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HARARY SPECTRUM OF GENERALIZED COMPOSITION OF GRAPHS AND HARARY EQUIENERGETIC GRAPHS

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Abstract. The Harary spectrum of a connected graph G is the collection of the eigenvalues of its Harary matrix. The Harary energy of a graph G is the sum of absolute values of its Harary eigenvalues. Harary equitable partition is defined and is used to obtain Harary spectrum of generalized composition of graphs. Harary equienergetic graphs have been constructed with the help of generalized composition through Harary equitable partition.

1. Introduction

Thoughout this paper we consider simple, undirected graphs. Let G be a graph on n vertices with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *degree* of a vertex v in a graph G is the number of edges incident to it. If all vertices of G have same degree equal to r, then G is said to be an *r-regular graph*. The *distance* between the vertices v_i and v_j is the length of a shortest path joining them in G and is denoted by $d_G(v_i, v_j)$. The *diameter* of a graph G is the maximum distance between any pair of vertices and is denoted by $diam(G)$. For graph theoretic terminology we refer the book [7].

The *adjacency matrix* of a graph G is the square matrix $A(G) = [a_{ij}]$ of order n, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise.

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The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of G, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ [3].

The *Harary matrix* [9] (also called as *reciprocal distance matrix* [10]) of a graph G is a square matrix $H(G) = [h_{ij}]$ of order n, where

$$h_{ij} = \begin{cases} \frac{1}{d_G(v_i,v_j)}, & \text{if} \quad i \neq j \\ 0, & \text{if} \quad i = j. \end{cases}$$

It was used in the study of molecules in QSPR (quantitative structure property relationship) models [9].

In mathematical chemistry, topological index, also known as molecular descriptor, is a single number that can be used to characterize some property of the graph of a molecule. So far, hundreds of indices of graphs are introduced for various purposes. Among these, indices of graphs that are defined on the basis of distances in graphs represent a large family of molecular descriptors. In recent years, characterizing the extremal graphs in a given set of graphs with respect to some distance based topological index has become an important direction in chemical graph theory [21].

The Harary index defined as the sum of the reciprocal of the distances between all pairs of vertices can be derived from the Harary matrix. It has a number of interesting properties and it has shown a modest success in structure-property correlations [13, 20].

The eigenvalues of $H(G)$ labeled as $\xi_1, \xi_2, \ldots, \xi_n$ are said to be the *Harary eigenvalues* of G and their collection is called as *Harary spectrum*. The Harary index and the largest eigenvalue of the Harary matrix are well connected [22].

Let $\phi(M) = \phi(M : \xi)$ denotes the characteristic polynomial in ξ of matrix M. The collection of the eigenvalues of a matrix M is called its spectrum and is denoted by $\sigma(M)$.

Two non-isomorphic graphs are said to be *Harary cospectral* if they have same Harary eigenvalues. The results on Harary eigenvalues of a graph are obtained in [2, 4, 8, 22].

The *Harary energy* of a connected graph G, denoted by $\mathcal{E}_H(G)$, is defined as [5]

$$\mathcal{E}_H(G) = \sum_{i=1}^{n} |\xi_i|. \tag{1.1}$$

The Harary energy is defined in full analogy with the *ordinary graph energy*, defined as the sum of the absolute values of the eigenvalues of its adjacency matrix [6, 11]. The ordinary graph energy has a relation with the total π-electron energy of a molecule in quantum chemistry [11]. Bounds for the Harary energy of a graph are reported in [1, 2, 5].

Two connected graphs G_1 and G_2 of same order are said to be *Harary equienergetic* if $\mathcal{E}_H(G_1) = \mathcal{E}_H(G_2)$. Trivially, the Harary cospectral graphs are Harary equienergetic. In [15] the Harary energy of line graphs of certain regular graphs is obtained and thus constructed the Harary equienergetic graphs. Construction of pairs of Harary equienergetic graphs is given in [16] for $n \geq 6$ vertices.

Stevanović [19] has studied distance spectrum of joined union (generalized composition) of graphs and constructed distance equienergetic graphs. Lu et al. [12] defined distance equitable partition and have studied distance spectrum.

The equitable partition of a adjacency matrix of graph is given in [18] and of distance matrix is given in [12]. The distance equienergetic graphs have been constructed using generalized composition through distance equitable partition [17]. In this paper we use similar technique of [17] to report the equitable partition for Harary matrix and obtain the Harary spectrum of generalized composition of graphs. Further we construct the regular as well as non regular Harary equienergetic graphs having different spectra.

Definition 1.1. [18] If G is a graph with vertices $v_1, v_2, \ldots, v_n$, then the generalized composition of graphs, denoted by $G[G_1, G_2, \ldots, G_n]$, is formed by taking the disjoint graphs $G_1, G_2, \ldots, G_n$ and then joining every point of G_i to every point of G_j whenever v_i is adjacent to v_j in G.

Let $N(v)$ denotes the set of vertices which are adjacent to v in G.

Definition 1.2. [18] A partition $V_1 \cup V_2 \cup \ldots \cup V_k$ of a vertex set $V(G)$ is equitable or (a coloration [14]) if for each i and for all $v_1, v_2 \in V_i$, $|N(v_1) \cap V_j| = |N(v_2) \cap V_j|$ for all j.

The partition of $V(G)$ into singletons is always equitable. In generalized composition if a graph G is regular then $V(G)$ can be taken as partite set in an equitable partition. If P is an equitable partition, we associate with it an $k \times k$ matrix $Q = [q_{ij}]$, where $q_{ij} = |N(v) \cap V_j|$ for any $v \in V_i$. Such a matrix is called a *equitable partition matrix* (or *coloration matrix*).

We denote $d_D(v, S) = \sum_{u \in S} d_G(u, v)$ where $v \in V(G)$ and S is a non empty subset of $V(G)$ not containing v.

Definition 1.3. [12] Let G be a connetced graph and $\Pi : V_1 \cup V_2 \cup \ldots V_k$ be vertex set partition of $V(G)$. Then Π is called a distance equitable partition if for any $v \in V_i$, $d_D(v, V_j) = b_{ij}$ is a constant depending only on i, j $(1 \leq i, j \leq k)$. The matrix $B^*_\Pi = (b_{ij})_{k \times k}$ is called distance divisor matrix of G with respet to Π.

Let K_n denotes the complete graph on n vertices.

Proposition 1.4. [16] *Let G_1 and G_2 be r-regular, Harary equienergetic graphs of same order n and $diam(G_i) \leq 2$, $i = 1, 2$. Then for any regular graph G with $diam(G) \leq 2$, $K_2[G_1, G]$ and $K_2[G_2, G]$ are Harary equienergetic graphs.*

2. Harary Equitable Partition and Harary Spectrum of Generalized Composition of Graphs

We denote $d_H(v, S) = \sum_{u \in S} \frac{1}{d_G(u,v)}$, where $v \in V(G)$ and S is a non empty subset of $V(G)$ not containing v. Harary equitable partition of a graph G in terms of $d_H(v, S)$ is defined in the following way.

Definition 2.1. Let G be a connected graph and $\Pi : V_1 \cup V_2 \cup \ldots \cup V_k$ be a partition of the vertex set $V(G)$. Then Π is said to be a Harary equitable partition if for any $v \in V_i$, $d_H(v, V_j) = b_{ij}$ is constant depending on i, j $(1 \leq i, j \leq k)$. The matrix corresponding to the equitable partition Π, denoted by $H^*_\Pi = (b_{ij})_{k \times k}$, is called Harary divisor matrix.

Lemma 2.2. *Let G be a connected graph with a Harary equitable partition $\Pi : V_1 \cup V_2 \cup \ldots \cup V_k$ of the vertex set $V(G)$. Then $HC = CH^*_\Pi$, where H is the Harary matrix of G, and C and H^*_Π are characteristic matrix and Harary divisor matrix with respect to Π, respectively.*

Proof. The (v, j)-th entry of HC is

$$(HC)_{vj} = \sum_{u \in V_j} \frac{1}{d_G(v, u)} = d_H(v, V_j) = b_{ij}$$

and $(CH^*_\Pi)_{vj} = b_{ij},$ where $v \in V_i$. Hence the result follows. □

Theorem 2.3. *Let G be a connected graph and $\Pi : V_1 \cup V_2 \cup \ldots \cup V_k$ be a Harary equitable partition of G. Then $\phi(H^*_\Pi)$ divides $\phi(H)$, where H is the Harary matrix of G and H^*_Π is the Harary divisor matrix with respect to Π.*

Proof. If C is the characteristic matrix with respect to Π then C has rank k. Choose a matrix C^* of order $n \times (n - k)$ so that $(C|C^*)$ is an invertible matrix of order n, where $n = |V(G)|$. Then there exists two matrices Y and Z such that $HC^* = CY + C^*Z$. This equation along with Lemma 2.2 yields

$$H(C|C^*) = (C|C^*) \begin{pmatrix} H^*_\Pi & Y \\ O & Z \end{pmatrix}.$$

Which implies that $\det(\xi I - H) = \det(\xi I - H^*_\Pi) \det(\xi I - Z),$ since $(C|C^*)$ is invertible. Hence the result follows. □

Corollary 2.4. *Let G be a connected graph of order n with Harary equitable partition Π and Harary divisor matrix H_Π^* of G with respect to Π. Then the largest eigenvalue of H_Π^* is same as the largest Harary eigenvalue of G.*

Proof. If μ is the largest eigenvalue of H_Π^* with eigenvector y then $H_\Pi^* y = \mu y$. Consider $x = Cy$, where C is the characteristic matrix with respect to Π. Therefore by Lemma 2.2, we have
$Hx = H(Cy) = (HC)y = (CH_\Pi^*)y = C(H_\Pi^* y) = C(\mu y) = \mu(Cy) = \mu x$, which implies x is an eigenvector of H. If $y > 0$, then $x > 0$. Hence by Perron-Frobenius Theorem, μ is the largest Harary eigenvalue of G. $\square$

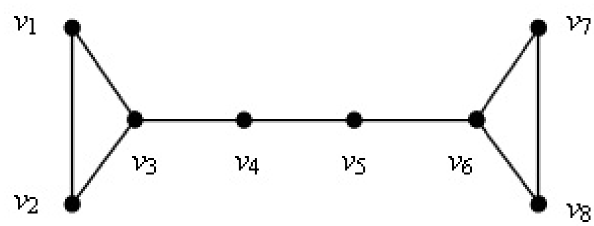

FIGURE 1.

Example 2.5. Consider the graph G as in Figure 1 with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. Consider the partition

$$\Pi : \{v_1, v_2\} \cup \{v_3\} \cup \{v_4\} \cup \{v_5\} \cup \{v_6\} \cup \{v_7, v_8\},$$

which constitute an equitable partition. The Harary divisor matrix corresponding to this equitable partition is

$$H_\Pi^* = \begin{pmatrix} 1 & 1 & 1/2 & 1/3 & 1/4 & 2/5 \\ 2 & 0 & 1 & 1/2 & 1/3 & 1/2 \\ 1 & 1 & 0 & 1 & 1/2 & 2/3 \\ 2/3 & 1/2 & 1 & 0 & 1 & 1 \\ 1/2 & 1/3 & 1/2 & 1 & 0 & 2 \\ 2/5 & 1/4 & 1/3 & 1/2 & 1 & 1 \end{pmatrix}.$$

The part of the spectrum due to H_Π^* is

$$\{3.9017, 1.3863, 0.0047, -0,7735, -1.1730, -1.3461\}$$

and that due to the partitions $\{v_1, v_2\}$ and $\{v_7, v_8\}$ is $\{-1, -1\}$. These two together form the Harary spectrum of G.

Theorem 2.6. *Let G be the connected graph of order n with vertices $u_1, u_2, \ldots, u_n$. For $i = 1, 2, \ldots, n$, let G_i be an r_i-regular graph of order k_i with eigenvalues $\mu_{i,1} = r_i \geq \mu_{i,2} \geq \cdots \geq \mu_{i,k_i}$ of adjacency matrix $A(G_i)$. Then the Harary spectrum of generalized composition $G[G_1, G_2, \ldots, G_n]$ consists of the eigenvalues $\frac{1}{2}(\mu_{i,j} - 1)$, $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, k_i$ and the eigenvalues of the Harary divisor matrix*

$$H^*_\Pi = \begin{bmatrix} \frac{k_1+r_1-1}{2} & \frac{k_2}{d_G(u_1,u_2)} & \frac{k_3}{d_G(u_1,u_3)} & \cdots & \frac{k_n}{d_G(u_1,u_n)} \\ \frac{k_1}{d_G(u_2,u_1)} & \frac{k_2+r_2-1}{2} & \frac{k_3}{d_G(u_2,u_3)} & \cdots & \frac{k_n}{d_G(u_2,u_n)} \\ \frac{k_1}{d_G(u_3,u_1)} & \frac{k_2}{d_G(u_3,u_2)} & \frac{k_3+r_3-1}{2} & \cdots & \frac{k_n}{d_G(u_3,u_n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k_1}{d_G(u_n,u_1)} & \frac{k_2}{d_G(u_n,u_2)} & \frac{k_3}{d_G(u_n,u_3)} & \cdots & \frac{k_n+r_n-1}{2} \end{bmatrix}.$$

Proof. The Harary matrix of the generalized composition $U = G[G_1, G_2, \ldots, G_n]$ is a block matrix

$$H(U) = \begin{bmatrix} \frac{J-I+A(G_1)}{2} & d_G(u_1,u_2)J & d_G(u_1,u_3)J & \cdots & d_G(u_1,u_n)J \\ d_G(u_2,u_1)J & \frac{J-I+A(G_2)}{2} & d_G(u_2,u_3)J & \cdots & d_G(u_2,u_n)J \\ d_G(u_3,u_1)J & d_G(u_3,u_2)J & \frac{J-I+A(G_3)}{2} & \cdots & d_G(u_3,u_n)J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_G(u_n,u_1)J & d_G(u_n,u_2)J & d_G(u_n,u_3)J & \cdots & \frac{J-I+A(G_n)}{2} \end{bmatrix},$$

where I and J are the identity and all one's matrices of suitable orders.

As G_i is an r_i-regular graph, r_i is its adjacency eigenvalue with eigenvector $\mathbf{j}$, where $\mathbf{j}$ is all one's vector, for $i = 1, 2, \ldots, n$ and remaining eigenvectors are orthogonal to $\mathbf{j}$. Any eigenvector y of G_i corresponding to an arbitrary eigenvalue $\mu \neq r_i$ satisfies $\mathbf{j}^T y = 0$. Then the eigenvector x of $H(U)$ corresponding to the eigenvalue $\frac{1}{2}(\mu - 1)$ is given by

$$x_w = \begin{cases} y_w, & w \in V_i \\ 0, & w \notin V_i \end{cases}$$

which can be justified as follows:

$$H(U)x = \begin{pmatrix} \frac{1}{d_G(u_1,u_i)}J \\ \vdots \\ \frac{1}{d_G(u_{i-1},u_i)}J \\ \frac{J-I+A(G_i)}{2} \\ \frac{1}{d_G(u_{i+1},u_i)}J \\ \vdots \\ \frac{1}{d_G(u_n,u_i)}J \end{pmatrix} y = \begin{pmatrix} \frac{1}{d_G(u_1,u_i)}Jy \\ \vdots \\ \frac{1}{d_G(u_{i-1},u_i)}Jy \\ \frac{Jy-Iy+A(G_i)y}{2} \\ \frac{1}{d_G(u_{i+1},u_i)}Jy \\ \vdots \\ \frac{1}{d_G(u_n,u_i)}Jy \end{pmatrix} = \frac{1}{2}(\mu - 1).$$

As G_i is an r_i-regular graph, the vertex set of G_i can be taken as a partite set. The remaining n eigenvalues can be obtained from the Harary divisor matrix with diagonal entries $\frac{k_i+r_i-1}{2}$ which is due to the fact that the graph is locally of diameter 2 in the generalized composition and non diagonal entries are $\frac{k_j}{d_G(u_i,u_j)}$, $i \neq j$. Hence the result follows. □

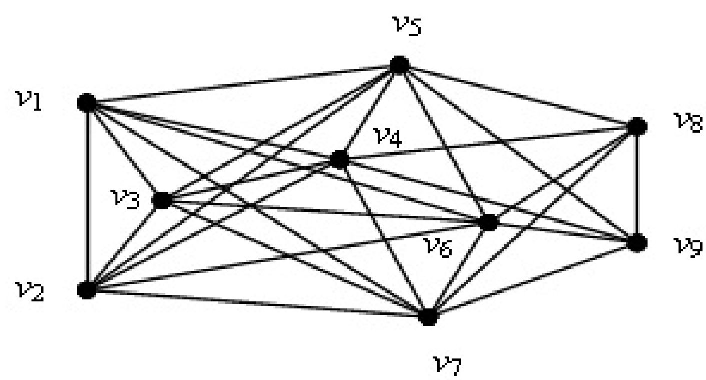

FIGURE 2. $P_3[K_3, C_4, K_2]$

Example 2.7. Consider the graph $G[H_1, H_2, H_3]$ with the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ as in Figure 2, where $G = P_3$, $H_1 = K_3$, $H_2 = C_4$ and $H_3 = P_2$. It is noted that each of H_1, H_2 and H_3 are regular connected components and can be taken as partite sets in the Harary equitable partition. The corresponding Harary divisor matrix is

$$H^*_\Pi = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 5/2 & 2 \\ 3/2 & 4 & 1 \end{pmatrix}.$$

The part of the spectrum due to H^*_Π is

$$\{7.1291, 0.2098, -1.8389\}$$

and that due to three partitions $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6, v_7\}$ and $\{v_8, v_9\}$ is

$$\{-1, -1, -1. -1/2, -1/2, -3/2\}.$$

These two together form the Harary spectrum of G.

Lemma 2.8. *Let G_1 be an r-regular graph of order n_1 and G_2 be a non regular graph of order n_2. Then the Harary spectrum of $K_2[G_1, G_2]$ is*

$$\sigma(H^*_\Pi)\bigcup\left(\sigma(H(G_1))\backslash\left\{\frac{n_1+r-1}{2}\right\}\right)$$

where H^*_Π is the Harary divisor matrix of $K_2[G_1, G_2]$.

Proof. Let $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$. As G_1 is a regular and G_2 is non regular, the partition $V(G_1)\bigcup(\bigcup_{i=1}^{n_2}\{v_i\})$ can be taken as Harary equitable partition and the Harary divisor matrix H^*_Π can be written as

$$\begin{pmatrix} \frac{n_1+r-1}{2} & J_{1\times n_2} \\ n_1 J_{n_2\times 1} & A(G_2)+\frac{1}{2}A(\overline{G_2}) \end{pmatrix}.$$

By Theorem 2.3,

$$\sigma(H(K_2[G_1, G_2])) = \sigma(H^*_\Pi)\bigcup\left(\sigma(H(G_1))\backslash\left\{\frac{n_1+r-1}{2}\right\}\right).$$

□

3. Harary equienergetic graphs

Theorem 3.1. *Let G be a connected graph of order n such that $\overline{G}$ also connected. Let G_{e_1} and G_{e_2} be an r-regular, Harary equienergetic graphs of same order k. If $G_1, G_2, \dots, G_i, \dots, G_n$ are any n graphs of orders $k_1, k_2, \dots, k_i, \dots, k_n$ respectively. Then $G[G_1, G_2, \dots, G_{e_1}, \dots, G_n]$ and $G[G_1, G_2, \dots, G_{e_2}, \dots, G_n]$ are Harary equienergetic, where G_i is replaced by G_{e_1} and G_{e_2} respectively for $1 \le i \le n$ in $G[G_1, G_2, \dots, G_i, \dots, G_n]$. In addition if $\overline{G_{e_1}}$ and $\overline{G_{e_2}}$ are Harary equienergetic, then $\overline{G[G_1, G_2, \dots, G_{e_1}, \dots, G_n]}$ and $\overline{G[G_1, G_2, \dots, G_{e_2}, \dots, G_n]}$ are also Harary equienergetic.*

Proof. If G_i is regular then $V(G_i)$ is a partite set and if G_i is non-regular then each vertex of G_i can be taken as a partite set for $1 \le i \le n$. These two kind of partite sets together form a Harary equitable partition of $G[G_1, G_2, \dots, G_i, \dots, G_n]$. The Harary divisor matrix corresponding to this Harary equitable partition of $G[G_1, G_2, \ldots, G_i, \ldots, G_n]$ for $G_i = G_{e_1}$ and $G_i = G_{e_2}$ is same and can be written as

$$H^*_\Pi = \begin{pmatrix} D_1 & \frac{b_{12}B_{12}}{d_G(u_1,u_2)} & \cdots & \frac{b_{1i}B_{1i}}{d_G(u_1,u_i)} & \cdots & \frac{b_{1n}B_{1n}}{d_G(u_1,u_n)} \\ \frac{b_{21}B_{21}}{d_G(u_2,u_1)} & D_2 & \cdots & \frac{b_{2i}B_{2i}}{d_G(u_2,u_i)} & \cdots & \frac{b_{2n}B_{2n}}{d_G(u_2,u_n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \frac{b_{i1}B_{i1}}{d_G(u_i,u_1)} & \frac{b_{i2}B_{i2}}{d_G(u_i,u_2)} & \cdots & D_i & \cdots & \frac{b_{in}B_{in}}{d_G(u_i,u_n)} \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \frac{b_{n1}B_{n1}}{d_G(u_n,u_1)} & \frac{b_{n2}B_{n2}}{d_G(u_n,u_2)} & \cdots & \frac{b_{ni}B_{ni}}{d_G(u_n,u_i)} & \cdots & D_n \end{pmatrix}, \tag{3.1}$$

where

$$D_i = \begin{cases} A(G_i) + \frac{1}{2}A(\overline{G_i}), & \text{if } G_i \text{ is a non-regular graph} \\ \frac{k_i+r_i-1}{2}, & \text{if } G_i \text{ is an } r_i\text{-regular graph,} \end{cases}$$

$$b_{ij} = \begin{cases} k_j & \text{if } G_i \text{ is non-regular and } G_j \text{ is regular} \\ k_j & \text{if } G_i \text{ and } G_j \text{ both are regular} \\ 1 & \text{if } G_i \text{ is regular and } G_j \text{ is non-regular} \\ 1 & \text{if } G_i \text{ and } G_j \text{ both are non-regular,} \end{cases}$$

and

$$B_{ij} = \begin{cases} J_{k_i\times 1} & \text{if } G_i \text{ is non-regular and } G_j \text{ is regular} \\ 1 & \text{if } G_i \text{ and } G_j \text{ both are regular} \\ J_{1\times k_j} & \text{if } G_i \text{ is regular and } G_j \text{ is non-regular} \\ J_{k_i\times k_j} & \text{if } G_i \text{ and } G_j \text{ both are non-regular.} \end{cases}$$

By Theorem 2.3,
$\sigma(H(G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]))$

$$= \left(\bigcup_{\substack{j=1 \\ j\neq i \\ \text{whenever } G_j \text{ is } r_j\text{-regular}}}^{n} \left[\sigma(H(G_j))\setminus\left\{\frac{k_j+r_j-1}{2}\right\}\right] \right)$$

$$\bigcup \left(\sigma(H(G_{e_1}))\setminus\{\frac{k+r-1}{2}\}\right) \bigcup \sigma(H^*_\Pi).$$

Therefore

$$\begin{aligned}
&\mathcal{E}_H(G[G_1,G_2,\ldots,G_{e_1},\ldots,G_n]) \\
&= \left(\sum_{\substack{j=1 \\ j\neq i \\ \text{whenever } G_j \text{ is } r_j\text{- regular}}}^{n} \left[\mathcal{E}_H(G_j) - \frac{k_j+r_j-1}{2} \right] \right) \\
&\quad + \ \mathcal{E}_H(G_{e_1})) - \frac{k+r-1}{2} + \mathcal{E}(H^*_\Pi), \qquad (3.2)
\end{aligned}$$

where $\mathcal{E}(H^*_\Pi)$ is the sum of the absolute values of the eigenvalues of H^*_Π.

Similarly

$$\begin{aligned}
&\mathcal{E}_H(G[G_1,G_2,\ldots,G_{e_2},\ldots,G_n]) \\
&= \left(\sum_{\substack{j=1 \\ j\neq i \\ \text{whenever } G_j \text{ is } r_j\text{- regular}}}^{n} \left[\mathcal{E}_H(G_j) - \frac{k_j+r_j-1}{2} \right] \right) \\
&\quad + \ \mathcal{E}_H(G_{e_2})) - \frac{k+r-1}{2} + \mathcal{E}(H^*_\Pi). \qquad (3.3)
\end{aligned}$$

Since G_{e_1} and G_{e_2} are Harary equienergetic, the result follows by Eqs. (3.2) and (3.3).

In addition if $\overline{G_{e_1}}$ and $\overline{G_{e_2}}$ are Harary equienergetic, then $\overline{G[G_1,G_2,\ldots,G_{e_1},\ldots,G_n]}$ and $\overline{G[G_1,G_2,\ldots,G_{e_2},\ldots,G_n]}$ are also Harary equienergetic. The proof follows with the same Harary equitable partition by incorporating the following changes in Eq. (3.1):

$$D_i = \begin{cases} A(\overline{G_i}) + \frac{1}{2}A(G_i) & \text{if } G_i \text{ is a non-regular graph} \\ k_i - \frac{r_i}{2} - 1 & \text{if } G_i \text{ is an } r_i\text{-regular graph.} \end{cases}$$

Hence

$$\begin{aligned}
&\mathcal{E}_H\left(\overline{G[G_1,G_2,\ldots,G_{e_1},\ldots,G_n]}\right) \\
&= \left(\sum_{\substack{j=1 \\ j\neq i \\ \text{whenever } G_j \text{ is } r_j\text{-regular}}}^{n} \left[\mathcal{E}_H(\overline{G_j}) - \left(k_j - \frac{r_j}{2} - 1\right) \right] \right) \\
&\quad + \ \mathcal{E}_H(\overline{G_{e_1}}) - \left(k - \frac{r}{2} - 1\right) + \mathcal{E}(H^*_\Pi) \qquad (3.4)
\end{aligned}$$

and

$$\mathcal{E}_H\left(\overline{G[G_1, G_2, \ldots, G_{e_2}, \ldots, G_n]}\right)$$

$$= \left(\sum_{\substack{j=1 \\ j\neq i \\ \text{whenever } G_j \text{ is } r_j\text{-regular}}}^{n} \left[\mathcal{E}_H(\overline{G_j}) - \left(k_j - \frac{r_j}{2} - 1\right)\right]\right) + \mathcal{E}_H(\overline{G_{e_2}}) - \left(k - \frac{r}{2} - 1\right) + \mathcal{E}(H^*_\Pi), \tag{3.5}$$

where $\mathcal{E}(H^*_\Pi)$ is the sum of the absolute values of the eigenvalues of H^*_Π.

Since $\overline{G_{e_1}}$ and $\overline{G_{e_2}}$ are Harary equienergetic, the result follows by Eqs. (3.4) and (3.5). □

Corollary 3.2. *Let G be a connected graph of order n. Let G_{e_1} and G_{e_2} be an r-regular, Harary equienergetic graphs of same order k. If $G_1, G_2, \ldots, G_i, \ldots, G_n$ are regular graphs of orders $k_1, k_2, \ldots, k_i, \ldots, k_n$ with degrees $r_1, r_2, \ldots, r_i, \ldots, r_n$ respectively. Then $G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]$ and $G[G_1, G_2, \ldots, G_{e_2}, \ldots, G_n]$ are Harary equienergetic, where G_i is replaced by G_{e_1} and G_{e_2} respectively for $1 \leq i \leq n$ in $G[G_1, G_2, \ldots, G_i, \ldots, G_n]$. In addition if $\overline{G_{e_1}}$ and $\overline{G_{e_2}}$ are Harary equienergetic, then $\overline{G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]}$ and $\overline{G[G_1, G_2, \ldots, G_{e_2}, \ldots, G_n]}$ are also Harary equienergetic.*

Proof. As $G_1, G_2, \ldots, G_i, \ldots, G_n$ are regular graphs, $\bigcup_{i=1}^n V(G_i)$ is a Harary equitable partition of $G[G_1, G_2, \ldots, G_i, \ldots, G_n]$. The Harary divisor matrix corresponding to this Harary equitable partition of $G[G_1, G_2, \ldots, G_i, \ldots, G_n]$ for $G_i = G_{e_1}$ and $G_i = G_{e_2}$ is same and can be written as

$$H^*_\Pi = \begin{pmatrix} \frac{k_1+r_1-1}{2} & \frac{k_2}{d_G(u_1,u_2)} & \cdots & \frac{k_i}{d_G(u_1,u_i)} & \cdots & \frac{k_n}{d_G(u_1,u_n)} \\ \frac{k_1}{d_G(u_2,u_1)} & \frac{k_2+r_2-1}{2} & \cdots & \frac{k_i}{d_G(u_2,u_i)} & \cdots & \frac{k_n}{d_G(u_2,u_n)} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \frac{k_1}{d_G(u_i,u_1)} & \frac{k_2}{d_G(u_i,u_2)} & \cdots & \frac{k_i+r_i-1}{2} & \cdots & \frac{k_n}{d_G(u_i,u_n)} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \frac{k_1}{d_G(u_n,u_1)} & \frac{k_2}{d_G(u_n,u_2)} & \cdots & \frac{k_i}{d_G(u_n,u_i)} & \cdots & \frac{k_n+r_n-1}{2} \end{pmatrix}. \tag{3.6}$$

By Theorem 2.3,
$\sigma(H(G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]))$

$$= \bigcup_{\substack{j=1 \\ j\neq i}}^{n} \left[\sigma(H(G_j)) \setminus \left\{\tfrac{k_j+r_j-1}{2}\right\}\right] \bigcup \left(\sigma(H(G_{e_1})) \setminus \left\{\tfrac{k+r-1}{2}\right\}\right) \bigcup \sigma(H^*_\Pi).$$

Therefore

$$\mathcal{E}_H\left(G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]\right) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[\mathcal{E}_H(G_j) - \frac{k_j + r_j - 1}{2}\right] + \mathcal{E}_H(G_{e_1}) - \frac{k+r-1}{2} + \mathcal{E}(H^*_\Pi). \quad (3.7)$$

Similarly

$$\mathcal{E}_H\left(G[G_1, G_2, \ldots, G_{e_2}, \ldots, G_n]\right) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[\mathcal{E}_H(G_j) - \frac{k_j + r_j - 1}{2}\right] + \mathcal{E}_H(G_{e_2}) - \frac{k+r-1}{2} + \mathcal{E}(H^*_\Pi). \quad (3.8)$$

Since G_{e_1} and G_{e_2} are Harary equienergetic, the result follows by Eqs. (3.7) and (3.8).

In addition if $\overline{G_{e_1}}$ and $\overline{G_{e_2}}$ are Harary equienergetic, then $\overline{G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]}$ and $\overline{G[G_1, G_2, \ldots, G_{e_2}, \ldots, G_n]}$ are also Harary equienergetic. The proof follows with the same Harary equitable partition by incorporating changes in the diagonal entries of the Harary divisor matrix (3.6) as $k_i - \frac{r_i}{2} - 1$ for $1 \leq i \leq n$. Hence

$$\mathcal{E}_H\left(\overline{G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]}\right) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[\mathcal{E}_H(\overline{G_j}) - \left(k_j - \frac{r_j}{2} - 1\right)\right] + \mathcal{E}_H(\overline{G_{e_1}}) - \left(k - \frac{r}{2} - 1\right) + \mathcal{E}(H^*_\Pi) \quad (3.9)$$

and

$$\mathcal{E}_H\left(\overline{G[G_1, G_2, \ldots, G_{e_2}, \ldots, G_n]}\right) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[\mathcal{E}_H(\overline{G_j}) - \left(k_j - \frac{r_j}{2} - 1\right)\right] + \mathcal{E}_H(\overline{G_{e_2}}) - \left(k - \frac{r}{2} - 1\right) + \mathcal{E}(H^*_\Pi). \quad (3.10)$$

Since $\overline{G_{e_1}}$ and $\overline{G_{e_2}}$ are Harary equienergetic, the result follows by Eqs. (3.9) and (3.10). $\square$

Corollary 3.3. *Let G be a connected graph of order n. Let G_{1i} and G_{2i} be Harary equienergetic, regular graphs of same order k_i and of same degree r_i, $i = 1, 2, \ldots, n$. Then $G[G_{11}, G_{12}, \ldots, G_{1n}]$ and $G[G_{21}, G_{22}, \ldots, G_{2n}]$ are Harary equienergetic.*

Proof. As G_{1i} and G_{2i} are regular graphs, $i = 1, 2, \ldots, n$, the partitions $\bigcup_{i=1}^{n} V(G_{1i})$ and $\bigcup_{i=1}^{n} V(G_{2i})$ are Harary equitable partitions of $G[G_{11}, G_{12}, \ldots, G_{1n}]$ and of $G[G_{21}, G_{22}, \ldots, G_{2n}]$ respectively. The Harary divisor matrix corresponding to these Harary equitable partitions is same and can be written as

$$H_{\Pi}^{*} = \begin{pmatrix} \frac{k_1+r_1-1}{2} & \frac{k_2}{d_G(u_1,u_2)} & \cdots & \frac{k_i}{d_G(u_1,u_i)} & \cdots & \frac{k_n}{d_G(u_1,u_n)} \\ \frac{k_1}{d_G(u_2,u_1)} & \frac{k_2+r_2-1}{2} & \cdots & \frac{k_i}{d_G(u_2,u_i)} & \cdots & \frac{k_n}{d_G(u_2,u_n)} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \frac{k_1}{d_G(u_i,u_1)} & \frac{k_2}{d_G(u_i,u_2)} & \cdots & \frac{k_i+r_i-1}{2} & \cdots & \frac{k_n}{d_G(u_i,u_n)} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \frac{k_1}{d_G(u_n,u_1)} & \frac{k_2}{d_G(u_n,u_2)} & \cdots & \frac{k_i}{d_G(u_n,u_i)} & \cdots & \frac{k_n+r_n-1}{2} \end{pmatrix}.$$

By Theorem 2.3,

$$\sigma(H(G[G_{11}, G_{12}, \ldots, G_{1n}])) = \bigcup_{i=1}^{n} \left[\sigma(H(G_{1i})) \setminus \left\{ \frac{k_i + r_i - 1}{2} \right\} \right] \bigcup \sigma(H_{\Pi}^{*}).$$

Therefore

$$\mathcal{E}_H \left(G[G_{11}, G_{12}, \ldots, G_{1n}] \right) = \sum_{i=1}^{n} \left[\mathcal{E}_H(G_{1i}) - \frac{k_i + r_i - 1}{2} \right] + \mathcal{E}(H_{\Pi}^{*}). \tag{3.11}$$

Similary

$$\mathcal{E}_H \left(G[G_{21}, G_{22}, \ldots, G_{2n}] \right) = \sum_{i=1}^{n} \left[\mathcal{E}_H(G_{2i}) - \frac{k_i + r_i - 1}{2} \right] + \mathcal{E}(H_{\Pi}^{*}). \tag{3.12}$$

Since $\mathcal{E}_H(G_{1i}) = \mathcal{E}_H(G_{2i})$, $i = 1, 2, \ldots, n$, by Eqs. (3.11) and (3.12), we have

$$\mathcal{E}_H \left(G[G_{11}, G_{12}, \ldots, G_{1n}] \right) = \mathcal{E}_H \left(G[G_{21}, G_{22}, \ldots, G_{2n}] \right).$$

□

Remark 3.4. In Corollary 3.2, if $G = K_n$ and $k_1 - r_1 = k_2 - r_2 = \cdots = k - r = \cdots = k_n - r_n = p$ and $k_1 + k_2 + \cdots + k + \cdots + k_n = s$, then $G[G_1, G_2, \ldots, G_{e_1}, \ldots, G_n]$ and $G[G_1, G_2, \ldots, G_{e_2}, \ldots, G_n]$ are of order s and of degree $t = s - p$ with $\mathcal{E}(H_{\Pi}^{*}) = t + (n-1)(s-t)$.

This way of construction enables to construct family of regular Harary equienergetic graphs.

Remark 3.5. If the conditions in the above Remark 3.4 are not satisfied, we get family of non-regular Harary equienergetic graphs.

Remark 3.6. The proposed method of construction leads to a family of co-spectral or non co-spectral Harary equienergetic graphs when a pair (G_{e_1}, G_{e_2}) of co-spectral or non co-spectral Harary equienergetic regular graphs of same order and of same degree are considered.

Remark 3.7. Proposition 1.4 is a particular case of Corollary 3.2.

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