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CONNECTIONS BETWEEN GRAPHS AND SHEAVES

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Abstract. In this paper, we discussed a method to construct a global sheaf space using graphs via Maximal compatibility blocks (MCB's) and we proposed the correspondence between graphs and sheaves. Further we discussed the sheaf constructions for various graphs using MCB's and vice-versa. We also presented some graph theoretical examples for the construction of sheaves.

1. Introduction

The concept of sheaves was introduced by Jean Leray in 1950's. Sheaves provide a mechanism for dealing with information at local and global levels. The theory has been successfully applied to areas like Coding Networks [8], Signal Processing [9]. Some applications of sheaves in computer science has been studied by Malcolm [3]. In particular the algebraic representations were studied extensively by researchers like Comer [10], Hofmann [6], Davey [2] and Swamy [11]. Swamy [11] and Wolf [1] gave mechanisms for construction of sheaves of a universal algebra based on Chinese Remainder Theorem. The effectiveness of the theory is visible in several mathematical disciplines. The theory of graphs is known for its versatility in applications. Recently, Joel Friedman[4] introduces a notion of a Sheaf vector spaces on a graph and studied homology theory for such sheaves. The authors studied construction of sheaves of sets [7] via tolerances and established connections with graphs. The motivation for the present work is to establish the connections between sheaves and graphs using maximal

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compatibility blocks as they play major role in establishing connections of graphs and also in practical applications of computer networks.
In this paper the sheaf terminology used is based on the theory developed by Swamy [11].
The motivation for the work is to establish a link between graphs and sheaf representations so that problems of one domain can be visualised in another domain and vice-versa to facilitate better understanding and better solutions.

2. Preliminaries

In this section we provide a fairly comprehensive overview of sheaf theory and present those parts of the theory that will be useful for our construction. We start with the following.

A sheaf (of sets) is a triple (S, π, X) where S, X are topological spaces and π is a surjective local homeomorphism of S onto X, for any $p \in X$, $\pi^{-1}(p)$ is a non empty set and is called the stalk at p, denote it by S_p and S is a disjoint union of all $S_p's$. For $Y \subseteq X$, a section on Y is a continuous map $f : Y \to S$ such that $\pi \circ f = id_Y$. Sections on X are called global sections. A global sheaf is a sheaf in which every element of the sheaf space is in the image of some global section. A sheaf (S, π, X) is said to be isomorphic with a sheaf (S', π', X') if there exists homeomorphisms $\alpha : S \to S'$ and $\beta : X \to X'$ such that $\pi' \circ \alpha = \beta \circ \pi$.
A relation R on X is said to be a tolerance or compatibility relation if it is reflexive and symmetric. Let X be a set and R be a compatibility relation on X. A subset $A \subseteq X$ is called a maximal compatibility block if any element of A is compatible to every other element of A and no element of $X - A$ is compatible to all the elements of A. We denote an MCB by $(abcd)$, where a, b, c, d are the elements of the set X.
A graph is an ordered pair $G = (V, E)$ comprising a set V of vertices together with a set E of edges, which are 2-element subsets of V. A graph is called a complete graph if every two vertices pair are joined by exactly one edge. A cycle graph is a graph that consists of a single cycle. A tree is a connected undirected graph with no cycles. There is a unique path between ever pair of vertices in G.

Proposition 2.1. *If (S, π, X) is a sheaf, then π is a continuous and an open map.*

Proof. Let (S, π, X) be a sheaf, which implies $\pi : S \to X$ is a surjection and local homeomorphism. Claim(1): $\pi : S \to X$ is continuous. Let W be an open set in X. Let $s \in \pi^{-1}(W)$. Since (S, π, X) is a sheaf, we can choose two open sets G and U in S and X respectively such that

$s \in G$, $\pi(s) \in U$ and $\pi|_G : G \to U$ is a homeomorphism. Since W is open in X and U is open in X, it follows that, $(\pi|_G)^{-1}(U \cap W)$ is open in G and hence it is open in S. Also $s \in G$ and $\pi|_G(s) = \pi(s) \in U \cap W$. Therefore, $s \in (\pi|_G)^{-1}(U \cap W) \subseteq \pi^{-1}(W)$. Thus $\pi^{-1}(W)$ is open in S. Hence $\pi : S \to X$ is continuous. Claim(2): $\pi : S \to X$ is open. Suppose H is an open set in S, it is enough to show that $\pi(H)$ is open in X. Let $p \in \pi(H)$. Choose $s \in H$ such that $\pi(s) = p$. Since (S, π, X) is a sheaf, we can choose two open sets G and U in S and X respectively such that $s \in G$, $\pi(s) = p \in U$ and $\pi|_G : G \to U$ is a homeomorphism. Since $(G \cap H)$ is open in G which implies $\pi|_G(G \cap H)$ is open in U and hence is open in X, since $s \in (G \cap H)$ we have $p = \pi(s) \in \pi|_G(G \cap H) \subseteq \pi(H)$ thus $\pi(H)$ is open in X. Hence $\pi : S \to X$ is open. ☐

Proposition 2.2. *Every section on an open set is an open map.*

Proof. Let (S, π, X) be a sheaf and $Y \subseteq X$. Suppose Y is open and $f : Y \to S$ is a section on Y.
Claim: $f : Y \to S$ is an open map. Let V be an open subset of Y. We have to prove that $f(V)$ is open in S. Let $s \in f(V)$ which implies there exists $p \in V$ such that $f(p) = s$. Choose two open sets G and U in S and X respectively such that $s \in G$, $\pi(s) \in U$ and $\pi|_G : G \to U$ is a homeomorphism. Now $(U \cap V)$ is open in U implies $H = (\pi|_G)^{-1}(U \cap V)$ is open in G and hence in S. Also $\pi(s) = \pi(f(p)) = (\pi \circ f)(p) = p \in U \cap V \Rightarrow s \in (\pi|_G)^{-1}(U \cap V) = H$. Further, if $t \in H$ then $f(\pi(t)) \in H \subseteq G$ and $\pi(t) = \pi(f(\pi(t))) \in H \subseteq G$ and since $\pi|_G : G \to U$ is one-one. Thus $s \in H \subseteq f(V)$ and H is open in S. Therefore $f(V)$ is open in S. ☐

3. Construction of Sheaves by Maximal Compatibility blocks

The construction of sheaf for an arbitrary set over a topological space using equivalence relations is discussed in [11]. M.P.K.Kishore et.al.,[7] discussed the construction of sheafs for an arbitrary set over a topological space using Tolerance relations. In this section we propose a method to construct a sheaf for an arbitrary set over a topological space using Maximal compatibility blocks.

Let X be a topological space and let A be a non-empty set. Let η be a tolerance relation on A. Let $Tol(A)$ denote the set of all tolerance relations on A. It can be observed that every tolerance relation generates a graph and denote the set of all maximal compatibility blocks with respect to η by $A//\eta$, that is,
$A//\eta = \{B | B \; is \; a \; Maximal \; Compatibility \; Block \; with \; respect \; to \; \eta\}$.

Suppose there is a map from X to $Tol(A)$ such that for each $p \in X$, η_p be the tolerance relation on A. Denote for each $a \in A$, $\eta_p(a) = B$ where B is a Maximal Compatibility Block with respect to η_p containing a, (if there are more than one MCB's arbitrarily select one of the MCB's). Let $S_p = A//\eta_p$ where $A//\eta_p = \{MCB's\ w.r.t.\ \eta_p\ \ on\ A\}$ and

$$\text{Define } S = \bigsqcup_{p \in X} S_p, \text{be the disjoint union of } S_p^{'}s.$$

For any $a \in A$, define $\hat{a} : X \to S$ by $\hat{a}(p) = \eta_p(a)$. Topologies S with the largest topology with respect to each $\hat{a}$ is continuous. Define $\pi : S \to X$ by $\pi(s) = \pi(\eta_p(a)) = p$ for all $s \in S_p$. Then (S, π, X) forms a triple.

Example 3.1. Let $G = (V, E)$ be the graph, where $V = \{a, b, c, d\}$, $E = \{(ab)(bc)(cd)(da)(bd)\}$, $X = \{p, q\}$ be a topological space and η_p $= \{(ab)(bc)(bd)(dc)(ad)\} \cup \{(ba)(cb)(db)(cd)(da)\} \cup \Delta$. For $a \in A$, $\eta_p(a) = (abd)$, $\eta_p(b) = (bcd)$ (select), $\eta_p(c) = (bcd)$, $\eta_p(d) = (abd)$ (select) and $S_p = A//\eta_p = \{(abd)(bcd)\}$, $\eta_q = \Delta = \{(aa)(bb)(cc)(dd)\}$ and $S_q = A//\eta_q = \{(a)(b)(c)(d)\}$ and

$$\text{Define } S = \bigsqcup_{p \in X} S_p, \text{be the disjoint union of } S_p^{'}s$$

$=\{((abd), p), ((bcd), p), ((a), q), ((b), q), ((c), q), ((d), q)\}$.

Now we prove a necessary and sufficient condition for the triple (S, π, X) to be a global sheaf. The proof of the following theorem is analogous to the similar proof given in [7], however in the following theorem $\eta_p(a) = \hat{a}(p)$ denote a Maximal Compatibility Block.

Theorem 3.2. *(S, π, X) is a global sheaf if and only if for any $a, b \in A$, $X(a, b) = \{p \in X | \hat{a}(p) = \hat{b}(p)\}$ is open in X [7].*

Proof. Let (S, π, X) be a global sheaf. First we prove that for $a \in A$, $\hat{a}$ is a global section. Continuity of $\hat{a}$ is clear from the definition. Also $\pi \circ \hat{a}(p) = \pi(\hat{a}(p)) = \pi(\eta_p(a)) = p$, for all $p \in X$. Therefore $\pi \circ \hat{a}$ is the identity and hence $\hat{a}$ is a global section. Now we claim that $X(a, b)$ is open in X. Let $p \in X(a, b)$ that is, $p \in X$ and $\hat{a}(p) = \hat{b}(p) = s(say)$, $s \in S$. By the definition of sheaf there exists open sets G and U in S and X respectively such that $s \in G$ and $\pi|_G : G \to U$ is a homeomorphism. Observe that $\pi(s) = \pi(\hat{a}(p)) = p$, $p \in U$. Now take $V = \hat{a}^{-1}(G) \cap \hat{b}^{-1}(G) \cap U$. Since $\hat{a}, \hat{b}$ are continuous and U is open, it follows that V is open in X and $p \in V$. Now for any $q \in V$, $\hat{a}(q)$,

$\hat{b}(q) \in G$ and $\pi(\hat{a}(q)) = \pi(\hat{b}(q))$. From the fact that $\pi|_G$ is a one-to-one map, it follows that $\hat{a}(q) = \hat{b}(q)$. Therefore $q \in X(a,b)$ and hence $X(a,b)$ is open.

Conversely assume that $X(a,b)$ is open in X. We now prove that (S,π,X) is a global sheaf. Let $s \in S$. Then there exist $p \in X$, $a \in A$ such that $s = \eta_p(a)$. Now since $\eta_p(a) = \hat{a}(p)$, $\hat{a}(p) \in \hat{a}(X)$ it follows that $s \in \hat{a}(X)$. We now prove that $\pi|_{\hat{a}(X)}$:$\hat{a}(X) \to X$ is a homeomorphism. Suppose $\pi|_{\hat{a}(X)}(\eta_p(a)) = \pi|_{\hat{a}(X)}(\eta_q(a))$. By definition of π, it follows that $p = q$. Thus, $\eta_p(a) = \eta_q(a)$ and hence $\pi|_{\hat{a}(X)}$ is one-one. Given $p \in X$, observe that $\pi|_{\hat{a}(X)}(\eta_p(a))$=$p$ for $a \in A$, $\eta_p(a) \in \hat{a}(X)$. Therefore $\pi|_{\hat{a}(X)}$ is onto. Let U be open in X and $s \in (\pi|_{\hat{a}(X)})^{-1}(U)$. Then $\pi|_{\hat{a}(X)}(s) \in U$. Now since $s \in S_p$ for some p, there exists $a \in A$ such that $s = \eta_p(a)$ and hence $\pi|_{\hat{a}(X)}(\eta_p(a)) \in U$. Since $\pi|_{\hat{a}(X)}(\eta_p(a))$=$p$, it follows that $p \in U$, clearly $\hat{a}(p) \in \hat{a}(U)$. From the fact that $\hat{a}$ is an open map, it is clear that $\hat{a}(U)$ is open in S. Let $s' \in \hat{a}(U)$. Then $s' = \hat{a}(q)(= \eta_q(a))$ for some $q \in U$.It can be observed that $\pi|_{\hat{a}(X)}(\eta_q(a)) \in U$ as $\pi(\eta_p(a)) = q$. Therefore $s' = \eta_q(a)$$\in (\pi|_{\hat{a}})^{-1}(U)$. Thus $\hat{a}(U) \subseteq (\pi|_{\hat{a}(X)})^{-1}(U)$ and hence $\pi|_{\hat{a}(X)}$ is continuous. Let H be an open set in $\hat{a}(X)$. By the subspace topology induced by S, there exists an open set G in S such that $H = \hat{a}(X) \cap G$. Let $s \in H$; then there exists $q \in X$ such that $s = \hat{a}(q)(= \eta_q(a))$, $s \in G$. Since $q \in \hat{a}^{-1}(G)$, consider $W = \hat{a}^{-1}(G) \cap X$. Clearly $q \in W$ and W is open in X. Now let $p \in W$, that is, $p \in \hat{a}^{-1}(G) \cap X$. Then $\hat{a}(p) \in G$ and since $\hat{a}(p) \in \hat{a}(X)$, it follows that $\hat{a}(p) \in \hat{a}(X) \cap G = H$. $p = \pi|_{\hat{a}(X)}(\hat{a}(p)) \in \pi|_{\hat{a}(X)}(H)$. Thus $\pi|_{\hat{a}(X)}$ is an open map. □

The proof of the following theorem is analogous to theorem (3.4) of [7]

Theorem 3.3. *For* $a,b \in A$. *Let* $< a,b > = \{\eta \in Tol(A) | \eta(a) = \eta(b)\ where \eta(a) denotes\ set\ of\ all\ MCB's\ containing\ a\}$. *Equip* $Tol(A)$ *with the topology for which* $\{< a,b > | a,b \in A\}$ *is a sub-base. Then* (S,π,X) *is a global sheaf if and only if* $f : p \mapsto \eta(p)$ *is continuous.*

I. **Construction of sheaf from the given graph:**
Let $G = (V,E)$ be a finite graph. That is V,E are finite and V is non-empty, $V = \{v_1, v_2, ..., v_n\}$. Let X denote discrete topology of sub graphs $G_i(= (V_i, E_i))$ of G such that $V(G) = V(G_i)$. That is $X_G = \{G_i | G_i$ is a sub graph of G and $V(G) = V(G_i)\}$. It can be observed that $E \cup \tilde{E} \cup \Delta$ is a tolerance relation on V, where $\tilde{E}$ denotes a converse of E and hence every graph can be uniquely identified with a tolerance relation on V. For each G_i, define the tolerance relation η_{G_i} by η_{G_i}=$E_i \cup \tilde{E}_i \cup \Delta$. Define $\eta_{G_i}(v)$=MCB containing v in G_i (if there

are more than one MCB containing v fix MCB that contains smallest indexed vertex v_j that immediately follows v). Every MCB shall be denoted as $(v_1v_2...v_k)$, where $v_1, v_2, ..., v_k$ are arranged in lexicographic order and shall be considered in cyclic order. That is v_k precedes v_1. Let $S_{G_i} = V//\eta_{G_i} = \{\eta_{G_i}(v)|v \in V\}$ and

$$\text{Define } S_G = \bigsqcup_{G_i \in X} S_{G_i}, \text{ be the disjoint union of } {S_{G_i}}'s.$$

For any $v \in V$, define $\hat{v}(G_i) = \eta_{G_i}(v)$. Topologies S with the largest topology for which each $\hat{v}$ is continuous. Define $\pi_G : S_G \to X_G$ by $\pi_G(s) = \pi_G(\eta_{G_i}(v)) = G_i$. Then (S_G, π_G, X_G) is a global sheaf, since for any $v_i, v_j \in V$, $\{G_i \in X|(v_i, v_j) \in \eta_{G_i}\}$ is open in X.

Example 3.4. Let $G = (V, E)$ be the given graph, where
$V = \{a, b, c, d\}$ be the vertices and $E = \{(ab), (ac), (bd)\}$ be the edges. Sub graphs of the above graph and the corresponding MCB's are as follows.
1. $G_1 = (V_1, E_1)$ where $V_1 = V$, $E_1 = \phi$ and the respective MCB's are $S_{G_1} = (a)(b)(c)(d)$
2. $G_2 = (V_2, E_2)$ where $V_2 = V$, $E_2 = \{(ab)\}$ and the respective MCB's are $S_{G_2} = (ab)(c)(d)$
3. $G_3 = (V_3, E_3)$ where $V_3 = V$, $E_3 = \{(ab)(bd)\}$ and the respective MCB's are $S_{G_3} = (ab)(bd)(c)$
4. $G_4 = (V_4, E_4)$ where $V_4 = V$, $E_4 = \{(ab)(ac)\}$ and the respective MCB's are $S_{G_4} = (ab)(ac)(d)$
5. $G_5 = (V_5, E_5)$ where $V_5 = V$, $E_5 = \{(ac)\}$ and the respective MCB's are $S_{G_5} = (ac)(b)(d)$
6. $G_6 = (V_6, E_6)$ where $V_6 = V$, $E_6 = \{(bd)\}$ and the respective MCB's are $S_{G_6} = (bd)(a)(c)$
7. $G_7 = (V_7, E_7)$ where $V_7 = V$, $E_7 = \{(ac)(bd)\}$ and the respective MCB's are $S_{G_7} = (ac)(bd)$
8. $G_8 = (V_8, E_8)$ where $V_8 = V$, $E_8 = \{(ab)(ac)(bd)\}$ and the respective MCB's are $S_{G_8} = (ab)(ac)(bd)$ and $S = \{\{(a), G_1\}, \{(b), G_1\}, \{(c), G_1\}, \{(d), G_1\}, \{(ab), G_2\},$
$\{(c), G_2\}, \{(d), G_2\}, \{(ab), G_3\}, \{(bd), G_3\}, \{(c), G_3\}, \{(ab), G_4\},$
$\{(ac), G_4\}, \{(d), G_4\}, \{(ac), G_5\}, \{(b), G_5\}, \{(d), G_5\}, \{(bd), G_6\},$
$\{(a), G_6\}, \{(c), G_6\}, \{(ac), G_7\}, \{(bd), G_7\}, \{(ab), G_8\}, \{(ac), G_8\},$
$\{(bd), G_8\}\}$. Now $X = \{G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8\}$ and the distinct MCB's are $\{(a), (b), (c), (d), (ab), (ac), (bd)\}$.
The sheaf space encapsulates the information of MCB's that are present in all possible subsets of G. Each $\hat{v}$ identifies the role of the vertex v with respect to the MCB's that it identifies in each of the sub graphs.

Each stalk S_{G_i} contains the information of the MCB's corresponding to the sub graph G_i and the projection map $\pi : S \to X$ identifies the sub graph to which the particular MCB belongs.

Observations on different types of graphs:
Observation(1): Null graph: Let $G = (V, E)$ be a null graph. That is $E = \phi$. let $X = \{\phi, G\}$ be the indiscrete topology.
$S_G = V//\eta_G = \{(v)|\ v \in V\}$, $S = S_G$. Each $\hat{v} : X \to S$ by $\hat{v}(G) = \eta_G(v) = (v), v \in V$. Define $\pi : S \to X$ by $\pi((v)) = \pi(\eta_G(v)) = G$. Therefore (S, π, X) is a sheaf with stalks consisting of MCB's with one element.
Observation(2): Complete graph: Let $G = (V, E)$ be the complete graph. Let $X = \{G_i|\ G_i$ is a sub graph of G and $V(G) = V(G_i)\}$ with discrete topology. X consists of $2^{(|V|\cdot(|V|-1))/2}$ sub graphs with discrete topology. There are $2^{(|V|\cdot(|V|-1))/2}$ number of stalks, where each stalk corresponds to an element of $P(E)$ ranging from null graph to complete graph. Hence there is a bijection between the number of elements in the stalks and the set of numbers $\{1, 2, 3, ..., 2^{|E|}\}$. Each $\hat{v} : X \to S$ by $\hat{v}(G_i)= \eta_{G_i}(v) = \{(v_1v_2...v_k)|v_{G_i} = (v_1v_2...v_k)\}$, for all $v \in V_i$. In this case (S, π, X) is a sheaf with $2^{|E_i|}$ number of elements.
Observation(3): Cycle graph: Let $G = (V, E)$ be the cycle graph such that $V = \{v_1, v_2, ..., v_n\}$ and $E = \{(v_i, v_{i+1})|1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$. Let X denotes $\{G_i|\ G_i$ is a sub graph of G and $V(G) = V(G_i)\}$ be a discrete topology. Then (S, π, X) is a sheaf with $2^{|V|}$ number of stalks such that every stalk contains MCB's contains two elements.
Observation(4): Tree: Let $G = (V, E)$ be a tree with $|V| = n$ and $|E| = |V| - 1$. The sheaf (S, π, X) consists of $2^{|V|-1}$ number of stalks with each stalk element contains at most two elements.
Observation: In general different topologies can be considered with sub graphs of G other than discrete topology.

Example 3.5. Let $G = (V, E)$ be the given graph, where $V = \{a, b, c, d\}$ be the vertices and $E = \{(ab), (bc), (cd), (ad), (bd)\}$ be the edges. The sub graphs are as follows. 1. $G_1 = \{(abd), (c)\}$ 2. $G_2 = \{(bcd), (a)\}$ 3. $G_3 = \{(bd), (a), (c)\}$ 4. $G_4 = \{(a), (b), (c), (d)\}$ 5. $G_5 = \{(abd), (bcd)\}$

Formula for number of stalks:$|E|_{C_0} + |E|_{C_1} + ...+ |E|_{C_{|E|}} = 2^{|E|}$

Definition 3.6. Sub-sheaf space : Let (S, π, X) be a sheaf space. A sheaf space (T, η, Y) is said to be sub-sheaf space of (S, π, X) if T is equipped with subspace topology of that of S and Y is equipped with subspace topology that of X and $\pi|_T = \eta$.

Observation: Let (S_G, π_G, X_G) be a sheaf space obtained for a graph G. Then for any sub graph G_i corresponding sheaf generated $(S_{G_i}, \pi_{G_i}, X_{G_i})$

is a sub-sheaf space of (S_G, π_G, X_G).
Observation: **Construction of a graph:** From the above constructed sheaf the original graph can be reconstructed by the following procedure. Step(i): Consider vertices corresponding to MCB's with single elements. Step(ii): Consider edges corresponding to remaining MCB's. That is construct edges between every pair of vertices in a given MCB.
II. **Construction of a graph from a sheaf of sets:** Let (S, π, X) be a sheaf of sets such that each element of the stalk S_p is a set. That is

$$S = \bigsqcup_{p\in X} S_p, [\text{ be the disjoint union of } {S_p}^{'}s$$

]and S_p=$\{s_1^p, s_2^p, ..., s_n^p\}$ where S_p is a sheaf of sets, and s_i^p =$\{a_{1_i}, a_{2_i}, ..., a_{l_i}\}$. For any $p \in X$ construct a graph $G_p = (V_p, E_p)$ where

$$V_p = \bigcup_{(1\leq i\leq n)} s_i^p,$$

$= \bigcup_{(1\leq i\leq n)} of \{a_{1_i}, a_{2_i}, ..., a_{l_i}\}$that is each a_{j_i} corresponds to a vertex and add an edge $(a_{m_i}a_{n_i}) \in E_p$ if $(a_{m_i}, a_{n_i}) \in s_i^p$ for some p. Now consider

$$G_S = \bigsqcup_{p\in X} G_p$$

where $G_S = (V_S, E_S)$,

$$V_S = \bigsqcup_{p\in X} V_p$$

and

$$E_S = \bigsqcup_{p\in X} E_p$$

Theorem 3.7. *Let (S, π, X) be a sheaf of finite sets such that each stalk $S_p, p \in X$, is a collection of finite sets. Let*

$$A = \bigcup_{p\in X} (\bigcup_{V\in S_p} V)$$

Let $\leq$ be an ordering on A such that $(A, \leq)$ is an ordered set. Let G_S be the graph constructed over the given sheaf as given in (II). Let Y be a topological space on sub graphs of G such that $\{G_p|\ p \in U,\ U$ is an open set in $X\}$ is open in Y. Let $(T,\ \eta,\ Y)$ be the sheaf constructed over G_S by the construction of a sheaf from the given graph as given in (I). Then $(S,\ \pi, X) \cong (T,\ \eta, Y)$.

Proof. Let $s \in S$, that is $s = s_i^p \in S_p$ for some $p \in X$ and $s_i^p = \{a_{1_i},\ a_{2_i}, ...,\ a_{r_i}\}$ define $\alpha : S \to T$ by $\alpha(s) = \eta_{G_p}(a_{k_i})$, where $a_{k_i} \in$

$s_i^p(= s)$, then choose a_{k_i} such that k_i is the least suffix. Define $\beta : X \to Y$ by $\beta(p) = G_p$.

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[d, "\alpha"'] & X \arrow[d, "\beta"] \\ T \arrow[r, "\theta"'] & Y \end{tikzcd}$$

FIGURE 1.

Observe that $(\beta \circ \pi)(s) = (\beta \circ \pi)(s_i^p) = \beta(\pi(s_i^p)) = \beta(p) = G_p$ and $(\theta \circ \alpha)(s) = (\theta \circ \alpha)(s_i^p) = \theta(\alpha(s_i^p)) = \theta(\eta_{G_p}(a_{k_i})) = G_p$ for a suitable a_{k_i} as given above. Hence $\beta \circ \pi = \theta \circ \alpha$.
Claim: β is an isomorphism. (i) β is one-one: Suppose $\beta(p) = \beta(q)$ which implies $G_p = G_q$ implies $S_p = S_q$ which implies $s_i^p = s_i^q$ so that $\pi(s_i^p) = \pi(s_i^q)$ which implies $p = q$ therefore β is one-one.(ii) β is onto: By the construction of Y, β is clearly onto. (iii) β is continuous: Let W be an open set in Y. $\beta^{-1}(W) = \{p \in X |\ G_p \in W\}$. By the construction of topology on Y, $\{G_p |\ p \in U\}$ is open in Y if and only if U is open in X. Clearly $\beta^{-1}(W)$ is open in X. Hence β is continuous, and hence β is an isomorphism.
Claim: $\alpha : S \to T$ be an isomorphism.(i) α is one-one: Let $\alpha(s) = \alpha(s')$ which implies $\eta_{G_p}(a_{k_i}) = \eta_{G_p}(a_{j_i})$ by the construction, the MCB of $\eta_{G_p}(a_{k_i}) =$ the MCB of $\eta_{G_p}(a_{j_i})$ implies the corresponding sets s, s' in (S, π, X) is same by the construction of the graph, implies $s = s'$. Therefore α is one-one. (ii) α is onto: Every element of T is in the form of $\eta_{G_p}(v)$ for some v, which corresponds to $s_i^p = \eta_{G_p}(v)$, by the construction of the graph and the sheaf. Hence α is onto. (iii) α is continuous: Let W be an open set in T. Since $\beta \circ \pi = \theta \circ \alpha$. Now $\alpha^{-1}(W) = \pi^{-1}(\beta^{-1}(\theta(w))$. Since $\pi,\ \beta,\ \theta$ being continuous and open, $\alpha^{-1}(W)$ is open in S. Hence α is continuous and open. Hence α is an isomorphism. □

Theorem 3.8. *Let $G = (V,\ E)$ be a graph and $(S_G,\ \pi_G,\ X_G)$ be the sheaf of sets constructed by (I). Let $G' = (V',\ E')$ be the graph constructed from $(S_G,\ \pi_G,\ X_G)$ by (II). Then G is isomorphic to G'.*

Proof. By the construction of the sheaf, every sub graph G_p contains all vertices of G and by the construction of the graph from the sheaf

$$V' = \bigcup_{p \in X} V'_p$$

where in

$$V'_p = \bigcup_{(1 \leq i \leq n)} s_i^P,$$

where each s_i^p corresponds to MCB of G_i which is a sub graph of G. Hence $V = V'$. Suppose $(v_i, v_j) \in E$. Let G_i be a sub graph containing E so that $(v_i, v_j) \in \eta_{G_i}$. By the construction of sheaf (v_i, v_j) belongs to some MCB $\eta_{G_i}(v)$ for some v. And by the construction of the graph given in (II) corresponding to every element in the sheaf space an MCB is constructed and hence $\eta_{G_i}(v)$ forms an MCB in the new graph G', as a result $(v_i, v_j) \in E'$. Then G is isomorphic to G'. □

Theorem 3.9. *Two graphs G, G' are isomorphic if and only if their corresponding sheaves constructed over discrete topology of sub graphs of G are isomorphic.*

Proof. Suppose G and G' are isomorphic. Let (S, π, X), (S', π', X') be two sheaves corresponding to G, G', and by the above construction of sheaves, since G, G' are isomorphic, the MCB's corresponding to G are isomorphic to MCB's corresponding to G', which implies (S, π, X) is isomorphic to (S', π', X'). Conversely suppose (S, π, X), (S', π', X') be two sheaves constructed corresponding to G, G' are isomorphic, X, X' are discrete topologies on G, G'. Since sheaves are isomorphic, the collection of MCB's on G, the collection of MCB's on G' are bijective. Thus G and G' are isomorphic. □

Example 3.10. Consider the graphs $G = (V, E)$ where $V = \{r, x, y, z, t\}$ be the vetices and $E = \{(rx), (xy), (yz), (zt), (tr), (ry), (yt)\}$ be the edges and another graph $G' = (V', E')$ where $V' = \{a, b, c, d, e\}$ be the vertices and $E' = \{(ab), (bc), (cd), (de), (ea), (ac), (ad)$ be the edges. The graphs G, G' are isomorphic. The MCB's of G are $\{(rty)(rxy)(ytz)\}$ and the MCB's of G' are $\{(ade)(adc)(abc)\}$.

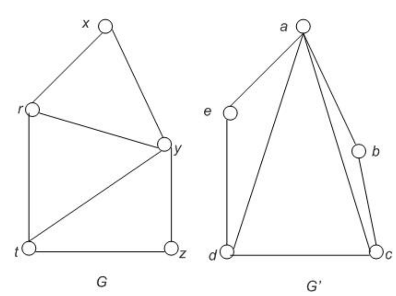

Figure 2.

Now Constructing new graphs H from G, H' from G' where each MCB becomes a vertex and whenever two MCB's share vertices add an edge between the corresponding vertices (MCB's) with an edge level of common vertices to both. The graphs $H = (V, E)$ where $V = \{(rxy), (rty), (ytz)\}$ be the vetices and $E = \{(ry), (yt), y\}$ be the edges and another graph $H' = (V', E')$ where $V' = \{ade, adc, abc\}$ be the vertices and $E' = \{(ad), (ac), a\}$ be the edges. Since both the graphs produce equivalent MCB's corresponding sheaves shall also be isomorphic when constructed on the same topological space.

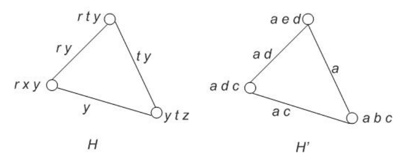

Figure 3.

Algorithm for testing isomorphism of graphs:
Let $G = (V, E)$, $G' = (V', E')$ be two graphs such that $|V| = |V'|$, $|E| = |E'|$
Step(i): Construct sheaves (S, π, X), (S', π', X') where X, X' are discrete topologies on G, G' respectively. If S is not isomorphic to S' then graphs are not isomorphic. Else
Step(ii): Construct a new graphs H from G, H' from G' where each MCB becomes a vertex and whenever two MCB's share vertices add an edge between the corresponding vertices (MCB's) with an edge level of common vertices to both.
Step(iii): Construct sheaves on H, H'. If the corresponding sheaves are not isomorphic, then the graphs G, G' are not isomorphic.
Step(iv): Repeat steps (i),(ii), (iii) until the multilevel sheafs constructed in successive steps are same.

4. Conclusion

In this paper, a method for construction of sheaves using maximal compatibility blocks is discussed. The one-to-one correspondence between graphs and sheaves is observed.

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