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SEMI n-ABSORBING IDEALS IN THE SEMIRING $\mathbb{Z}_0^+$

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ABSTRACT. In this paper, all principal (m, n)-closed ideals and principal semi n-absorbing ideals in the semiring of non-negative integers are investigated.

1. INTRODUCTION

The concept of 2-absorbing ideals in a commutative ring R with $1 \neq 0$ was introduced by Ayman Badawi [2] and extended to n-absorbing ideals in R by Anderson and Badawi [3]. Chaudhari [4] introduced the concept of 2-absorbing ideals in commutative semiring R with $1 \neq 0$, which is a generalization of prime ideals in R. All 2-absorbing ideals in the semiring of non-negative integers are investigated by Chaudhari [5]. Chaudhari and Ingale [8] have introduced the notion of n-absorbing ideals in commutative semiring R with $1 \neq 0$ and investigated all n-absorbing ideals in the semiring $(\mathbb{Z}_0^+, gcd, lcm)$ and all n-absorbing principal ideals in the semiring of non-negative integers. Several other authors used these concepts and some other relative concepts which are generalizations of prime ideals. Anderson and Ayman Badawi [1] introduced the concept of semi-n-absorbing ideal and (m, n)-closed ideal in a commutative ring R with $1 \neq 0$ which are generalizations of n-absorbing ideals in R. Chaudhari and Ingale [7] have characterized prime ideals, semi prime ideals, irreducible k-ideals and irreducible principal T-ideals in the ternary semiring of non-positive integers. For

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the definition of semiring we refer [9]. We assume throughout that all semirings are commutative with $1 \neq 0$.

Denote the sets of all non-negative integers, positive integers and non-negative real numbers respectively by $\mathbb{Z}_0^+$, $\mathbb{N}$ and $\mathbb{R}_0^+$. Then under usual addition and multiplication of nonnegative integers, $\mathbb{Z}_0^+$ forms a commutative semiring with identity 1 but it is not a ring.

In this paper we introduce the concept of (m, n)-closed ideal and semi n-absorbing ideal in commutative semiring R with $1 \neq 0$ and study some generalizations of n-absorbing ideals in the semiring $\mathbb{Z}_0^+$.

Throughout this paper we use the following notations:
$a \mid b$ ($a \nmid b$): a divides b (a does not divide b) where $a, b \in \mathbb{Z}_0^+$.
$\langle a \rangle$: the principal ideal generated by a where $a \in \mathbb{Z}_0^+$.
$\langle m_1, m_2, \cdots, m_k \rangle$: the ideal generated by $m_1, m_2, \cdots, m_k$ in $\mathbb{Z}_0^+$, where $m_1 < m_2 < \cdots < m_k$ and $m_i \nmid m_j$ for all $i < j$.
$(m_1, m_2, \cdots, m_k)$: the gcd of $m_1, m_2, \cdots, m_k$ in $\mathbb{Z}_0^+$, where $m_1 < m_2 < \cdots < m_k$.
$[x]_l$: the largest integer $\leq x$, where $x \in \mathbb{R}_0^+$.
$[x]_s$: the smallest integer $\geq x$, where $x \in \mathbb{R}_0^+$.
$a_1 a_2 \cdots \widehat{a_i} \cdots a_n$: the term a_i is excluded from the product $a_1 a_2 \cdots a_i \cdots a_n$

If $a \in \mathbb{Z}_0^+$ and $a \geq 2$, then $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime power factorization (ppf) of a where $p_1, p_2, \cdots, p_k$ are pair wise distinct primes, $r_i \geq 1$, $k \geq 1$ and $p_1 < p_2 < \cdots < p_k$.

Definition 1.1. A proper ideal I of a semiring R is called semi-n-absorbing ideal of R, if $x^{n+1} \in I$ implies $x^n \in I$, where $n \in \mathbb{N}, x \in R$.

Clearly an n-absorbing ideal of a semiring R is a semi-n-absorbing ideal of R and a semi-1-absorbing ideal of R is just a semi prime ideal of R. The following example shows that the converse is not true.

Example 1.2. Let $I = 18\mathbb{Z}_0^+ = \langle 2 \cdot 3^2 \rangle$. Then I is a semi-2-absorbing ideal of $\mathbb{Z}_0^+$ but not a 2-absorbing ideal of $\mathbb{Z}_0^+$ as $2 \times 3 \times 3 = 18 \in I$ and $2 \times 3 = 6 \notin I$, $3 \times 3 = 9 \notin I$.

Example 1.3. Let $I = 4\mathbb{Z}_0^+ = \langle 4 \rangle$. Then I is a semi-2-absorbing ideal of $\mathbb{Z}_0^+$ but not a semiprime ideal of $\mathbb{Z}_0^+$ as $2^2 = 4 \in I$ and $2 \notin I$.

Clearly an n-absorbing ideal of a semiring R is also an $(n+1)$-absorbing ideal of R but this may not be true for semi n-absorbing ideals of R.

Example 1.4. Let $I = 16\mathbb{Z}_0^+ = \langle 16 \rangle$. Then I is a semi-2-absorbing ideal of $\mathbb{Z}_0^+$ but it is not a semi 3-absorbing ideal of $\mathbb{Z}_0^+$ as $2^4 = 16 \in I$ and $2^3 = 8 \notin I$.

Now the following theorem gives a characterization of non-zero principal semi n-absrobing ideals of the type $\langle p^k\rangle$ where p is a prime number and $k \in \mathbb{N}$, in the semirng $\mathbb{Z}_0^+$.

Theorem 1.5. *Let $I = \langle p^k\rangle$ where p is a prime number and $k \in \mathbb{N}$. Then I is a semi-n-absorbing ideal of $\mathbb{Z}_0^+$ if and only if $k = (n+1)a+r$ where a, r are integers such that $a \geq 0, 1 \leq r \leq n$ and $a + r \leq n$.*

Proof. Let $I = \langle p^k\rangle$ be a semi-n-absorbing ideal of $\mathbb{Z}_0^+$, where p is a prime number and $k \in \mathbb{N}$. By applying division algorithm to k and $n + 1$ there exist integers a and r such that $a \geq 0$, $0 \leq r \leq n$ and $k = (n + 1)a + r$. If $r = 0$, then $k = (n + 1)a$. Therefore $a > 0$ as $k > 0$ and $n + 1 > 0$. Therefore $(p^a)^{n+1} = (p^{n+1})^a = p^{(n+1)a} = p^k \in I$ and hence $(p^a)^n = p^{na} \in I$, since I is semi-n-absorbing ideal. It is a contradiction as $n < n + 1 \Rightarrow na < (n + 1)a = k$. Therefore $r \neq 0$. Hence $1 \leq r \leq n$ and $n + 1 < k$ as $k = (n + 1)a + r$. Choose the smallest positive integer d such that $p^{d(n+1)} \in I$. Now $(n+1)(a+1) = (n+1)a+(n+1) = k-r+n+1 = k+n+1-r > k$ as $r \leq n < n+1$. So choose $d = a+1$. Now $d = a+1$ is the smallest positive integer such that $p^{d(n+1)} \in I$. That is $(p^{a+1})^{n+1} \in I$. Now $p^{(a+1)n} = (p^{a+1})^n \in I$, since I is a semi-n-absorbing ideal. Therefore $(a + 1)n = na + n \geq k = (n + 1)a + r$ and hence $na + n \geq na + a + r$. Therefore, $n \geq a + r$. Thus $k = (n + 1)a + r$ where a, r are integers such that $a \geq 0$, $1 \leq r \leq n$ and $a + r \leq n$.

Conversely, suppose that $k = (n+1)a+r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$ and $a + r \leq n$. To show that $I = \langle p^k\rangle$ is a semi-n-absorbing ideal of $\mathbb{Z}_0^+$. Let $x^{n+1} \in I$.
Case (I): $a = 0$. Then $k = r$ and hence $1 \leq k \leq n$. Now, $x^{n+1} \in I = \langle p^k\rangle \Rightarrow p \mid x$. So $p^k \mid x^k$ and hence $p^k \mid x^n$ as $k \leq n$. Thus $x^n \in \langle p^k\rangle = I$.
Case (II): $a \neq 0$. Then $a > 0$. Now we have $p^k|x^{n+1}$ as $x^{n+1} \in I = \langle p^k\rangle$. If $p^k|x$, then $p^k|x^n$ and hence $x^n \in I$. Assume that $p^k \nmid x$. Choose the largest positive integer i such that $p^i|x, 1 \leq i < k$. Then $(n + 1)i$ is the largest positive integer such that $p^{(n+1)i}|x^{n+1}$. Now $x^{n+1} \in I = \langle p^k\rangle \Rightarrow n + 1 \geq k$. Therefore $(n + 1)i \geq n + 1 \geq k$. This implies $0 \geq k - (n + 1)i = (n + 1)a + r - (n + 1)i = (n + 1)(a - i) + r$. Therefore $i > a$, since $1 \leq r \leq n$. Thus $i = a + b$ for some $b \geq 1$. Then $k = (n + 1)a + r$ gives $\frac{k}{n} = \frac{(n+1)a+r}{n} = \frac{na+a+r}{n} = a + \frac{a+r}{n} \leq a + 1$ as $a + r \leq n$. Since $b \geq 1$, we have $i = a + b \geq a + 1 \geq \frac{k}{n}$. Therefore $ni \geq k$. Thus $p^{ni}|x^n$ as $p^i|x$ and hence $p^k|x^n$ as $ni \geq k$. Thus $x^n \in I$ and hence I is a semi-n-absorbing ideal. ∎

Definition 1.6. Let $m, n \in \mathbb{N}$. A proper ideal I of a semiring R is called an (m,n)-closed ideal of R if $x^m \in I$ where $x \in R$ implies that $x^n \in I$.

Thus an ideal I of a semiring R is a semi n-absorbing ideal of R if and only if it is a $(n+1,n)$-closed ideal of R and I is a semiprime ideal of R if and only if it is a $(2,1)$-closed ideal of R. Clearly, every proper ideal of R is an (m,n)-closed ideal for $1 \leq m \leq n$. Thus we generally assume that $1 \leq n < m$. Clearly if I is an n-absorbing ideal of R, then it is (m,n)-closed for every $m \in \mathbb{N}$.

Now the following theorem gives a characterization of non-zero principal (m,n)-closed ideals of the type $\langle p^k \rangle$ where p is a prime number and $k \in \mathbb{N}$, in the semiring $\mathbb{Z}_0^+$.

Theorem 1.7. *Let $I = \langle p^k \rangle$ be an ideal in $\mathbb{Z}_0^+$, where p is a prime number and $k \in \mathbb{N}$. Let $1 \leq n < m$. Then I is an (m,n)-closed ideal if and only if $k = ma + r$, where $a, r \in \mathbb{Z}_0^+, 1 \leq r \leq n$ and $ac + r \leq n$, where $c \equiv m(mod n)$. Further if $a \neq 0$, then $m = n + c$ where $1 \leq c \leq n-1$.*

Proof. Let $I = \langle p^k \rangle$ be an (m,n)-closed ideal, $k \in \mathbb{N}$ and p is a prime number. By division algorithm, $k = ma + r, a \in \mathbb{Z}_0^+$ and $0 \leq r < m$.

If $r = 0, a > 0$ as $k \in \mathbb{N}$. Now $(p^a)^m = p^{ma} = p^k \in I$ implies $(p^a)^n \in I$, since I is an (m,n)-closed ideal. Therefore, $p^{na} \in I = \langle p^k \rangle$ implies $na \geq k$ a contradiction as $n < m \Rightarrow na < ma = k$. Therefore, $r \neq 0$ and hence $1 \leq r \leq m-1$. Choose the smallest positive integer d such that $(p^d)^m \in I$. Then $m(a+1) = ma + m = k - r + m > k$ as $r < m$. Also, $ma < k$ as $r > 0$. Therefore $ma < k < ma + m = m(a+1)$. Thus $d = a+1$ is the smallest positive integer such that $p^{m(a+1)} = (p^{a+1})^m \in I$ implies $(p^{a+1})^n \in I$ as I is an (m,n)-closed ideal. Therefore $n(a+1) = na + n \geq k = ma + r$. This implies $n \geq ma + r - na = a(m-n) + r \geq r$ as $a(m-n) \geq 0$. Thus $1 \leq r \leq n$. Now, since $n < m$, by division algorithm, we have $m = bn + c$ where $b \geq 1, 0 \leq c \leq n-1$. Therefore $n \geq a(bn+c-n)+r = a(b-1)n+ac+r$ where $ac + r \geq 1$. Since $n \geq a(b-1)n + ac + r$ and $ac + r \geq 1$, we have $a(b-1) = 0$. For if $a(b-1) \neq 0$, $b > 1$, then $ac + r \geq 1$ implies $a(b-1)n + ac + r \geq a(b-1)n + 1$ and this implies $n \geq a(b-1)n + 1$ which is not true. Thus $a(b-1) = 0$ and hence $n \geq ac + r$ where $c \equiv m(mod n)$. Now if $a \neq 0$, $b - 1 = 0$. i.e. $b = 1$. Thus $m = n + c$ where $a \leq c \leq n-1$ as $n < m$.

Conversely, assume that $k = ma+r, a \in \mathbb{Z}_0^+, 1 \leq r \leq n$ and $ac+r \leq n$ where $c \equiv m(mod n)$. Also, assume that if $a \neq 0$, then $m = n+c$ where $1 \leq c \leq n-1$. To show that I is an (m,n)-closed ideal. Let $x^m \in I$

Case (I): $a = 0$. Then $k = r, 1 \leq r \leq n$. Therefore $1 \leq k \leq n$. Now, $x^m \in I = \langle p^k \rangle \Rightarrow p|x$ as p is a prime number. Therefore $p^k|x^k$. This implies $p^k|x^n$ as $k \leq n$. Therefore $x^n \in I$.
Case (II): $a \neq 0$. We have $x^m \in I = \langle p^k \rangle$. So that $p^k|x^m$ implies $p|x$. If $p^k|x$, then $p^k|x^n$ and hence $x^n \in I$. Assume that $p^k \nmid x$. Choose the largest positive integer i such that $p^i|x, 1 \leq i < k$. Then mi is the largest positive integer such that $p^{mi}|x^m$. Therefore $mi \geq k$ i.e. $0 \geq k - mi = ma + r - mi = m(a - i) + r$. Therefore $a < i$, thus $i = a + b$ for some integer $b \geq 1$. Now, $k = ma + r$ and $m = n + c$ gives $k = (n + c)a + r = na + ca + r$. Therefore $\frac{k}{n} = a + \frac{ca+r}{n} \leq a + 1$ as $ca + r \leq n$. Therefore $i = a + b \geq a + 1 \geq \frac{k}{n}$ as $b \geq 1$. Therefore $ni \geq k$. Now, $p^i|x \Rightarrow p^{ni}|x^n \Rightarrow p^k|x^n$ as $ni \geq k$ and hence $x^n \in I = \langle p^k \rangle$. Therefore I is an (m, n)-closed ideal of $\mathbb{Z}_0^+$. □

Theorem 1.8. *Let $I = \langle p^k \rangle$ be an ideal in $\mathbb{Z}_0^+$, where p is a prime number and $k \in \mathbb{N}$. Then following statements are equivalent:*

(1) *I is an (m, n)-closed ideal*
(2) *Exactly one of the following statemants holds*
   (i) $1 \leq k \leq n$,
   (ii) *There is a positive integer a such that $k = ma+r = na+d$ for integers r and d with $1 \leq r, d \leq n - 1$,*
   (iii) *There is a positive integer a such that $k = ma+r = n(a+1)$ for an integer r with $1 \leq r \leq n - 1$.*

Proof. (1) ⇒ (2) Suppose that I is an (m, n)-closed ideal of $\mathbb{Z}_0^+$. Then by Theorem 1.7, $k = ma+r$, where $a, r \in \mathbb{Z}_0^+, 1 \leq r \leq n$ and $ac+r \leq n$, where $c \equiv m(mod n)$. Further if $a \neq 0$, then $m = n + c$ with $1 \leq c \leq n - 1$.
Thus, if $a = 0$, then $k = r$ and thus $1 \leq k \leq n$. This proves (i).
If $a \neq 0, a > 0$ and $k = ma + r$. Also, since $c \equiv m(mod n), c \neq 0$ as $n < m$. Next $ac + r \leq n, 1 \leq r \leq n$. Now, $k = ma + r$ and $m = n + c$ $\Rightarrow k = (n + c)a + r = na + ca + r = na + d$ where $d = ac + r \leq n$. If $d < n$, then $k = ma + r = na + d$ with $1 \leq r, d \leq n - 1$. This proves (ii).
Now if $d = n$, then $k = ma+r = na+n = n(a+1)$ with $1 \leq r \leq n-1$. This proves (iii).
(2) ⇒ (1) First suppose that $1 \leq k \leq n$. Let $x^m \in \langle p^k \rangle = I$. Therefore $p^k|x^m \Rightarrow p|x \Rightarrow p^k|x^n$ as $k \leq n$. Therefore $x^n \in \langle p^k \rangle = I$, and hence I is an (m, n)-closed ideal of $\mathbb{Z}_0^+$.

Now, suppose that $a \geq 1$ such that $k = ma + r = na + d, 1 \leq r, d \leq n-1$. Then $ma = na+d-r$ or $m = n+\left(\frac{d-r}{a}\right) = n+c$, where $c = \left(\frac{d-r}{a}\right)$ is an integer with $1 \leq c \leq n - 1$. Thus, $m = n + c$ with $1 \leq c \leq n - 1$. Therefore by Theorem 1.7, I is (m, n)-closed ideal of $\mathbb{Z}_0^+$.

Finally, suppose that there is an integer $a \geq 1$ such that $k = ma+r = n(a+1)$, where $1 \leq r \leq n-1$. Now, $m = \frac{n(a+1)-r}{a} = n + \frac{n-r}{a} = n + c$ for an integer $c = \frac{n-r}{a} \leq n-1$ as $a \geq 1$ and hence by theorem 1.7, I is an (m, n)-closed ideal. □

Now we give the following lemma which will be used in the subsequent theorem.

Lemma 1.9. *Intersection of finite number of (m, n)-closed ideals in the semirng R is an (m, n)-closed ideal.*

Proof. Trivial. □

Now the following theorem gives a characterization of non-zero principal (m, n)-closed ideals in the semiring $\mathbb{Z}_0^+$.

Theorem 1.10. *Let $I = \langle p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \rangle$ be an ideal in $\mathbb{Z}_0^+$ and let $1 \leq n < m$,where $n, m \in \mathbb{N}$, $p_1, p_2, ..., p_k$ are prime numbers such that $p_1 < p_2 < \cdots < p_k$ and $r_1, r_2, ..., r_k$ are positive integers. Then the following statements are equivalent:*

(1) *I is an (m, n)-closed ideal of $\mathbb{Z}_0^+$,*
(2) *$\langle p_j^{r_j} \rangle$ is an (m, n)-closed ideal of $\mathbb{Z}_0^+$, for every $1 \leq j \leq k$.*

Proof. (1) ⇒ (2)

Suppose that I is an (m, n)-closed ideal of $\mathbb{Z}_0^+$. Let $x^m \in \langle p_j^{r_j} \rangle$ where $x \in \mathbb{Z}_0^+$. Let $y = xp_1^{r_1} p_2^{r_2} ... \widehat{p_j^{r_j}} ... p_k^{r_k}$. Then $y^m = x^m \left(p_1^{r_1} p_2^{r_2} ... \widehat{p_j^{r_j}} ... p_k^{r_k} \right)^m \in I$. Since I is an (m, n)-closed ideal, $y^n \in I$. Therefore $x^n \left(p_1^{r_1} p_2^{r_2} ... \widehat{p_j^{r_j}} ... p_k^{r_k} \right)^n \in I = \langle p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \rangle \Rightarrow p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \mid x^n \left(p_1^{r_1} p_2^{r_2} ... \widehat{p_j^{r_j}} ... p_k^{r_k} \right)^n \Rightarrow p_j^{r_j} | x^n \Rightarrow x^n \in \langle p_j^{r_j} \rangle$. Therefore $\langle p_j^{r_j} \rangle$ is an (m, n)-closed ideal, $1 \leq j \leq k$.

(2) ⇒ (1)

Now suppose that each $\langle p_j^{r_j} \rangle$ is an (m, n) closed ideal of $\mathbb{Z}_0^+$, $1 \leq j \leq k$. By Lemma 1.9, $\langle p_1^{r_1} \rangle \cap \langle p_2^{r_2} \rangle \cap ... \cap \langle p_k^{r_k} \rangle$ is an (m, n)-closed ideal and hence $I = \langle p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \rangle$ is an (m, n)- closed ideal of $\mathbb{Z}_0^+$. □

Lemma 1.11. *Let I be a semi-n-absorbing ideal in the semiring $\mathbb{Z}_0^+$. If $a \in \mathbb{Z}_0^+$ and m is the smallest positive integer such that $a^m \in I$, then $m \in \mathbb{N} \setminus \{rn + t : r \geq 1, 1 \leq t \leq r\}$.*

Proof. Let I be a semi-n-absorbing ideal in the semiring $\mathbb{Z}_0^+$. Let $a \in \mathbb{Z}_0^+$ and m be the smallest positive integer such that $a^m \in I$. Suppose that $m \in \{rn + t : r \geq 1, 1 \leq t \leq r\}$. Now $a^m \in I$. Therefore, $m = rn + t$ for some $r \geq 1$ and $1 \leq t \leq r$. So $rn + t \leq rn + r$. Now $a^{r(n+1)} \in I$ as $a^{rn+t} \in I$. Since I is a semi n-absorbing ideal $a^{rn} \in I$, a contradiction to $rn < rn + t = m$ and m is the smallest such that $a^m \in I$. Hence $m \in \mathbb{N} \setminus \{rn + t : r \geq 1, 1 \leq t \leq r\}$. □

Lemma 1.12. *Let $I = \langle a^m \rangle$ be a principal ideal in the semiring $\mathbb{Z}_0^+$. If I is a semi n-absorbing ideal, then $m \in \mathbb{N} \setminus \{rn + t : r \geq 1, 1 \leq t \leq r\}$.*

Proof. Since m is the least positive integer such that $a^m \in I$, by Lemma 1.11, $m \in \mathbb{N} \setminus \{rn + t : r \geq 1, 1 \leq t \leq r\}$. □

Corollary 1.13. *Let I be a semi-3-absorbing ideal in the semiring $\mathbb{Z}_0^+$. If $a \in \mathbb{Z}_0^+$ and m is the smallest positive integer such that $a^m \in I$, then $m \in \{1, 2, 3, 5, 6, 9\}$.*

Proof. Let I be a semi-3-absorbing ideal in the semiring $\mathbb{Z}_0^+$. Let $a \in \mathbb{Z}_0^+$ and m be the smallest positive integer such that $a^m \in I$. Suppose that $m \notin \{1, 2, 3, 5, 6, 9\}$.
Case i): $m = 4$. Now $a^4 \in I$ but $a^3 \notin I$, a contradiction.
Case ii): $m = 7$. Now $a^7 \in I$. Then $a^{12} = (a^3)^4 \in I$ but $(a^3)^3 \notin I$, a contradiction.
Case iii): $m = 8$. Now $a^8 = (a^2)^4 \in I$ but $(a^2)^3 = a^6 \notin I$, a contradiction.
Case iv): $m = 10$. Now $a^{10} \in I$. Then $a^9 \in I$, a contradiction.
Case v): $m = 11$. Now $a^{11} \in I$. Then $a^{12} = (a^3)^4 \in I$ but $(a^3)^3 \notin I$, a contradiction.
Case vi): If $m \geq 12$ and $4 \mid m$, then $m = 4t$ for some $t \geq 3$. Take $b = a^{\frac{m}{4}} = a^t$. Now $b^4 = (a^{\frac{m}{4}})^4 = a^m \in I \Rightarrow b^3 = (a^{\frac{m}{4}})^3 = a^{\frac{3m}{4}} \in I$ as I is a semi-3-absorbing ideal, a contradiction, since $\frac{3m}{4} < m$.
Case vii): $m > 12$ and $4 \nmid m$, then $m = 4t + r$ with $r = 1, 2, 3$, $t \geq 3$. Clearly, $\left[\frac{m}{4}\right]_l = t$ Take $b = a^{t+1}$. Now $b^4 = (a^{t+1})^4 = a^{4t+4} \in I \Rightarrow b^3 = (a^{t+1})^3 = a^{3(t+1)} \in I$ as I is a semi-3-absorbing ideal, a contradiction, since $3(t + 1) < m$. □

Theorem 1.14. *Let $I = \langle p^m \rangle$ be an ideal in the semiring $\mathbb{Z}_0^+$ where p is a prime number and $m \in \mathbb{N}$. Then I is a semi-n-absorbing ideal if and only if $\left[\frac{m}{n}\right]_s = \left[\frac{m}{n+1}\right]_s$.*

Proof. First suppose that $\left[\frac{m}{n}\right]_s = \left[\frac{m}{n+1}\right]_s$. Let $x^{n+1} \in I$ for some $x \in \mathbb{Z}_0^+$. Now $p^m \mid x^{n+1}$. Therefore $p \mid x^{n+1}$ as p is a prime number. Choose largest $r \in \mathbb{N}$ such that $p^r \mid x$. Then $x = p^r y$ where $y \in \mathbb{Z}_0^+$ and y is relatively prime to p. Now $p^m \mid x^{n+1} \Rightarrow p^m \mid (p^r y)^{n+1} \Rightarrow m \leq r(n+1)$ $\Rightarrow \frac{m}{n+1} \leq r \Rightarrow \left[\frac{m}{n+1}\right]_s \leq r$. Now $\frac{m}{n} \leq \left[\frac{m}{n}\right]_s = \left[\frac{m}{n+1}\right]_s \leq r$. Therefore $m \leq rn$. Therefore $p^m \mid (p^r y)^n$. Now $p^m \mid x^n$. Hence I is an semi n-absorbing ideal. Conversely, suppose that I is an semi n-absorbing ideal and suppose that $\left[\frac{m}{n}\right]_s > \left[\frac{m}{n+1}\right]_s$. Take $b = \left[\frac{m}{n+1}\right]_s$ and $x = p^b$. Now

$\frac{m}{n+1} \leq \left[\frac{m}{n+1}\right]_s = b \Rightarrow m \leq (n+1)b \Rightarrow x^{n+1} \in I$. Now $\left[\frac{m}{n+1}\right]_s < \left[\frac{m}{n}\right]_s \Rightarrow b+1 = \left[\frac{m}{n+1}\right]_s + 1 \leq \left[\frac{m}{n}\right]_s$ and $bn + n \leq \left[\frac{m}{n}\right]_s n < m + n$. This shows that $bn < m$, so $p^m \nmid x^n$, a contradiction to I is an semi n-absorbing ideal. Hence $\left[\frac{m}{n}\right]_s = \left[\frac{m}{n+1}\right]_s$. □

Theorem 1.15. *Let $I = \langle a \rangle$ be an ideal in the semiring $\mathbb{Z}_0^+$ and $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the ppf of a. Then I is a semi-n-absorbing ideal if and only if $\left[\frac{r_i}{n}\right]_s = \left[\frac{r_i}{n+1}\right]_s$ for all i.*

Proof. First suppose that $\left[\frac{r_i}{n}\right]_s = \left[\frac{r_i}{n+1}\right]_s$ for all i. Let $x^{n+1} \in I$ for some $x \in \mathbb{Z}_0^+$. Now $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid x^{n+1} \Rightarrow p_1 p_2 \cdots p_k \mid x^{n+1} \Rightarrow p_1 \mid x$, $p_2 \mid x$, $\cdots$ $p_k \mid x$ as each p_i is a prime number. Therefore $x = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \cdot y$ for some $y \in \mathbb{Z}_0^+$ such that y is relatively prime to each p_i. Now $a \mid x^{n+1}$ $\Rightarrow p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \mid (p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k} \cdot y)^{n+1} \Rightarrow r_i \leq (n+1)\beta_i \Rightarrow \frac{r_i}{n+1} \leq \beta_i \Rightarrow$ $\left[\frac{r_i}{n+1}\right]_s \leq \beta_i$, for all i. Now $\frac{r_i}{n} \leq \left[\frac{r_i}{n}\right]_s = \left[\frac{r_i}{n+1}\right]_s \leq \beta_i$. Therefore $r_i \leq n\beta_i$, for all i. Therefore $p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \mid (p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k} \cdot y)^n$. Now $a \mid x^n$. Hence I is an semi n-absorbing ideal. Conversely suppose that I is an semi n-absorbing ideal and suppose that $\left[\frac{r_i}{n}\right]_s > \left[\frac{r_i}{n+1}\right]_s$ for some i. Take $b_i = \left[\frac{r_i}{n+1}\right]_s$ and $x = p_i^{b_i} \prod\limits_{j \neq i} p_j^{r_j}$. Now $\frac{r_i}{n+1} \leq \left[\frac{r_i}{n+1}\right]_s = b_i$ implies $r_i \leq (n+1)b_i$, and hence $x^{n+1} \in I$. Now $\left[\frac{r_i}{n+1}\right]_s < \left[\frac{r_i}{n}\right]_s \Rightarrow$ $b_i + 1 = \left[\frac{r_i}{n+1}\right]_s + 1 \leq \left[\frac{r_i}{n}\right]_s$ and $b_i n + n \leq \left[\frac{r_i}{n}\right]_s n < r_i + n$. This shows that $b_i n < r_i$, so $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \nmid x^n$, a contradiction to I is an semi n-absorbing ideal. Hence $\left[\frac{r_i}{n}\right]_s = \left[\frac{r_i}{n+1}\right]_s$ for all i. □

Theorem 1.16. *Let I be an ideal of the semiring $\mathbb{Z}_0^+$ and $I = \langle p^m \rangle$ where p is a prime number and $m \in \mathbb{N}$. Then I is a semi n-absorbing ideal if and only if $m \in \mathbb{N} \setminus \{rn + t : r \geq 1, 1 \leq t \leq r\}$.*

Proof. Let I be a semi n-absorbing ideal of $\mathbb{Z}_0^+$ and $I = \langle p^m \rangle$ where p is a prime number and $m \in \mathbb{N}$. By Lemma 1.12, $m \in \mathbb{N} \setminus \{rn + t : r \geq 1, 1 \leq t \leq r\}$. Conversely, let $I = \langle p^m \rangle$ where $m \in \mathbb{N} \setminus \{rn + t : r \geq 1, 1 \leq t \leq r\}$. If $m = 1, 2, 3, \cdots n$, then I is a n-absorbing ideal (Theorem 2.5, [8]) and hence it is a semi n-absorbing ideal. Now assume that $m = r'n + t'$ where $1 \leq r' \leq n-1$ and $r'n \leq t' \leq (r'+1)n$. Then $\left[\frac{m}{n}\right]_s = r' + 1 = \left[\frac{m}{n+1}\right]_s$ and hence I is a semi n-absorbing ideal of R. □

Theorem 1.17. *(Theorem 2.4, [6]) Let I be a non-zero principal ideal in the semiring $\mathbb{Z}_0^+$. Then I is an irreducible ideal if and only if $I = \langle p^m \rangle$ for some prime number p and some $m \in \mathbb{N}$.*

From Theorem 1.16 and Theorem 1.17, we have the following corollary in which a characterization of principal irreducible semi n-absorbing ideals in the semiring $\mathbb{Z}_0^+$ is obtained.

Corollary 1.18. *Let I be a non-zero principal ideal in the semiring $\mathbb{Z}_0^+$. Then following statements are equivalent:*

1) *I is irreducible and semi n-absorbing ideal;*
2) *$I = \langle p^m \rangle$ for some prime number p where $m \in \mathbb{N} \setminus \{rn+t : r \geq 1, 1 \leq t \leq r\}$.*

Now the following Theorem gives a characterization of principal semi n-absorbing ideals in the semiring $\mathbb{Z}_0^+$.

Theorem 1.19. *A principal ideal I of $\mathbb{Z}_0^+$ is semi n-absorbing if and only if $I = \{0\}$ or $I = \langle m \rangle$ where $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the ppf of m and $r_i \in \mathbb{N} \setminus \{rn+t : r \geq 1, 1 \leq t \leq r\}$ for all i.*

Proof. Let I be a principal semi n-absorbing ideal of $\mathbb{Z}_0^+$ and $I \neq \{0\}$. Let $I = \langle m \rangle$ where $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the ppf of m. Suppose that $r_i \notin \mathbb{N} \setminus \{rn+t : r \geq 1, 1 \leq t \leq r\}$ for some i. We may assume that $r_1 \notin \mathbb{N} \setminus \{rn+t : r \geq 1, 1 \leq t \leq r\}$
Case i): $r_1 = rn + t$ where $1 \leq r \leq n-1$ and $1 \leq t \leq r$. Now $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \in \mathbb{Z}_0^+$ is such that $a^{n+1} \in I$ but $a^n \notin I$, a contradiction.
Case ii): $n^2 < r_1 \leq n(n+1)$. Now $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \in \mathbb{Z}_0^+$ is such that $a^{n(n+1)} = (a^n)^{n+1} \in I$ but $(a^n)^n = a^{n^2} \notin I$, a contradiction.
Case iii): If $r_1 > n(n+1)$ and $(n+1) \mid r_1$, then $r_1 = (n+1)t$. Now $a = p_1^{t} p_2^{r_2} \cdots p_k^{r_k} \in \mathbb{Z}_0^+$ is such that $a^{n+1} \in I$ but $a^n \notin I$, a contradiction.
Case iv): If $r_1 > n(n+1)$ and $(n+1) \nmid r_1$, then $r_1 = (n+1)t + r$ where $1 \leq r \leq n$ and $t \geq n$. Clearly $[\frac{r_1}{n+1}]_l = t$. Now $(n+1)([\frac{r_1}{n+1}]_l + 1) = (n+1)(t+1) = (n+1)t + (n+1) > r_1$ and $n([\frac{r_1}{n+1}]_l + 1) = n(t+1) = nt + n < nt + t + 1 = (n+1)t + 1 \leq r_1$. Then $a = p_1^{[\frac{r_1}{n+1}]_l + 1} p_2^{r_2} \cdots p_k^{r_k} \in \mathbb{Z}_0^+$ is such that $a^{n+1} = (p_1^{[\frac{r_1}{n+1}]_l + 1})^{n+1} p_2^{(n+1)r_2} \cdots p_k^{(n+1)r_k} = p_1^{(n+1)([\frac{r_1}{n+1}]_l + 1)} p_2^{(n+1)r_2} \cdots p_k^{(n+1)r_k} \in I$ as $(n+1)([\frac{r_1}{n+1}] + 1) > r_1$ but $a^n = (p_1^{[\frac{r_1}{n+1}]_l + 1})^n p_2^{nr_2} \cdots p_k^{nr_k} = p_1^{n([\frac{r_1}{n+1}]_l + 1)} p_2^{nr_2} \cdots p_k^{nr_k} \notin I$ as $n([\frac{r_1}{n+1}]_l + 1) < r_1$, a contradiction. Thus in any case we get a contradiction. Hence $r_i \in \mathbb{N} \setminus \{rn+t : 1 \leq r \leq n-1, 1 \leq t \leq r\}$ for all i. Conversely, if $I = \{0\}$, then clearly I is a semi n-absorbing ideal. Now suppose that $I = \langle m \rangle$ where $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the ppf of m and $r_i \in \mathbb{N} \setminus \{rn+t : 1 \leq r \leq n-1, 1 \leq t \leq r\}$ for all i. If $r_i \in \{1, 2, 3, \cdots n\}$, then $\left[\frac{r_i}{n}\right]_s = 1 = \left[\frac{r_i}{n+1}\right]_s$. Now we may assume that $r_i = ln + m$ where $1 \leq l \leq n-1$ and $l + 1 \leq m \leq (l+1)m$.

Then $\left[\frac{r_i}{n}\right]_s = l + 1 = \left[\frac{r_i}{n+1}\right]_s$. Thus $\left[\frac{r_i}{n}\right]_s = \left[\frac{r_i}{n+1}\right]_s$ for all i and hence by theorem 1.14, I is a semi n-absorbing ideal of R. □

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