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CHARACTERIZATION OF $\hat{\phi}$-AMENABILITY AND $\hat{\phi}$-MODULE AMENABILITY OF SEMIGROUP ALGEBRAS

S. GRAILOO TANHA *

Abstract. For every inverse semigroup S with subsemigroup E of idempotents, necessary and sufficient conditions are obtained for the semigroup algebra $l^1(S)$ to be $\hat{\phi}$-amenable and $\hat{\phi}$-module amenable. Also, we characterize the character amenability of semigroup algebra $l^1(S)$.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra, $\Delta(\mathcal{A})$ be the chracter space of $\mathcal{A}$ and $\phi \in \Delta(\mathcal{A})$. Kaniuth and Lau and Pym [7] have recently introduced and studied the interesting notion of ϕ-amenability. Specifically a Banach algebra $\mathcal{A}$ is called left ϕ-amenable if all continuous derivation from $\mathcal{A}$ into dual Banach $\mathcal{A}$-module X for which the left module action is given by

$$a \cdot x = \phi(a)x \ (a \in \mathcal{A}, x \in X),$$

to be inner. Right ϕ-amenability is defined similarly by considering dual Banach $\mathcal{A}$-module X for which the right module action is given by

$$x \cdot x = \phi(a)x \ (a \in \mathcal{A}, x \in X),$$

and $\mathcal{A}$ is called ϕ-amenable if it is both left and right ϕ-amenable.

More recently, Monfared [9] has introduced and studied the notion of character amenability. Throughout, a Banach algebra $\mathcal{A}$ is called

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*Corresponding author .

character amenable if it has a bounded approximate identity and it is ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$. Just as for amenability, there are many characterization of ϕ-amenability of Banach algebras. For example in [6], the authors characterized ϕ-amenability of Banach algebras in terms of the existence of bounded approximate ϕ-diagonals and ϕ-virtual diagonals. It is proved in [9] that character amenability of $L^1(G)$ is equivalent to the amenability of the underlying group G. The character amenability and ϕ-amenability of some inverse semigroup algebras investigated in [4]. They characterized character amenability of $l^1(S)$, for Brandt semigroup S.

M. Amini[1] introduced the notion of module amenability for a class of Banach algebras which could be considered as a generalization of the Johnson's amenability. In particular for an inverse semigroup S with the set of idempotent E, he showed that $l^1(S)$ is module amenable, as a Banach $l^1(E)$-module, if and only if S is amenable. In this case, $l^1(S)$ is considered as a $l^1(E)$-module with actions:$\alpha \cdot a = \hat{\phi}_S(\alpha)a, a \cdot \alpha = a * \alpha$, which ϕ_S is augmentation character on S and $*$ is natural multiplication of $l^1(S)$.

In this paper, we characterize ϕ-amenability and character amenability of the semigroup algebra $l^1(S)$, where S is an inverse semigroup. Also, we consider $l^1(S)$ as a $l^1(E)$-module with actions:$\alpha \cdot a = \hat{\phi}(\alpha)a, a \cdot \alpha = a * \alpha$, which ϕ is a character on S and $*$ is natural multiplication of $l^1(S)$ and we show that how module amenability of $l^1(S)$ affects the structure of S.

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \ (ab) \cdot \alpha = a(b \cdot \alpha)$$

for all $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$. Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \ (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a)$$

for all $a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X$ and similarly for the right and two-sided actions. Then, we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}$ and $x \in X$, then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Indeed, if $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A $\mathfrak{A}$-module derivation is a bounded $\mathfrak{A}$-module map $D : \mathcal{A} \longrightarrow X$ satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha,$$

for each $a, b \in \mathcal{A}$.

One should note that D is not necessarily linear, but its boundedness (defined as the existence of $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for all $a \in \mathcal{A}$) still implies its continuity, as it preserves subtraction. When X is commutative, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

These are called *inner* $\mathfrak{A}$-module derivations. The Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each $\mathfrak{A}$-module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner [1]. Note that if $\mathfrak{A} = \mathbb{C}$, then the module amenability will absolutely overlap with Johnson's amenability [8] for a Banach algebra.

Consider the closed ideal J of $\mathcal{A}$ generated by elements of the form $(a \cdot \alpha)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Then, J is an $\mathcal{A}$-submodule and an $\mathfrak{A}$-submodule of $\mathcal{A}$. Also, $\mathcal{A}/J$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with the compatible actions when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically.

An inverse semigroup is a semigroup S so that, for each $s \in S$, there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. The element s^* is termed the inverse of s. The set $E(S)$ (or briefly, E) of idempotents of S is a commutative subsemigroup; it is ordered by $e \leq f$ if and only if $ef = e$. With this ordering $E(S)$ is a meet semilattice with the meet given by the product; see [5, Theorem 5.1.1]. We recall that a semigroup S is a *semilattice* if S is commutative and $E = S$. The order on E extends to S as the so-called natural partial order by putting $s \leq t$ if $s = et$ for some idempotent e (or equivalently $s = tf$ for some idempotent f). This is equivalent to $s = ts^*s$ or $s = ss^*t$. If $e \in E$, then the set $\mathcal{G}_e = \{s \in S | ss^* = e = s^*s\}$ is a group, called the *maximal subgroup* of S at e.

Let S be a (discrete) inverse semigroup with the set of idempotents E. We recall that the subsemigroup E of S is a semilattice, and so $l^1(E)$ could be regarded as a commutative subalgebra of $l^1(S)$. Thus, $l^1(S)$ is a Banach algebra and a Banach $l^1(E)$-module with compatible actions [1].

A semi-chracter on S is a nonzero homomorphism $\phi : S \to \bar{\mathbb{D}}$. The space of semi-character on S is denoted by Φ_S. The semi-character $\phi_S : S \to \bar{\mathbb{D}}$, defined by

$$\phi_S(t) = 1 \ (t \in S),$$

is called the augmentation character on S. For each $\phi \in \Phi_S$, we associate the map $\hat{\phi} : l^1(S) \to \mathbb{C}$ defined by

$$\hat{\phi}(f) = \Sigma_{s \in S} \phi(s) f(s) \ (f \in l^1(S)).$$

It is easily verified that $\hat{\phi} \in \Delta(l^1(S))$ and every character on $l^1(S)$ arises in this way. Indeed, we have

$$\Delta(l^1(S)) = \{\hat{\phi} : \phi \in \Phi_S\}.$$

Let $\phi \in \Phi_S$. We consider the following actions of $l^1(E)$ on $l^1(S)$:

$$\delta_e \cdot \delta_s = \hat{\phi}(\delta_e)\delta_s, \ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \qquad (s \in S, e \in E). \tag{1.1}$$

If $l^1(S)$ is $l^1(E)$-module amenable with actions (1.1), we say that $l^1(S)$ is $\hat{\phi}$-module amenable. Note that it follows from [1] that $l^1(S)$ is $\hat{\phi}_S$-module amenable if and only if S is amenable. In this case, the ideal J_ϕ is the closed linear span of $\{\delta_{set} - \hat{\phi}(\delta_e)\delta_{st} : \ s,t \in S, e \in E\}$. We consider an equivalence relation on S such that $s \sim_\phi t$ if and only if $\delta_s - \delta_t \in J_\phi$ for $s,t \in S$. It is shown in [2] that the quotient $S/{\sim_{\phi_S}}$ is a discrete group (see also [2]). Indeed, $S/{\sim_{\phi_S}}$ is homomorphic to the maximal group homomorphic image $\mathcal{G}_S$ of S. Moreover, S is amenable if and only if $\mathcal{G}_S = S/{\sim_{\phi_S}}$ is amenable ([3]).
Next proposition is a generalization of theorem 3.1 of [1].

Proposition 1.1. *$l^1(S)$ is $\hat{\phi}$-module amenable if and only if $S\backslash ker\{\phi\}$ is amenable.*

Proof. We firstly suppose that $l^1(S)$ is $\hat{\phi}$-module amenable. Consider $l^1(E_{S\backslash ker\{\phi\}})$ acts on $l^1(S_{S\backslash ker\{\phi\}})$ with the following module actions:

$$\delta_e \cdot \delta_s = \delta_s = \hat{\phi}(\delta_e)\delta_s, \ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \ (s \in S\backslash ker\{\phi\}, e \in E_{S\backslash ker\{\phi\}}).$$

Suppose that X is a commutative Banach $l^1(S\backslash ker\{\phi\})$- $l^1(E_{S\backslash ker\{\phi\}})$-module. Consider $l^1(E)$ acts on $l^1(S)$ as 2.1. Then X is a commutative $l^1(S)$-$l^1(E)$-module such that for each $f \in l^1(ker\{\phi\}), \alpha \in l^1(E_{ker\{\phi\}})$, we have:

$$f \cdot x = x \cdot f = 0, \alpha \cdot x = x \cdot \alpha = 0.$$

If $D : l^1(S\backslash ker\{\phi\}) \to X^*$ is a module derivation, then

$$\tilde{D} = D(f|_{l^1(S\backslash ker\{\phi\})}) : l^1(S) \to X^*$$

is well-defined. For each $f \in l^1(S\backslash ker\{\phi\}), g \in l^1(ker\{\phi\})$ we have $\tilde{D}(fg) = 0$. On the other hand, since $g \in l^1(ker\{\phi\})$, $g \cdot x = 0$. Thus $\tilde{D}(f) \cdot g = 0$. By definition of $\tilde{D}$, $\tilde{D}(g) = 0$ and so $\tilde{D}(f) \cdot g + f \cdot \tilde{D}(g) = 0 = \tilde{D}(fg)$. Hence from the fact D is a module derivation and $\tilde{D}|_{l^1(ker\{\phi\})} = 0$, we conclude that $\tilde{D}(fg) = \tilde{D}(f) \cdot g + f \cdot \tilde{D}(g)$. Now for $f \in l^1(S), \alpha \in l^1(E_{ker\{\phi\}})$ we have $f \cdot \alpha \in l^1(ker\{\phi\})$ and

$$\tilde{D}(f \cdot \alpha) = \tilde{D}(f\alpha) = 0 = \tilde{D}(f) \cdot \alpha.$$

It is easy to see that

$$\tilde{D}(\alpha \cdot f) = \tilde{D}(\hat{\phi}(\alpha)f) = 0 = \alpha \cdot \tilde{D}(f).$$

Also if $\alpha \in l^1(E_{S\backslash ker\{\phi\}})$, we have

$$\tilde{D}(\alpha \cdot f) = D(\alpha \cdot f|_{S\backslash ker\{\phi\}}) = \alpha \cdot D(f|_{S\backslash ker\{\phi\}}) = \alpha \cdot \tilde{D}(f).$$

Similarly $\tilde{D}(f \cdot \alpha) = \tilde{D}(f) \cdot \alpha$ and $\tilde{D}$ is a module derivation. By assumption, $\tilde{D}$ is inner and so D is inner. Thus $l^1(S\backslash ker\{\phi\})$ is $\phi_{S\backslash ker\{\phi\}}$-module amenable and by theorem 3.1 of [1] $S\backslash ker\{\phi\}$ is amenable. Conversely, suppose that $S\backslash ker\{\phi\}$ is amenable. If μ is a right invariant mean on $S\backslash ker\{\phi\}$ and M is defined on $l^\infty(S \times S)$ by

$$M(f) = \int_{S\backslash ker\{\phi\}} f(t^*, t) d\mu(t).$$

Clearly, M is a bounded linear functional such that $M(1 \otimes 1) = \mu(1) = 1$. Also for each $s \in S$ and $f \in l^\infty(S \times S)$ we have

$$\begin{aligned} s \cdot M(f) = M(f.s) &= \int_{S\backslash ker\{\phi\}} f(st^*, t) d\mu(t) \\ &= \int_{S\backslash ker\{\phi\}} f(ss^*t^*, ts) d\mu(t) \\ &= \int_{S\backslash ker\{\phi\}} f((tss^*)^*, (tss^*)^* s) d\mu(t) \\ &= \int_{S\backslash ker\{\phi\}} f(t^*, ts) d\mu(t) \\ &= M(s \cdot f) = M \cdot s(f). \end{aligned}$$

Also, for each $s \in S$ and $f \in J^\perp$ we have

$$\begin{aligned} w^{**}(M) \cdot s(f) = w^{**}(M) \cdot (f \cdot s) &= M(w^*(f \cdot s)) \\ &= \int_{S\backslash ker\{\phi\}} w^*(f \cdot s)(t^*, t) d\mu(t) \\ &= \int_{S\backslash ker\{\phi\}} f \cdot s(t^*t) d\mu(t) \\ &= \int_{S\backslash ker\{\phi\}} f(st^*t) d\mu(t) \\ &= f(s) \int_{S\backslash ker\{\phi\}} d\mu(t) = f(s). \end{aligned}$$

□

2. ϕ-AMENABILITY OF SEMIGROUP ALGEBRAS

Any statement about left ϕ-amenability and left character amenability turns in to an analogous statement about right ϕ-amenability and right character amenability.

Theorem 2.1. *If $l^1(S)$ is $\hat{\phi}$-amenable, then $S\backslash ker\{\phi\}$ is amenable and $ker\{\phi\}$ satisfies condition D_k for some k.*

Proof. Let $T = S \cup \{1\}$. It follows from lemma 3.2 of [7] that $l^1(T)$ is $\hat{\phi_1}$-amenable which ϕ_1 is the unique extension of ϕ to an element of $\Phi(T)$. By corollary 2.7 of [9], $ker\hat{\phi_1}$ has a bounded approximate identity. Put $\psi : ker\hat{\phi_1} \cup \delta_1 \to \mathbb{C}$ defined by

$$\psi(\delta_1) = 1, \psi(f) = 0 \ (f \in ker\hat{\phi_1}).$$

By corollary 2.7 of [9], $ker\hat{\phi_1} \cup \delta_1$ is ψ-amenable. Since $ker\hat{\phi_1} \oplus_1 \mathbb{C}\delta_1$ is l^1-direct sum of $l^1(ker\phi) \oplus_1 \mathbb{C}\delta_1$ and E which $E = \{f \in l^1(T\backslash ker\phi) : \Sigma_{t\in T\backslash ker\phi} f(t) = 0\}$. Now from proposition 3.1 of [4], we have $l^1(ker\phi) \oplus_1 \mathbb{C}\delta_1$ is $\psi|_{l^1(ker\phi)\oplus_1\mathbb{C}\delta_1}$-amenable and by corollary 2.7 of [9], $l^1(ker\phi)$ has a bounded approximate identity and so $ker\phi$ satisfies condition D_k for some k. Now by 1.1, it is suffices to show that $\hat{\phi}$-amenablity of $l^1(S)$ implies $\hat{\phi}|_{S\backslash ker\{\phi\}}$-amenablity of $l^1(S\backslash ker\{\phi\})$ and amenability of $S\backslash ker\{\phi\}$ follows from Theorem 3.1 of [7].
Let X be a $l^1(S\backslash ker\{\phi\})$-module and $D : l^1(S\backslash ker\{\phi\}) \to X^*$ be a derivation. Clearly X is a $l^1(S)$-module with the actions:

$$f \cdot x = \hat{\phi}(f)x, \ x \cdot f = x \cdot f|_{l^1(S\backslash ker\{\phi\})}, \ (f \in l^1(S), x \in X).$$

Consider $\hat{D} : l^1(S) \to X^*$ defined by $\hat{D}(f) = D(f|_{l^1(S\backslash ker\{\phi\})})$ for all $f \in l^1(S)$. It is clear that $\hat{D}$ is a derivation and by assumption is inner. Hence $l^1(S\backslash ker\{\phi\})$ is $\hat{\phi}|_{S\backslash ker\{\phi\}}$-amenable and so $S\backslash ker\{\phi\}$ is amenable. □

Corollary 2.2. *Let S be an inverse semigroup. Then $l^1(S)$ is $\hat{\phi}$-amenable for each $\phi \in \Phi(S)$ if and only if I satisfies condition D_k for some k and $S\backslash I$ is amenable, for each ideal I of S such that $S\backslash I$ is a subsemigroup of S.*

Proof. It follows from above theorem and theorem 2.6 of [9]. □

Now next corollary chracterize character amenability of $l^1(S)$ based on the structure of S.

Corollary 2.3. *Let S be an inverse semigroup such that satisfies condition D_k for some k. Then $l^1(S)$ is character amenable if and only* *if I satisfies condition D_k for some k and $S\backslash I$ is amenable, for each ideal I of S such that $S\backslash I$ is a subsemigroup of S.*

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Somaye Grailoo Tanah
Esfarayen University of Technology, Esfarayen, North Khorasan, Iran
Email: grailotanha@gmail.com