(*)$ condition. Since $(M_1 \cap B) \cap (M_2 \cap B) = (M_1 \cap M_2) \cap B \ll_e B$, then $(M_1 \cap B) \cap (M_2 \cap B) = 0$. Thus we get $(M_1 \cap M_2) \cap B = 0$. Next we have, $(M_1 \cap M_2) = (M_1 \cap M_2) \cap M = (M_1 \cap M_2) \cap (A \oplus B) = A \oplus ((M_1 \cap M_2) \cap B) = A$. So $M_1 \cap M_2$ is a direct summand of M. □

The following theorem gives a decomposition of any e-lifting module.

Theorem 2.18. *Let M be a e-lifting module. Then $M = M_1 \oplus M_2 \oplus M_3$, where*

(1) *M_1 is semisimple.*
(2) *M_2 is e-lifting with $Rad(M_2)$ e-small and essential in M_2.*
(3) *M_2 is e-lifting with $Rad(M_3) = M_3$.*

Proof. Let M be a e-lifting module, then by [8, Proposition 3], we have a decomposition $M = M_1 \oplus A$ where M_1 is semisimple and $Rad_e(A) \leq_e A$. Also by [8, Lmma 3], A is e-lifting. Hence $A = M_2 \oplus M_3$, $M_3 \subseteq Rad_e(A)$ and $Rad_e(A) \cap M_2 \ll_e M_2$. But $M_2 \cap Rad_e(A) = M_2 \cap (Rad_e(M_2) \oplus Rad_e(M_3)) = Rad_e(M_2)$. So $Rad_e(M_2) \ll_e M_2$. Now, since $Rad_e(A) = Rad_e(M_2) \oplus Rad_e(M_3) \leq_e M_2 \oplus M_3$, then by [1, Proposition 5.20], $Rad_e(M_2) \leq_e M_2$. Also $M = M_1 \oplus A = M_1 \oplus M_2 \oplus M_3$, then M_3 is a direct summand of M and e-lifting. But $M_3 \subseteq Rad_e(A)$, therefore $M_3 = M_3 \cap Rad_e(A) = M_3 \cap (Rad_e(M_2) \oplus Rad_e(M_3)) = M_3 \cap Rad_e(M_3) = Rad_e(M_3)$. □

3. FI-e-lifting modules

Recall that a submodule K of M is called *fully invariant* if $\varphi(K) \subseteq K$ for all $\varphi \in End_R(M)$. In this section, we shall introduce a new generalization of FI-lifting modules. We call a module M is FI-e-lifting, if for fully invariant submodule N of M, there exsits a direct summand D of M, such that $N/D \ll_e M/D$.

The following Lemma contains some basic properties of fully invariant submodule which we use this section.

Lemma 3.1. *Let M be a module. Then:*

(1) *Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M (in fact the fully invariant submodules form a complete modular sublattice of the lattice of submodules of M).*

(2) *If $X \subseteq Y \subseteq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.*

(3) *If $M = \oplus_{i\in I} X_i$ and S is a fully invariant submodule of M, then $S = \oplus_{i\in I}\pi_i(S) = \oplus_{i\in I}(X_i \cap S)$, where π_i is the i-th projection homomorphism of M.*

(4) *If $X \leq Y \leq M$ such that X is a fully invariant submodule of M and Y/X is a fully invariant submodule of M/X, then Y is a fully invariant submodule of M.*

Proof. See [2, Lemma 1.1]. □

The following Proposition introduces an equivalent condition for a FI-e-lifting module.

Proposition 3.2. *Let M be an R-module. Then the following are equivalent:*

(1) *M is FI-e-lifting;*

(2) *Every fully invariant submodule A of M can be written as $A = B \oplus S$, where B is a direct summand of M and $S \ll_e M$;*

Proof. (1) $\Longrightarrow$ (2) Let A be a fully invariant submodule of M. Since M is FI-e-lifting, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ e-small in M_2. Therefore $A = M_1 \oplus (A \cap M_2)$.

(2) $\Longrightarrow$ (1) Assume that every fully invariant submodule has the stated decomposition. Let A be a fully invariant submodule of M. By hypothesis, there exists a direct summand N of M and a e-small submodule S of M such that $A = N \oplus S$. Now let $M = N \oplus N'$ for some submodule N' of M. Consider the natural epimorphism $\pi : M \longrightarrow M/N$. Then $\pi(S) = (S + N)/N = A/N \ll_e M/N$. Therefore, M is FI-e-lifting. □

Clearly, every e-lifting module is FI-e-lifting. It follows that every lifting module is FI-e-lifting. Also, every FI-lifting module is FI-δ-lifting and FI-e-lifting. So, by definitions, we have the following diagram:

$$\begin{array}{ccccc} lifting & \Rightarrow & \delta - lifting & \Rightarrow & e - lifting \\ \Downarrow & & \Downarrow & & \Downarrow \\ FI - lifting & \Rightarrow & FI - \delta - lifting & \Rightarrow & FI - e - lifting \end{array}$$

Example 3.3. The module $\mathbb{Z}$ as a $\mathbb{Z}$-module is not FI-δ-lifting, and since $\mathbb{Z}$ is nonsingular, then by Proposition 2.4, $\mathbb{Z}$ as a $\mathbb{Z}$-module is not FI-e-lifting.

The next example shows that every FI-e-lifting module is not e-lifting.

Example 3.4. The only fully invariant submodules of $\mathbb{Z}$-module $\mathbb{Q}$ are 0 and $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is FI-e-lifting. But it is not δ-lifting and since $\mathbb{Z}$-module $\mathbb{Q}$ is nonsingular, so $\mathbb{Z}$-module $\mathbb{Q}$ is not e-lifting.

The next result characterizes indecomposable FI-e-lifting modules.

Proposition 3.5. *Let M be an indecomposable module. Then the following conditions are equivalent:*

(1) *M is FI-e-lifting;*
(2) *Every proper fully invariant submodule of M is e-small in M;*

Proof. (i) $\Rightarrow$ (ii) Let A be a proper fully invariant submodule of M. By assumption, there exists a direct summand K of M such that A/K is e-small in M/K. Since M is indecomposable, we have $K = 0$. Hence A is e-small in M.

(ii) $\Rightarrow$ (i) It is clear. □

Corollary 3.6. *Let M be an indecomposable R-module. If M is FI-e-lifting, then for every fully invariant submodule A of M, $Rad_e(A) \ll_e M$.*

Proof. Let A be a fully invariant submodule of M. Since $Rad_e(A)$ is a fully invariant submodule of A, then $Rad_e(A)$ is a fully invariant submodule of M, by Proposition 3.5, $Rad_e(A) \ll_e M$. □

Proposition 3.7. *Let M be a FI-e-lifting module and let N be a fully invariant direct summand of M. Then N is a FI-e-lifting module.*

Proof. Let N' be a submodule of M such that $M = N \oplus N'$. Let A be a fully invariant submodule of N. Then A is a full invariant submodule of M since N is fully invariant in M. As M is FI-e-lifting, there exists a direct summand B of M such that $A/B \ll_e M/B$. It is easily seen that $M/B = N/B \oplus ((N' + B)/B)$. Therefore $A/B \ll_e N/B$ by Lemma 2.1. Note that B is a direct summand of N. It follows that N is a FI-e-lifting module. □

Proposition 3.8. *Let M be an R-module with $Rad_e(M) = 0$. Then M is FI-e-lifting if and only if every fully invariant submodule of M is a direct summand of M.*

Proof. Suppose that M is FI-e-lifting and let A be a fully invariant submodule of M. Then by Proposition, $A = X \oplus S$, where X is a direct summand of M and $S \ll_e M$. But $Rad_e(M) = 0$, therefore $S = 0$. Thus $A = X$ and hence A is a direct summand of M. The converse is true. □

We show every finite direct sum of FI-e-lifting modules is FI-e-lifting module.

Theorem 3.9. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of FI-e-lifting modules. Then M is FI-e-lifting.*

Proof. Let N be a fully invariant submodule of M. Then $N = \bigoplus_{i=1}^{n}(N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i. Since each M_i is FI-e-lifting, by Proposition 3.2, $N \cap M_i = L_i \oplus S_i$ where L_i is a direct summand of M_i and $S_i \ll_e M_i$. Set $L = \bigoplus_{i=1}^{n} L_i$ and $S = \bigoplus_{i=1}^{n} S_i$. Then $N = L \oplus S$ where L is a direct summand of M and $S \ll_e M$. □

Corollary 3.10. *If M is a finite direct sum of lifting modules, then M is FI-e-lifting.*

Example 3.11. (1) Let K be the quotient field of a discrete valuation domain R which is not complete. Set $M = K \oplus K$. We know that K is a hollow module. Therefore M is FI-e-lifting by Corollary 3.10. On the other hand, M is not lifting by [3, Example 23.7].

(2) Let p be any prime integer and consider the $\mathbb{Z}$-module $M = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$. Since any hollow module is lifting and so FI-e-lifting, Corollary 3.10 implies M is FI-e-lifting. But M is not e-lifting by [8, Example 1]. And so M is not a lifting module ([3, Example 23.5]).

Proposition 3.12. *Let R be a ring and M be FI-e-lifting. Then every fully invariant submodule of the module $M/Rad_e(M)$ is a direct summand.*

Proof. Let $N/Rad_e(M)$ be a fully invariant submodule of $M/Rad_e(M)$. Then N is fully invariant submodule of M by Lemma 3.1. By hypothesis, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll_e M_2$. Since $M_2 \cap N$ is also e-small in M, $N \cap M_2 \leq Rad_e(M)$. Thus $M/Rad_e(M) = (N/Rad_e(M)) \oplus ((M_2 + Rad_e(M))/Rad_e(M))$. □

Let M be an R-module. We say, a pair (P, p) is a projective e-cover of M in case P is a projective R-module and $P \xrightarrow{p} M \to 0$ is epimorphism and $Kerp \ll_e P$.

Theorem 3.13. *Let P be a projective module. Then P is FI-e-lifting if and only if P/A has a projective e-cover for every fully invariant submodule A of P.*

Proof. Suppose P is a projective FI-e-lifting module and A is a fully invariant submodule of P. Then $A = X \oplus S$ where X is a direct summand of P and $S \ll_e P$. Suppose $P = X \oplus Y$. As $S \ll_e P$, $(X+S)/X \ll_e P/X$. Hence the natural map $f : P/X \to P/(X+S) = P/A$ is a projective e-cover. Conversely, suppose P/A has a projective e-cover for every fully invariant submodule A of P. Let $f : Q \to P/A$ be a projective e-cover of P/A. Then there exists a map $h : P \to Q$ such that $fh = \varphi$ where $\varphi : P \to P/A$ is the natural map. As $Kerf \ll_e Q$ and φ is an epimorphism, h is an epimorphism and hence h splits. Suppose $P = Kerh \oplus B$. Then $A = Kerh \oplus (A \cap B)$ and $A \cap B \ll_e P$. Thus P is FI-e-lifting. □

Corollary 3.14. *Let R be a ring. Then R_R is FI-e-lifting if and only if R/I has a projective e-cover for every two sided ideal I of R.*

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